This is partly a review and many people have contributed to the subject
Some recent work on spectral dimension with Bergfinnur Durhuus and Thordur Jonsson
1. From quantum gravity to graphs

2. Large scale structure

3. Some graph ensembles
   - Combs
   - Trees
   - Triangulations

4. Open questions
Gravity’s dynamical degree of freedom is the metric $g_{\mu\nu}(x,t)$

Classically $g_{\mu\nu}(x,t)$ obeys Einstein’s equations:

$$g_{\mu\nu}(x,0) \rightarrow g_{\mu\nu}(x,t)$$

Quantum mechanics is different:

$$\langle g^b(x), t=T \mid g^a(x), t=0 \rangle \sim \text{space}$$

Probability amplitude for evolution from $g^a$ to $g^b$
How is $\Gamma$ defined?

In the discretized approach by triangulation, in 2d...

1. Unconstrained -- Planar Random Graphs

2. Constrained -- Causal Triangulations

$$g_{\mu\nu}(x,t) \rightarrow \text{geodesic distance} \sim a \times \text{graph distance } R$$

continuum $R \rightarrow \infty$, $a \rightarrow 0$
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Physics depends on large scale properties
2. Large scale structure

In fact many interesting physical systems can be expressed in terms of ensembles of graphs generated by local rules eg

- Percolation clusters
- Generic random trees
- Planar random graphs
- Causal dynamical triangulations

A simple way to characterize the typical large scale properties of graphs in these ensembles is through the notion of \textit{dimension}.
Hausdorff dimension $d_H$ -- we assume $\infty$ graphs

1. Choose a point $r_0$
2. Find all points $B_R(r_0)$ within graph distance $R$ of $r_0$
3. $|B_R(r_0)| \sim R^{d_H}$ as $R \to \infty$, independent of $r_0$

$d_H$ tells us about the volume distribution but is blind to some sorts of connectivity eg

\[ d_H = 2 \text{ for } \mathbb{Z}^2 \]  

and GRT
Spectral dimension $d_s$

1. Choose a point $r_0$

2. Random walker leaves $r_0$ at time 0 and returns at time $t$ with probability $q_G(t;r_0)$

$$\text{prob} = \sigma^{-1} \quad q_G(t;r_0) \sim t^{-d_s/2} \text{ as } t \to \infty$$

Random walk sees connectivity:

$d_s = 2$ for $\mathbb{Z}^2$ but $4/3$ for GRT
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Recurrence

\[ q_G(t;r_0) = \bullet + \bullet + \bigcirc + \bigcirc + \ldots \]

\[ Q_G(x) = 1 + \sum_{t=2}^{\infty} q_G(t;r_0) (1-x)^{t/2} \]

\[ = \frac{1}{1 - P_G(x)} \]

1. If \( d_S > 2 \) then \( Q_G(0) \) finite \( \Rightarrow 1 - P_G(0) > 0 \), walker can escape, graph is non-recurrent.

2. If \( Q_G(0) \) infinite \( \Rightarrow 1 - P_G(0) = 0 \), walker always comes back, graph is recurrent, and \( d_S \leq 2 \)
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3. Some graph ensembles: Combs

Tooth length i.i.d. $\pi(n) \sim n^{-a}$, $n \to \infty$

$\langle | B_R(r_0) | \rangle \sim R^{d_H}$ with $d_H = 3 - a$, $1 < a \leq 2$
and $1$ if $a > 2$

$\langle q_G(t;r_0) \rangle \sim t^{-d_S/2}$ with $d_S = 2 - a/2$, $1 < a \leq 2$
and $1$ if $a > 2$

Intuition? It is the very long teeth which matter....

Generic Random Tree  eg binary tree

\[ Z = g + g Z^2 \]

so

\[ Z = \frac{1 - (1 - 4g^2)^{1/2}}{2g} \]

At \( g = \frac{1}{2} \) we get a Critical Galton Watson ensemble

Special case of

\[ f(x) = \sum p_n x^n \]

CGW if

\[ f(1) = f'(1) = 1, \quad f''(1) < \infty \]
Generic Random Trees are the $\infty$ trees, measure $\mu_\infty$

\[
\langle | B_R(r_0) | \rangle_{\mu_\infty} \sim R^{d_H} \quad \text{with} \quad d_H = 2
\]

\[
\langle q_G(t;r_0) \rangle_{\mu_\infty} \sim t^{-d_S/2} \quad \text{with} \quad d_S = 4/3
\]
So far we have discussed ensemble average quantities.

Sometimes there are stronger results:
There may be a subset of graphs which appear with measure 1 and all have the same property eg for the GRT

\[ | B_R(r_0) | \sim R^2 \text{ up to } \log R \text{ factors} \]

Clearly there are infinite trees for which \( d_H \neq 2 \)
but they are rare -- they have measure 0

- \( d_H = 2 \) a.s.
- \( d_S = 4/3 \) a.s.
3. Some graph ensembles: Causal Triangulations

arXiv:0908.3643
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\[ w_G = \prod_{v \in G} g_{v+1} \]

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Critical at \( g_c = 1/2 \)

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Critical at \( g_c = 1/2 \)

at \( g_c \) the trees are CGW with offspring probability

\[ p_n = (1/2)^{n+1} \]

\( \mu(\infty \text{ CDT}) \Leftrightarrow \mu(\text{URT}) \)

Uniform RT is a particular GRT
• Every vertex in a CT appears in the associated URT so 
  \( d_H = 2 \) a.s.

• First return probability \( P_G(0) = 1 \) a.s. so recurrent and 
  \( d_S \leq 2 \) a.s.

• Very weak lower bound from deleting links until only the 
  URT remains

  \( d_S \geq 4/3 \) a.s.

-- but expect loops to be important so consider .....
This has a chain structure and (trivial) loops. It is recurrent a.s. and has

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CT results don't depend on URT -- for every GRT law there is a local action for the CT
\[ w_G = \prod_{v \in G} \tau_v \]
e.g. CT+dimer model of Di Francesco et al
4. Open questions

- Do CTs have $d_S = 2$ a.s.?
- Are PRGs recurrent a.s., what is $d_S$?
- What do other probes e.g., Ising spins show on CTs?
- Can the corresponding annealed systems be controlled?
- What can be said about higher dimensional CTs?
Theorem: 2d CDTs are a.s. recurrent

Nash-Williams criterion: if electrical resistance to infinity is infinite, $G$ is recurrent

$$\mu(L_n > K) = \frac{K + 2n - 1}{2n - 1} \left(1 - \frac{1}{2n}\right)^K$$

Resistance of $G \geq \sum_n \frac{1}{L_n}$

so if $K \gg n$, then $\mu$ very small

$L_n$ distribution determined by $\mu_\infty$
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$G = \{1, 2, \ldots\}$ superconductor

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\[ \text{Prob}(L_n > 2a \ n \log(n)) \leq (1 + 2a \log(n))n^{-a} \]
\[ \text{Prob}(L_n > 2a \ n \log(n) \text{ for at least one } n > N) \]
\[ \leq \sum (1 + 2a \log(n))n^{-a} \]
\[ \leq C \ N^{1-a} \log(N) \]

But
\[ \sum_{N}^{\infty} N^{1-a} \log(N) < \infty \text{ if } a > 2 \]

• Borel-Cantelli lemma
\[ \text{Prob}(L_n > 2a \ n \log(n)) \leq (1 + 2a \log(n))n^{-a} \]
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\[ \bullet \text{ Borel-Cantelli lemma} \quad \Rightarrow \quad \text{with measure 1 } \exists \ N: n > N \quad L_n < 2a \ n \log(n) \]
Theorem: 2d CDTs are a.s. recurrent

Nash-Williams criterion: if electrical resistance to infinity is infinite, \( G \) is recurrent

\[
G = \sum_{n=N}^{\infty} \frac{1}{2a n \log(n)} \geq \sum_{n=N}^{\infty} \frac{1}{L_n} \geq \mu_{\infty}
\]

\( L_n \) distribution determined by \( \mu_{\infty} \)
Theorem: 2d RCDT has \( d_s=2 \) a.s.

\[
P_G(x; n-1) = \frac{(1-x)(1-p_n)}{1 - p_n \ P_G(x; n)}
\]

Iterating out to \( n=N \) gives

\[
Q_G(x; 1) \leq L_1 \left( \frac{1}{x L_N} + \sum_{k=1}^{N} \frac{1}{L_k} \right)
\]

We only need

\[
\langle Q_G(x; 1) \rangle \leq c \left( \frac{1}{x N} + \sum_{k=1}^{N} \frac{1}{k} \right)
\]

Choosing \( N=x^{-1} \)

\[
\sim c \ | \log x |
\]
• Recurrence $Q_G(x;1)$ a.s. diverges as $x \to 0$

• $\langle Q_G(x;1) \rangle$ diverges only as $\log x$

• So $\nexists$ a subset of graphs with non-zero measure: $Q_G(x;1)$ diverges faster than $\log x$ as $x \to 0$

• So $Q_G(x;1)$ a.s. diverges logarithmically as $x \to 0$

• $d_S=2$ a.s.