Invariant prime ideals in the quantum grassmannian

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• Joint work at various times with: Karel Casteels, Ken Goodearl, Ann Kelly, Stéphane Launois, Laurent Rigal, and Ewan Russell

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Quantum 2×2 matrices

The coordinate ring of quantum 2×2 matrices

$\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is generated by four indeterminates a, b, c, d subject to the following rules:

$$ab = qba, \qquad cd = qdc$$

$$ac = qca, \qquad bd = qdb$$

$$bc = cb, \qquad ad - da = (q - q^{-1})cb.$$

The **quantum determinant** $ad - qbc$ is a central element

The algebra of $m \times p$ quantum matrices

- $R = O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right) := \mathbb{C}\left[\begin{array}{lll} X_{1,1} & \dots & X_{1,p} \\ \vdots & \cdots & \vdots \\ X_{m,1} & \dots & X_{m,p} \end{array}\right],$

where each 2×2 sub-matrix is a copy of $O_q\left(\mathcal{M}_2(\mathbb{C})\right)$.

- $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ is an iterated Ore extension and so is a noetherian integral domain.

- In the square case

$$D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)} \cdots X_{n,\sigma(n)}$$

is the **quantum determinant**, a central element.

Quantum minors of $R = \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

Quantum minors are quantum determinants of square submatrices of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

If I and J are row and column sets of the same size then the **quantum minor,** $[I \mid J]$, is the quantum determinant of the quantum matrix subalgebra formed using rows I and columns J

For example,

$$[12|23] = X_{12}X_{23} - qX_{13}X_{22}$$

is the quantum minor of R associated with rows 1 and 2, and columns 2 and 3.

• There is an action of the **torus** $\mathcal{H} = (\mathbb{C}^*)^{m+p}$ on $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$ given by multiplication of each row or column by a nonzero scalar

• Quantum minors are $\mathcal{H}$-eigenvectors

Example: With $h = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$,

$$\begin{aligned} h \cdot [12|23] &= h \cdot (X_{12}X_{23} - qX_{13}X_{22}) \\ &= (\alpha_1\beta_2X_{12}).(\alpha_2\beta_3X_{23}) - q(\alpha_1\beta_3X_{1,3}).(\alpha_2\beta_2X_{22}) \\ &= \alpha_1\alpha_2\beta_2\beta_3(X_{12}X_{23} - qX_{13}X_{22}) \end{aligned}$$

From now on, assume that the deformation parameter q is a nonroot of unity

Quantum matrices fall into a general class of algebras, known as **CGL extensions**, or **quantum nilpotent algebras**, for which there is a general strategy, known as the **Goodearl-Letzter stratification theory** for studying the prime spectrum of algebras.

In such algebras, there is an **action of a torus**, $\mathcal{H}$, and understanding the $\mathcal{H}$-invariant prime ideals is key to understanding the whole of the prime spectrum.

• The prime ideals in quantum matrices are all **completely prime**; that is, R/P is an integral domain (Goodearl-Letzter)

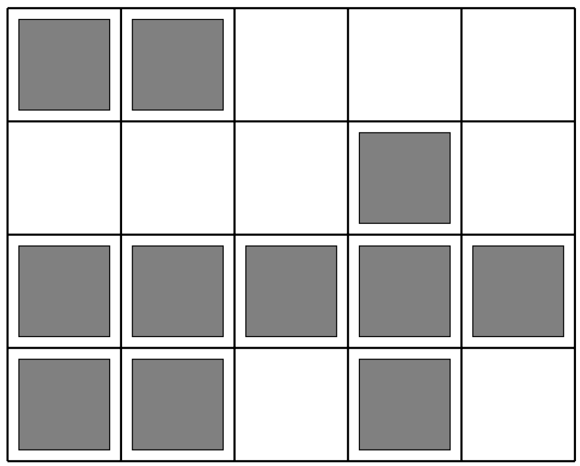
• There are only **finitely many** $\mathcal{H}$-prime ideals (Goodearl-Letzter)

• The $\mathcal{H}$-prime ideals are in bijection with **Cauchon diagrams** (definition to come soon) (Cauchon)

• The $\mathcal{H}$-prime ideals are each **generated by the quantum minors that they contain** (Goodearl-Lenagan, Launois, Yakimov, Casteels)

Cauchon diagrams (Total nonegativitists: Le-diagrams)

A Young diagram with entries coloured black or white is said to be a **Cauchon diagram** if it satisfies the following rule: if there is a black in a given square then either each square to the left is also coloured black or each square above is also coloured black



Cauchon diagrams

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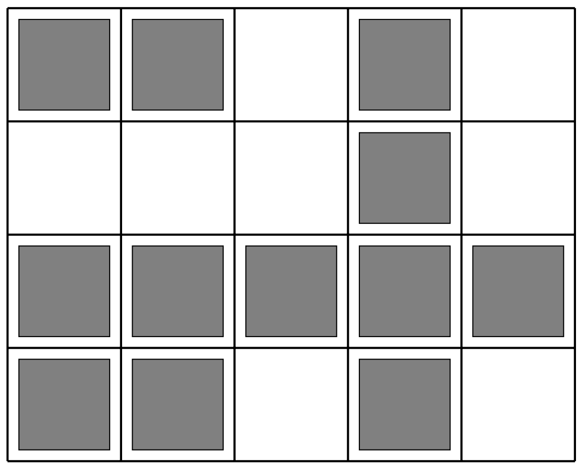


Goodearl, Launois, Lenagan There is a very close connection between the behaviour of totally nonnegative cells in the space of totally nonnegative matrices and the $\mathcal{H}$-prime spectrum of quantum matrices

A set of minors is the set of minors that are zero on elements in a totally nonnegative cell if and only if the corresponding set of quantum minors is the set of quantum minors in an $\mathcal{H}$-prime ideal of quantum matrices

Example For quantum 2×2 matrices, there are 16 black/white fillings of the 2×2 Young diagram, and only two fail the Cauchon test; so there are 14 $\mathcal{H}$-prime ideals in quantum 2×2 matrices

• These 14 $\mathcal{H}$-prime ideals can easily be found by hand, and each is generated by quantum minors

• For example, the ideal generated by the 2×2 quantum determinant $ad - qbc$ is an $\mathcal{H}$-prime ideal

The quantum grassmannian $\mathcal{O}_q(G(k,n))$

• The **quantum grassmannian** $\mathcal{O}_q(G(k,n))$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}(k,n))$ generated by the maximal $k \times k$ quantum minors.

• Denote by $[I]$ the quantum minor $[1 \dots k|I]$.

• There is a torus action of $\mathcal{H} = (\mathbb{C}^*)^n$ given by column multiplication

• The $k \times k$ quantum minors are the **quantum Plücker coordinates** of the quantum grassmannian

• The quantum grassmannian is a deformation of the homogeneous coordinate ring of the classical grassmannian

• We will see that the behaviour of the $\mathcal{H}$-prime spectrum of the quantum grassmannian mirrors the behaviour of the cell structure of the totally nonnegative grassmannian

Example $\mathcal{O}_q(G(2,4))$ is generated by the six quantum minors

$$[12], \quad [13], \quad [14], \quad [23], \quad [24], \quad [34]$$

Most quantum minors $q^\bullet$-commute, for example,

$$[14]\,[23] = [23]\,[14]\,, \quad [12]\,[13] = q\,[13]\,[12]\,, \quad [12]\,[34] = q^2\,[34]\,[12]$$

However,

$$[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$$

and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Aim: Describe $\mathcal{H} - \text{Spec}(\mathcal{O}_q(G(k,n)))$

Snag: Goodearl-Letzter theory can't be used directly since $\mathcal{O}_q(G(k,n))$ is not usually CGL extension (or a factor of one)

Nevertheless, one might hope that:

- There are only finitely many $\mathcal{H}$-primes

- All $\mathcal{H}$-primes are completely prime

- We can specify the quantum minors in a given $\mathcal{H}$-prime

- Each $\mathcal{H}$-prime is generated by the quantum minors that it contains

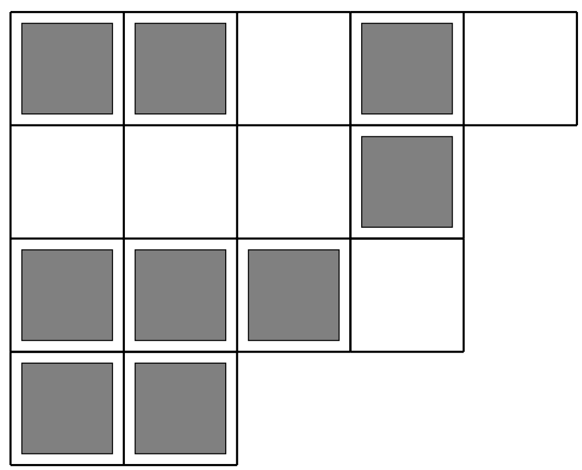
- We can describe the containments between $\mathcal{H}$-primes

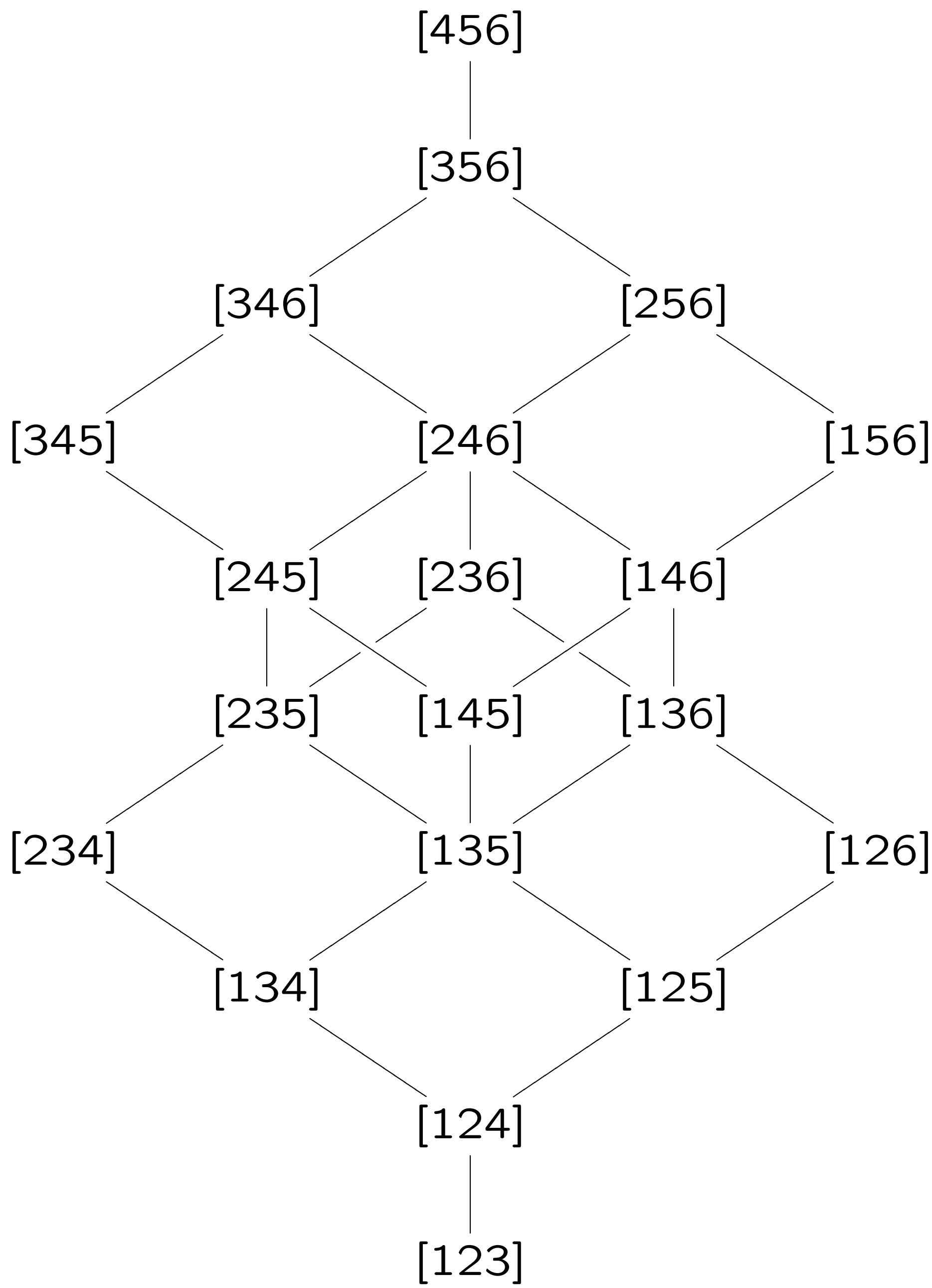
Launois, Lenagan and Rigal There is a bijection between $\mathcal{H}-$ Spec($\mathcal{O}_q(G(k,n))$) (ignoring the irrelevant ideal) and Cauchon-Le diagrams on Young diagrams that fit inside a $k \times (n-k)$ array

The theorem is proved by defining quantum algebras with a straightening law, quantum Schubert varieties, quantum Schubert cells, partition subalgebras of quantum matrices and using a noncommutative version of dehomogenisation.

Cauchon-Le diagrams

A Young diagram with entries coloured black or white is said to be a **Cauchon-Le diagram** if it satisfies the following rule: if there is a black in a given square then either each square to the left is also coloured black or each square above is also coloured black

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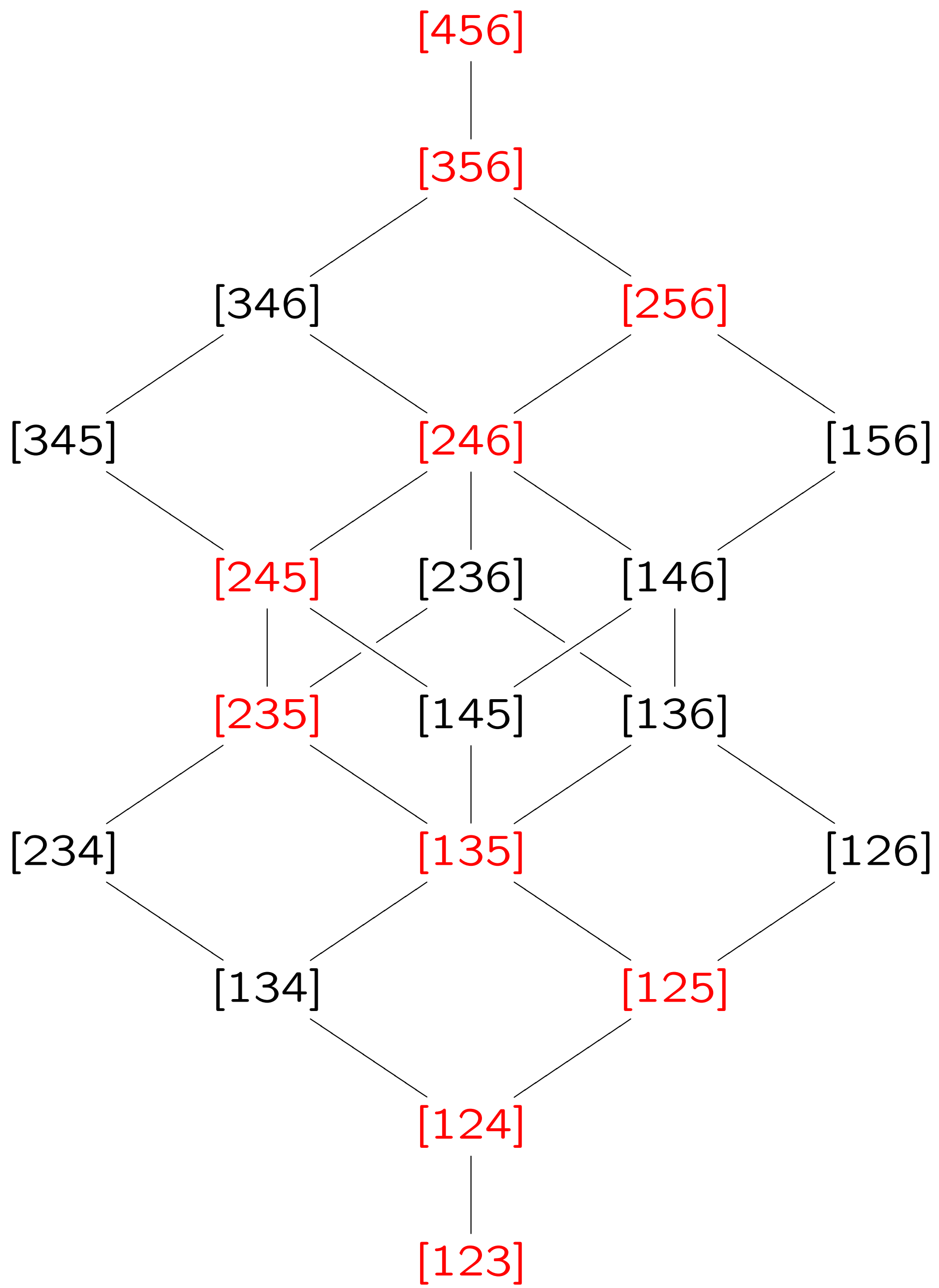
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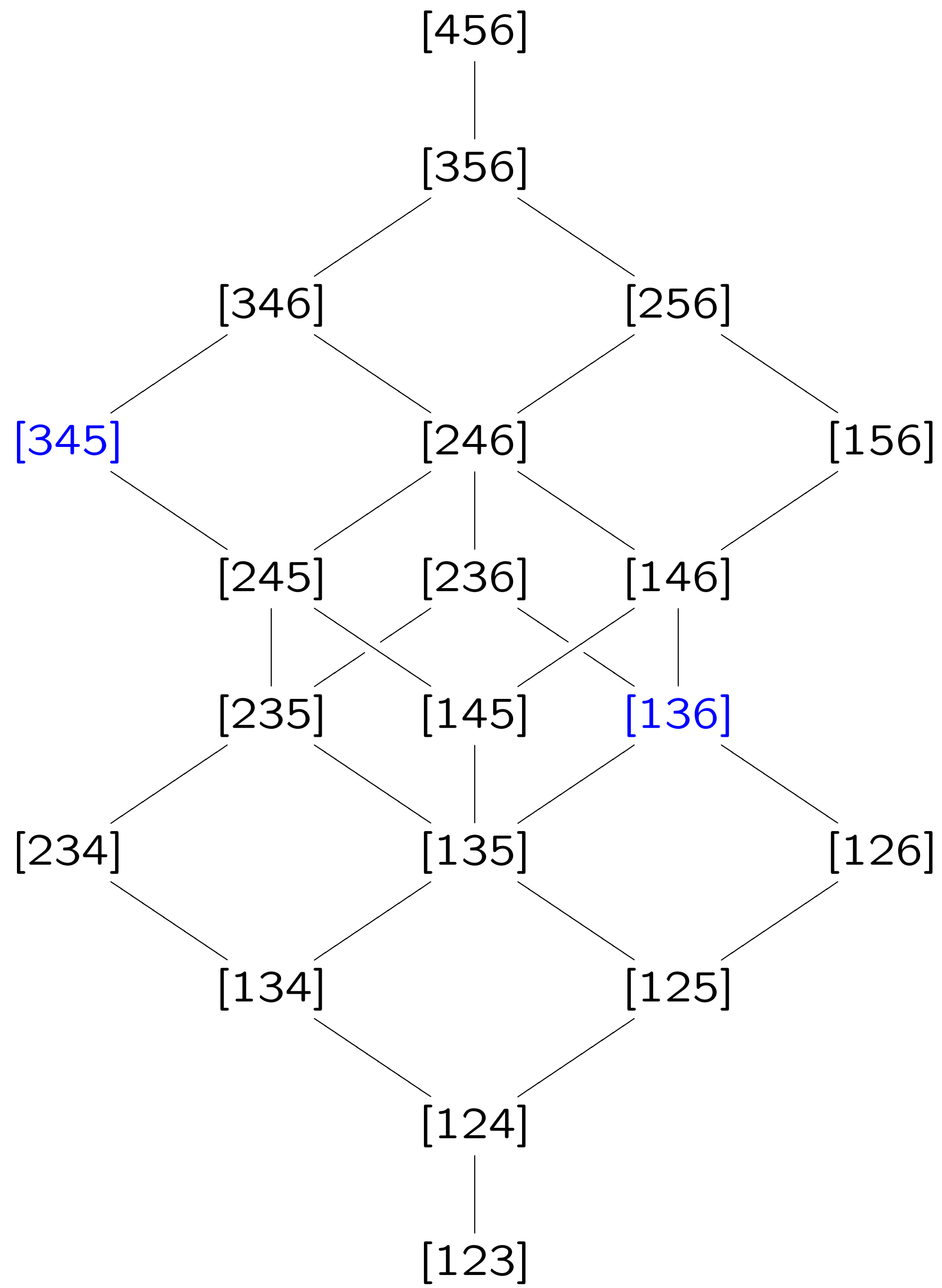
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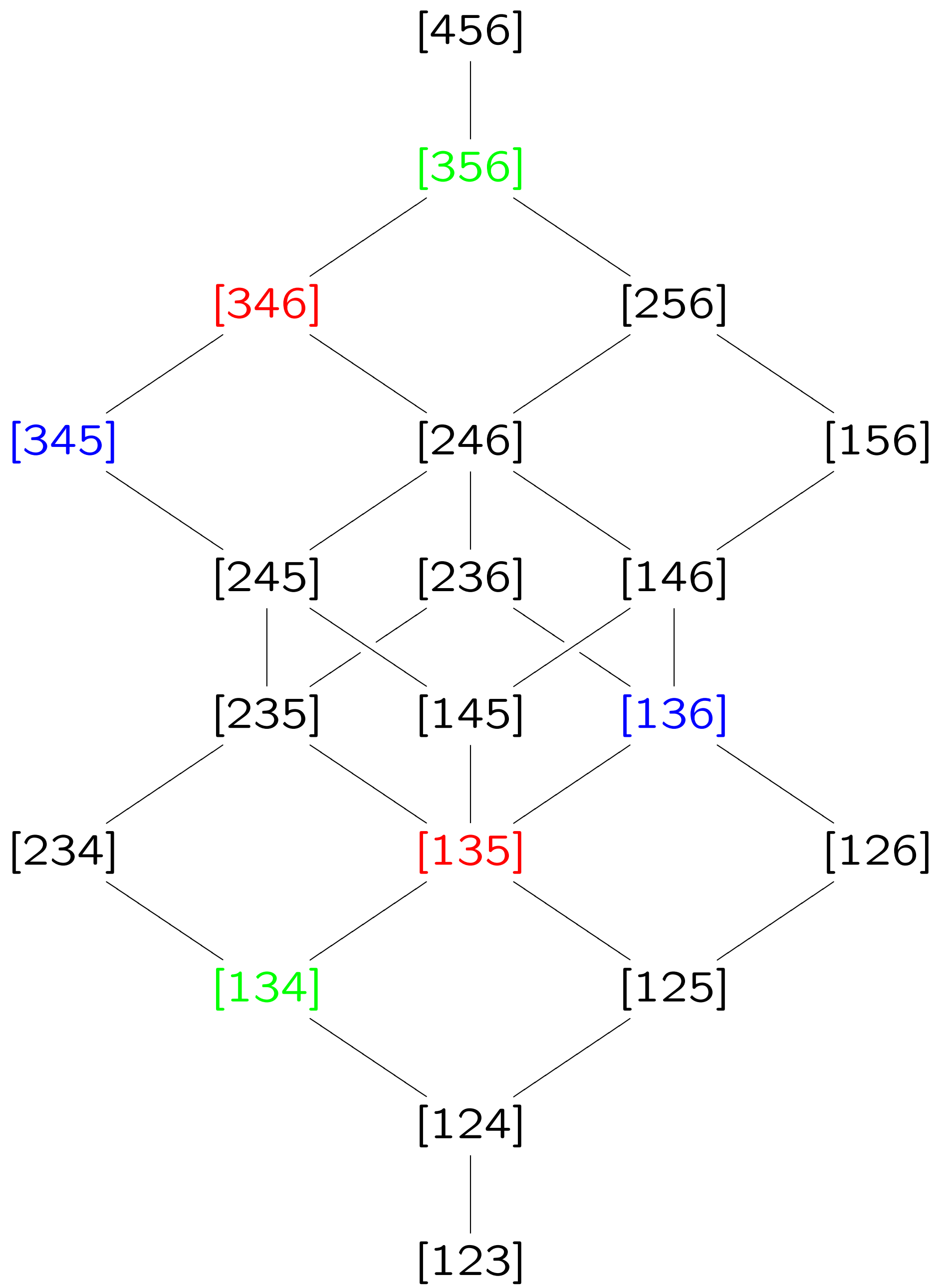
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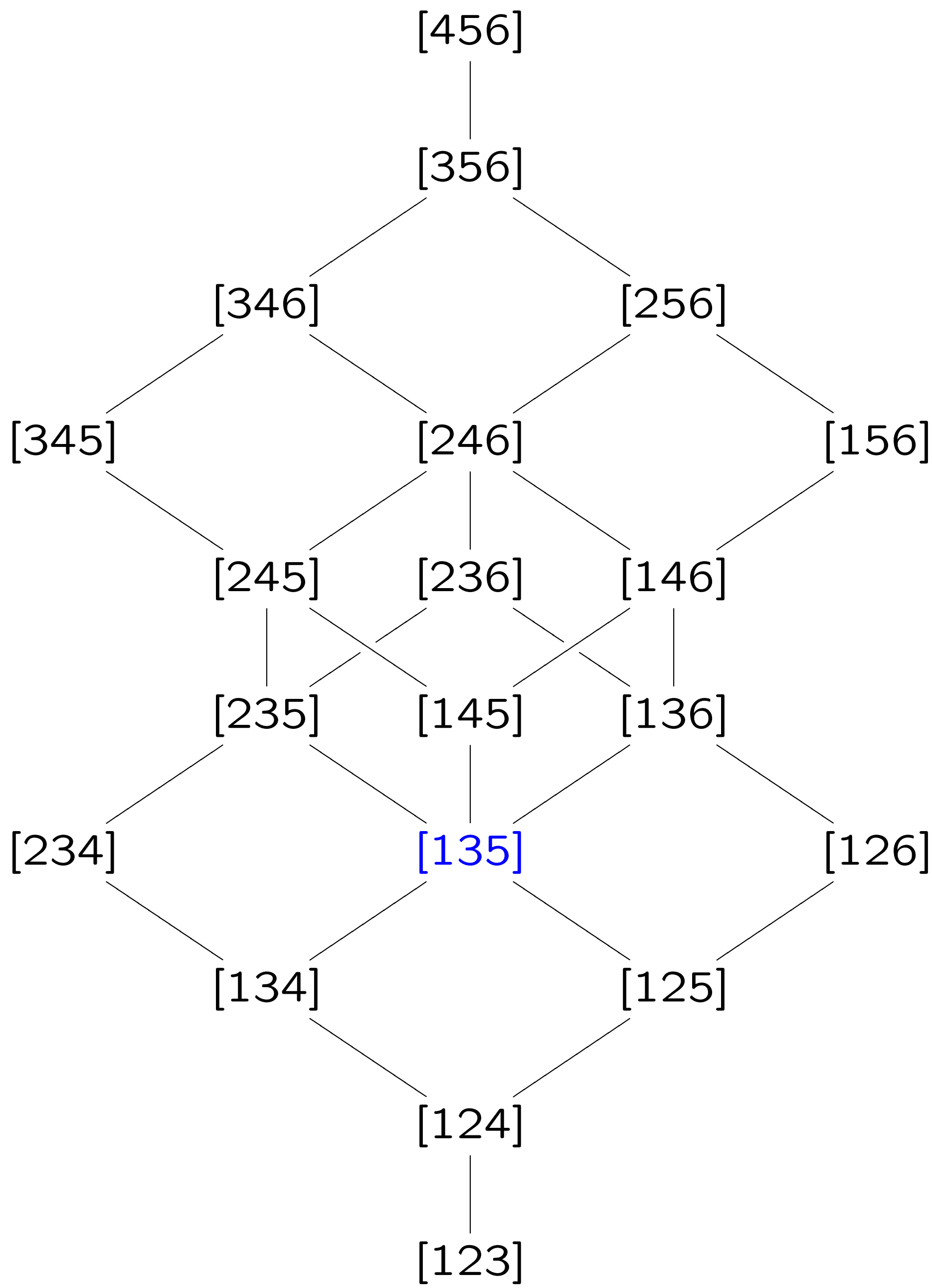
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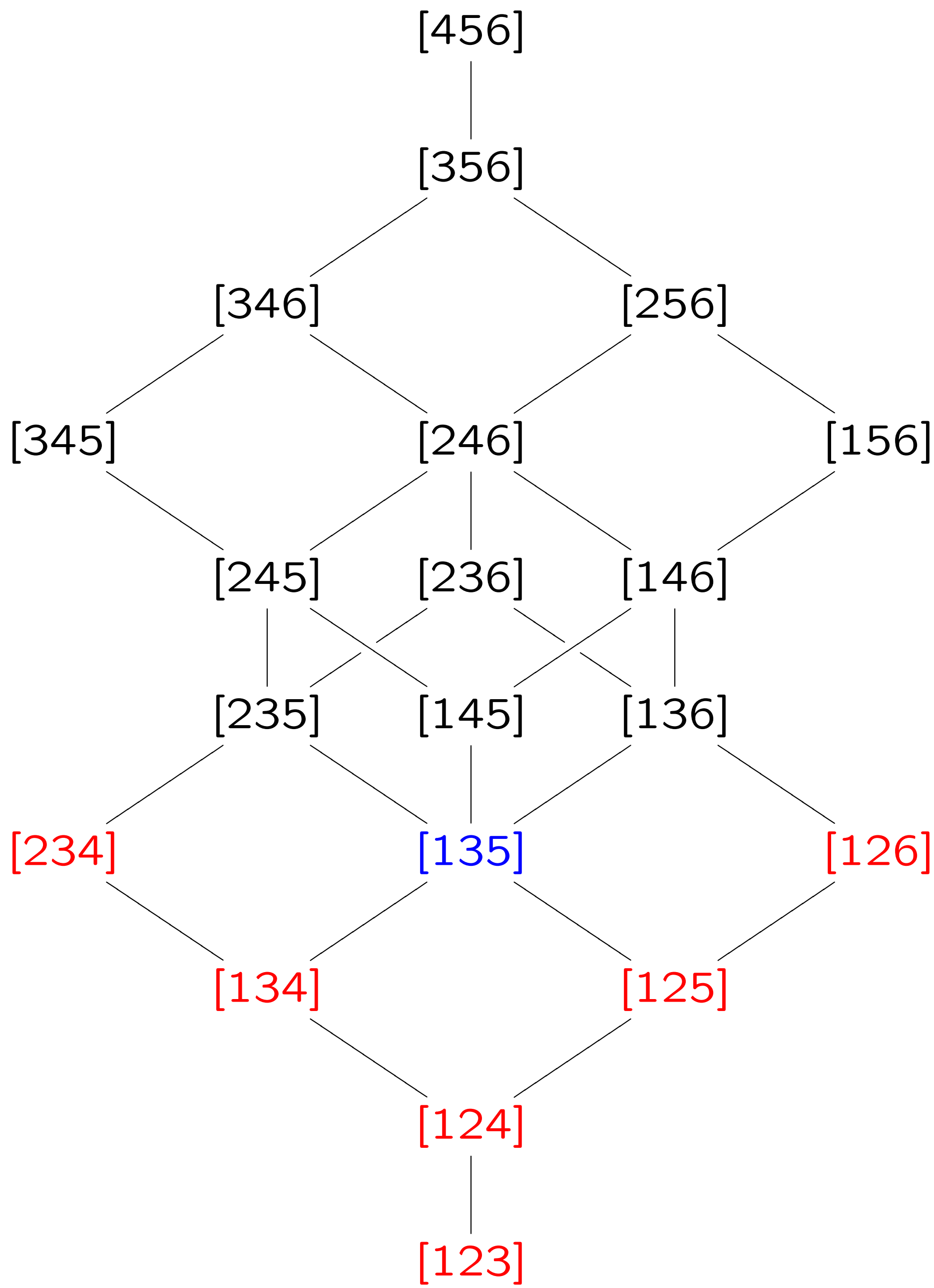
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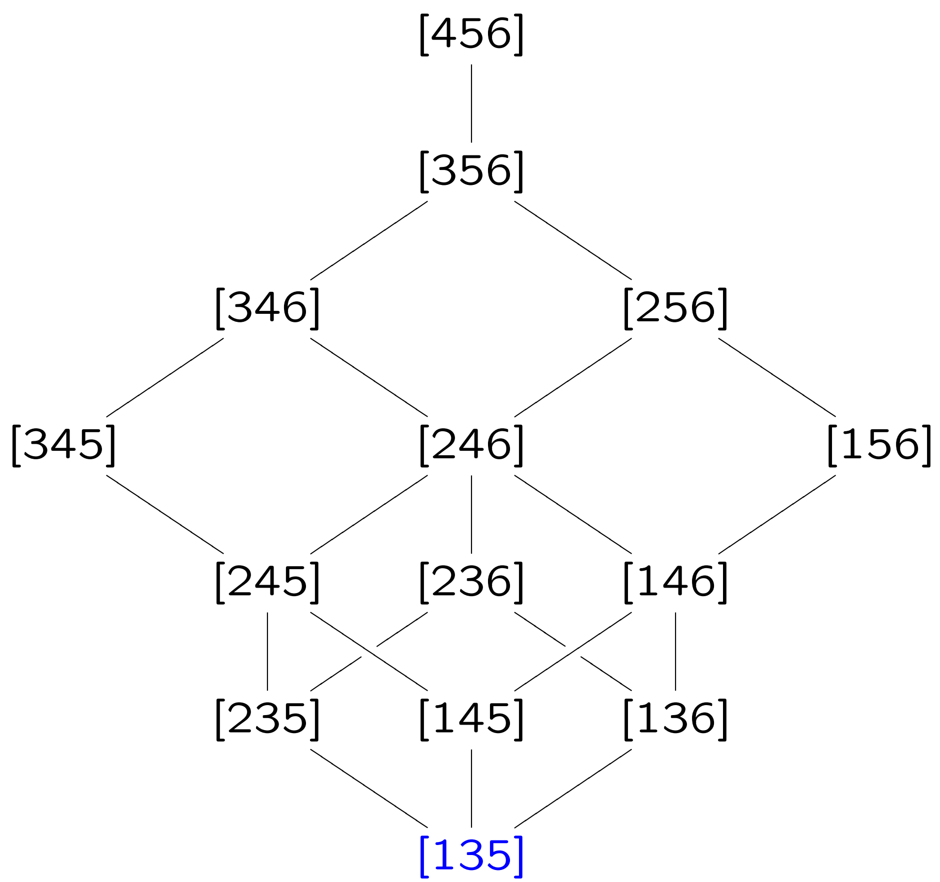
Quantum Schubert variety corresp to [135]



q-**Schubert cell**: use noncommutative dehomogenisation at [135]

Noncommutative dehomogenisation

Suppose that $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$ is an $\mathbb{N}$-graded algebra and that $u \in A_1$ is a nonzero normal element (ie. $uA = Au$).

We can invert u to obtain a $\mathbb{Z}$-graded algebra $A[u^{-1}]$. The zero component $A[u^{-1}]_0$ is the **noncommutative dehomogenisation of A at u**, written $\mathrm{Dhom}_u(A)$

Theorem $A[u^{-1}] \cong \mathrm{Dhom}_u(A)[x^{\pm 1}; \sigma]$, where σ is the automorphism of A given by the commutation rule for u.

Example Let u be the Plücker coordinate $u = [12\ldots k]$ then

$$\mathcal{O}_q(G(k,n))[u^{-1}] \cong O_q\left(\mathcal{M}_{k,n-k}(\mathbb{C})\right)[x^{\pm 1}; \sigma]$$

• For each invariant prime ideal P of the quantum grassmannian, there is a **unique** quantum Plücker coordinate $[I]$ such that

$$[I] \notin P, \text{ while } [J] \in P \text{ for all } J \not\geq I$$

(this follows from the fact that the quantum grassmannian is a **quantum algebra with a straightening law**)

• The invariant prime ideal P then belongs to the quantum Schubert cell corresponding to the quantum Plücker coordinate $[I]$

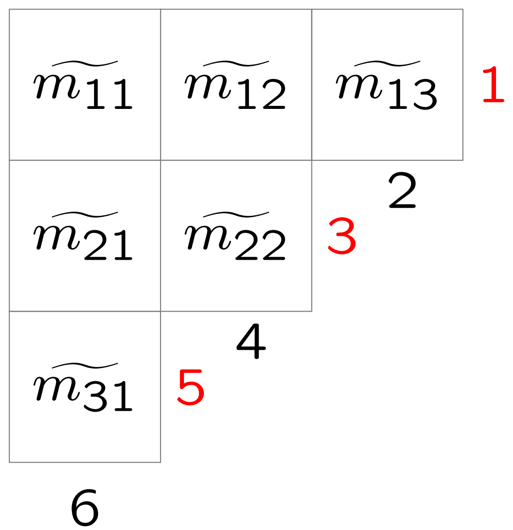
• In more detail, if we denote the quantum Schubert cell by $R_{[I]}$, and the ideal generated by quantum Plücker coordinates J with $J \ngeq I$ by $\Pi_{[I]}$ then

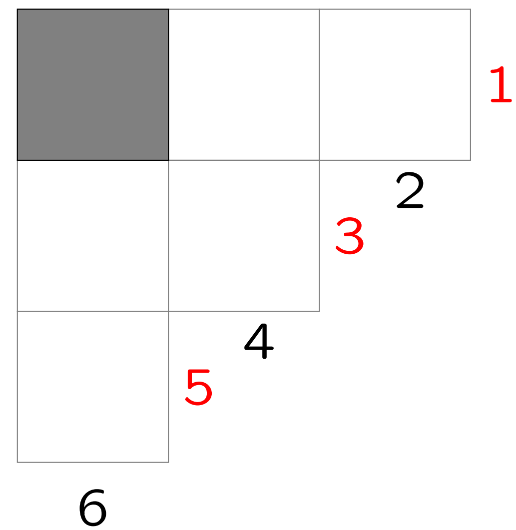
$$\frac{\mathcal{O}_q(G(k,n))}{\Pi_{[I]}}[[I]^{-1}] \quad \cong \quad R_{[I]}[x^{\pm};\sigma]$$

and the prime P passes through this isomorphism to the quantum Schubert cell $R_{[I]}$

• As $\Pi_{[I]}$ is generated by $\mathcal{H}$-eigenvectors, there is an induced action of the torus $\mathcal{H}$ on the quantum Schubert cells. So, to understand the $\mathcal{H}$-prime spectum of $\mathcal{O}_q(G(k,n))$ we need to understand the $\mathcal{H}$-prime spectum of quantum Schubert cells, and this is where Cauchon-Le diagrams come into play

Schubert cell for [135] $\mathcal{H}$**-prime in Schubert cell** [135]

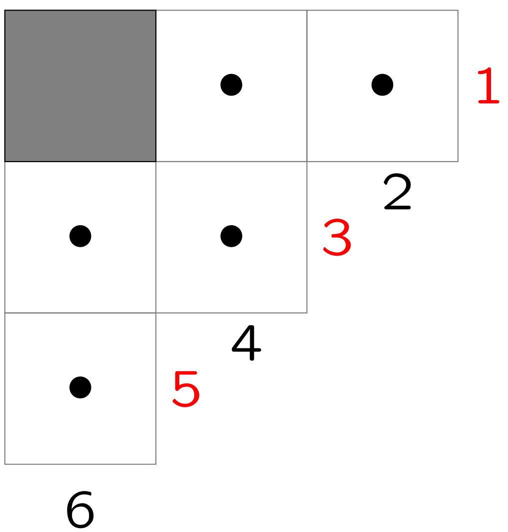


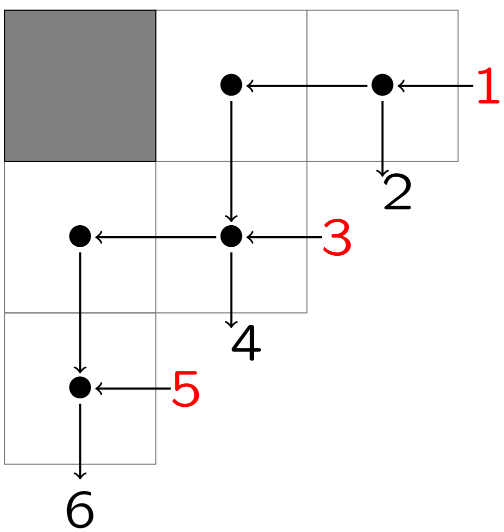


$\widetilde{m_{ij}} := \overline{[???]} \cdot \overline{[135]}^{-1}$ obey quantum matrix rules

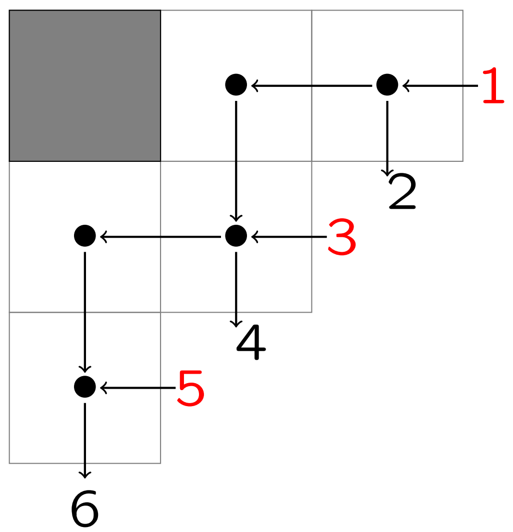
(eg. $\widetilde{m_{11}} = \overline{[356]} \cdot \overline{[135]}^{-1}$)

Postnikov graph





Speculation

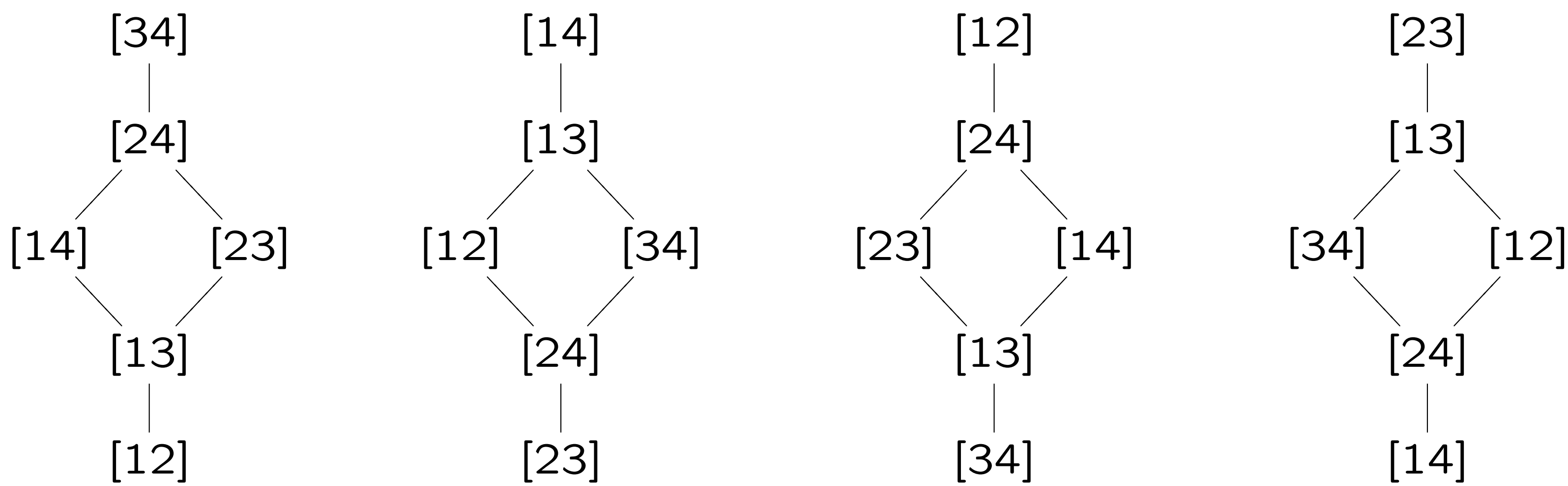


There is a vertex disjoint set of paths from $\{1, 3\}$ to $\{2, 4\}$ so [245] is not in the prime.

There is no vertex disjoint set of paths from $\{1, 3\}$ to $\{4, 6\}$ so [456] is in the prime.

- We hope to prove this conjecture by using the path methods that Casteels developed in the quantum matrices setting

The i-order: $i \leq_i i+1 \leq_i \ldots \leq_i n \leq_i 1 \leq_i \ldots \leq_i i-1$



The four orderings on $\mathcal{O}_q(G(2,4))$

- The quantum grassmannian is a quantum algebra with a straightening law with respect to each of the n orderings

Fix an invariant prime P in $\mathcal{O}_q(G(k,n))$

• For each i-order there is a unique quantum minor $[I_i]$ such that $[I_i] \notin P$ but $[J] \in P$ for each $J \not\geq_i I_i$

Let $\Pi_i(P)$ denote $\{[J] \mid J \not\geq_i I_i\}$. Then

$$\Pi(P) := \bigcup_{i=1}^{n} \Pi_i(P) \quad \subseteq P$$

Conjecture: $\Pi(P)$ is the set of quantum minors belonging to P, and P is generated as an ideal by $\Pi(P)$

• We hope to prove this conjecture by using the path methods that Casteels developed in the quantum matrices setting

Continuing with the notation on the previous slide:

• The quantum minors $I_1, I_2, \ldots, I_n$ form a **Grassmann necklace,** Neck(P)

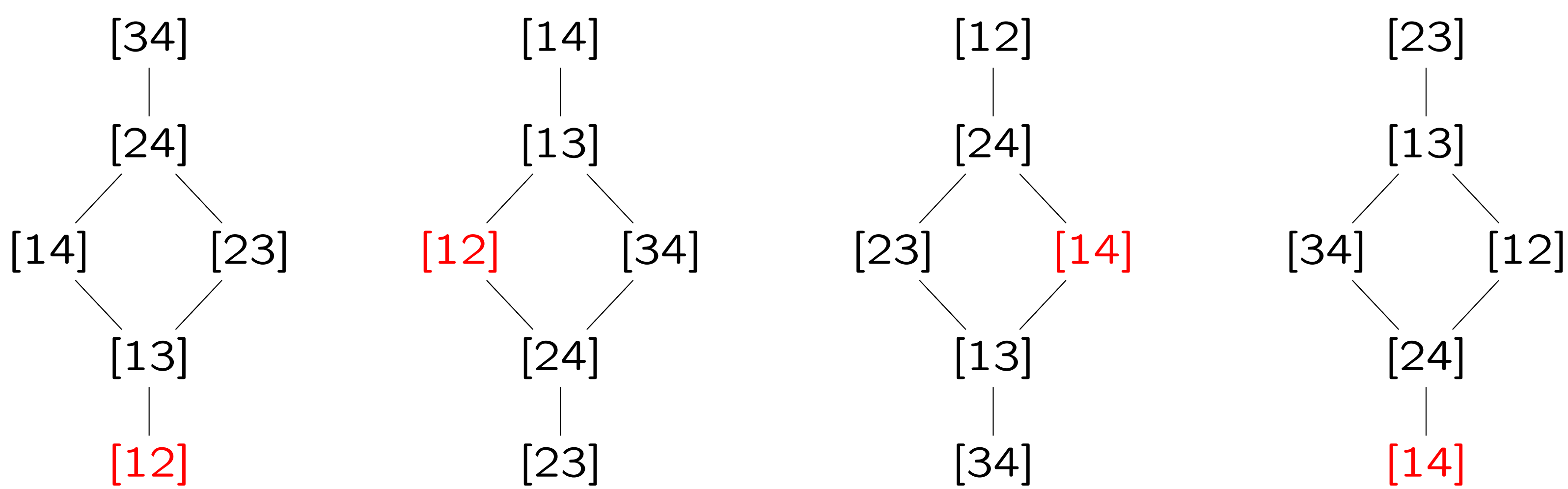
• Given a Cauchon diagram for an invariant prime P, we can construct Neck(P)

• If $P' \subseteq P$ then Neck(P') $\leq$ Neck(P)

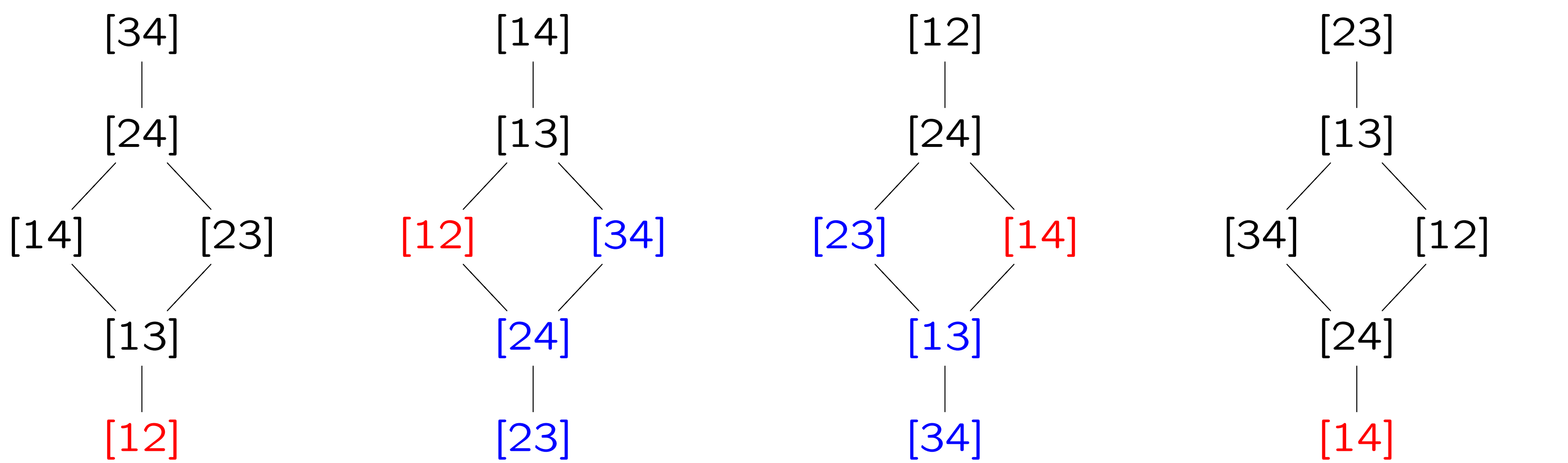
Conjecture The converse is true

- In $\mathcal{O}_q(G(2,4))$ consider the Grassmann necklace

$$(I_1, I_2, I_3, I_4) = (12, 12, 14, 14)$$



• **Grassmann necklace**: $(I_1, I_2, I_3, I_4) = (12, 12, 14, 14)$



• The $\mathcal{H}$-prime P with this necklace is $P = \langle [13], [23], [24], [34] \rangle$

• Note that $\mathcal{O}_q(G(2,4))/P \cong \mathbb{C}\Big[[12], [14]\Big]$ is a quantum plane, so P is prime.