Invariant prime ideals in the quantum grassmannian

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Quantum $2 \times 2$ matrices

The coordinate ring of quantum $2 \times 2$ matrices

\[ \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})) := \mathbb{C} \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \] is generated by four indeterminates $a, b, c, d$ subject to the following rules:

\[
\begin{align*}
ab &= qba, & cd &= qdc \\
ac &= qca, & bd &= qdb \\
bc &= cb, & ad - da &= (q - q^{-1})cb.
\end{align*}
\]

The quantum determinant $ad - qbc$ is a central element.
The algebra of $m \times p$ quantum matrices

• $R = O_q (\mathcal{M}_{m,p}(\mathbb{C})) := \mathbb{C} \begin{bmatrix} X_{1,1} & \cdots & X_{1,p} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,p} \end{bmatrix}$,

where each $2 \times 2$ sub-matrix is a copy of $O_q (\mathcal{M}_2(\mathbb{C}))$.

• $O_q (\mathcal{M}_{m,p}(\mathbb{C}))$ is an iterated Ore extension and so is a noetherian integral domain.

• In the square case

\[ D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)} \cdots X_{n,\sigma(n)} \]

is the quantum determinant, a central element.
Quantum minors of $R = O_q(M_{m,p}(\mathbb{C}))$

Quantum minors are quantum determinants of square submatrices of $O_q(M_{m,p}(\mathbb{C}))$

If $I$ and $J$ are row and column sets of the same size then the quantum minor, $[I \mid J]$, is the quantum determinant of the quantum matrix subalgebra formed using rows $I$ and columns $J$

For example,

$$[12|23] = X_{12}X_{23} - qX_{13}X_{22}$$

is the quantum minor of $R$ associated with rows 1 and 2, and columns 2 and 3.
• There is an action of the torus $\mathcal{H} = (\mathbb{C}^*)^{m+p}$ on $O_q(M_{m,p}(\mathbb{C}))$ given by multiplication of each row or column by a nonzero scalar.

• Quantum minors are $\mathcal{H}$-eigenvectors.

**Example:** With $h = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$,

$$ h \cdot [12|23] = h \cdot (X_{12}X_{23} - qX_{13}X_{22}) $$

$$ = (\alpha_1 \beta_2 X_{12}).(\alpha_2 \beta_3 X_{23}) - q(\alpha_1 \beta_3 X_{1,3}).(\alpha_2 \beta_2 X_{22}) $$

$$ = \alpha_1 \alpha_2 \beta_2 \beta_3(X_{12}X_{23} - qX_{13}X_{22}) $$
From now on, assume that the deformation parameter $q$ is a nonroot of unity.

Quantum matrices fall into a general class of algebras, known as CGL extensions, or quantum nilpotent algebras, for which there is a general strategy, known as the Goodearl-Letzter stratification theory for studying the prime spectrum of algebras.

In such algebras, there is an action of a torus, $\mathcal{H}$, and understanding the $\mathcal{H}$-invariant prime ideals is key to understanding the whole of the prime spectrum.
• The prime ideals in quantum matrices are all completely prime; that is, \( R/P \) is an integral domain (Goodearl-Letzter)

• There are only finitely many \( \mathcal{H} \)-prime ideals (Goodearl-Letzter)

• The \( \mathcal{H} \)-prime ideals are in bijection with Cauchon diagrams (definition to come soon) (Cauchon)

• The \( \mathcal{H} \)-prime ideals are each generated by the quantum minors that they contain (Goodearl-Lenagan, Launois, Yakimov, Casteels)
Cauchon diagrams  (Total nonegativitists: Le-diagrams)

A Young diagram with entries coloured black or white is said to be a **Cauchon diagram** if it satisfies the following rule: if there is a black in a given square then either each square to the left is also coloured black or each square above is also coloured black.

![Diagram of a Cauchon diagram]
Cauchon diagrams

A Young diagram with entries coloured black or white is said to be a **Cauchon diagram** if it satisfies the following rule: if there is a black in a given square then either each square to the left is also coloured black or each square above is also coloured black.
Goodearl, Launois, Lenagan There is a very close connection between the behaviour of totally nonnegative cells in the space of totally nonnegative matrices and the $\mathcal{H}$-prime spectrum of quantum matrices

A set of minors is the set of minors that are zero on elements in a totally nonnegative cell if and only if the corresponding set of quantum minors is the set of quantum minors in an $\mathcal{H}$-prime ideal of quantum matrices
Example For quantum $2 \times 2$ matrices, there are 16 black/white fillings of the $2 \times 2$ Young diagram, and only two fail the Cauchon test; so there are 14 $\mathcal{H}$-prime ideals in quantum $2 \times 2$ matrices.

• These 14 $\mathcal{H}$-prime ideals can easily be found by hand, and each is generated by quantum minors.

• For example, the ideal generated by the $2 \times 2$ quantum determinant $ad - qbc$ is an $\mathcal{H}$-prime ideal.
The quantum grassmannian $O_q(G(k,n))$

- The quantum grassmannian $O_q(G(k,n))$ is the subalgebra of $O_q(M(k,n))$ generated by the maximal $k \times k$ quantum minors.

- Denote by $[I]$ the quantum minor $[1 \ldots k | I]$.

- There is a torus action of $\mathcal{H} = (\mathbb{C}^*)^n$ given by column multiplication

- The $k \times k$ quantum minors are the quantum Plücker coordinates of the quantum grassmannian.
• The quantum grassmannian is a deformation of the homogeneous coordinate ring of the classical grassmannian

• We will see that the behaviour of the $\mathcal{H}$-prime spectrum of the quantum grassmannian mirrors the behaviour of the cell structure of the totally nonnegative grassmannian
Example $O_q(G(2,4))$ is generated by the six quantum minors

$$[12], \ [13], \ [14], \ [23], \ [24], \ [34]$$

Most quantum minors $q^\bullet$-commute, for example,

$$[14][23] = [23][14], \ [12][13] = q[13][12], \ [12][34] = q^2[34][12]$$

However,

$$[13][24] = [24][13] + (q - q^{-1})[14][23]$$

and there is a quantum Plücker relation

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$
**Aim:** Describe $\mathcal{H} - \text{Spec}(\mathcal{O}_q(G(k,n)))$

**Snag:** Goodearl-Letzter theory can't be used directly since $\mathcal{O}_q(G(k,n))$ is not usually CGL extension (or a factor of one)
Nevertheless, one might hope that:

- There are only finitely many $\mathcal{H}$-primes
- All $\mathcal{H}$-primes are completely prime
- We can specify the quantum minors in a given $\mathcal{H}$-prime
- Each $\mathcal{H}$-prime is generated by the quantum minors that it contains
- We can describe the containments between $\mathcal{H}$-primes
There is a bijection between $\mathcal{H} - \text{Spec}(\mathcal{O}_q(G(k,n)))$ (ignoring the irrelevant ideal) and Cauchon-Le diagrams on Young diagrams that fit inside a $k \times (n-k)$ array.

The theorem is proved by defining quantum algebras with a straightening law, quantum Schubert varieties, quantum Schubert cells, partition subalgebras of quantum matrices and using a noncommutative version of dehomogenisation.
Cauchon-Le diagrams

A Young diagram with entries coloured black or white is said to be a **Cauchon-Le diagram** if it satisfies the following rule: if there is a black in a given square then either each square to the left is also coloured black or each square above is also coloured black.
Quantum Schubert variety corresp to [135]

\[ q\text{-Schubert cell} \]: use noncommutative dehomogenisation at [135]
Noncommutative dehomogenisation

Suppose that \( A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \) is an \( \mathbb{N} \)-graded algebra and that \( u \in A_1 \) is a nonzero normal element (ie. \( uA = Au \)).

We can invert \( u \) to obtain a \( \mathbb{Z} \)-graded algebra \( A[u^{-1}] \). The zero component \( A[u^{-1}]_0 \) is the noncommutative dehomogenisation of \( A \) at \( u \), written \( \text{Dhom}_u(A) \).

**Theorem** \( A[u^{-1}] \cong \text{Dhom}_u(A)[x^\pm 1; \sigma] \), where \( \sigma \) is the automorphism of \( A \) given by the commutation rule for \( u \).

**Example** Let \( u \) be the Plücker coordinate \( u = [12 \ldots k] \) then

\[
\mathcal{O}_q(G(k,n))[u^{-1}] \cong \mathcal{O}_q \left( \mathcal{M}_{k,n-k}(\mathbb{C}) \right) [x^\pm 1; \sigma]
\]
For each invariant prime ideal $P$ of the quantum grassmannian, there is a **unique** quantum Plücker coordinate $[I]$ such that

$$[I] \not\in P,$$

while $[J] \in P$ for all $J \not\supset I$.

(this follows from the fact that the quantum grassmannian is a **quantum algebra with a straightening law**)

The invariant prime ideal $P$ then belongs to the quantum Schubert cell corresponding to the quantum Plücker coordinate $[I]$.
• In more detail, if we denote the quantum Schubert cell by $R[I]$, and the ideal generated by quantum Plücker coordinates $J$ with $J \not\supset I$ by $\Pi[I]$ then

$$\frac{\mathcal{O}_q(G(k,n))}{\Pi[I]}[[I]^{-1}] \cong R[I][x^\pm; \sigma]$$

and the prime $P$ passes through this isomorphism to the quantum Schubert cell $R[I]$

• As $\Pi[I]$ is generated by $\mathcal{H}$-eigenvectors, there is an induced action of the torus $\mathcal{H}$ on the quantum Schubert cells. So, to understand the $\mathcal{H}$-prime spectrum of $\mathcal{O}_q(G(k,n))$ we need to understand the $\mathcal{H}$-prime spectrum of quantum Schubert cells, and this is where Cauchon-Le diagrams come into play

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Schubert cell for $[135]$  

$\mathcal{H}$-prime in Schubert cell $[135]$  

$$
\begin{array}{ccc}
\tilde{m}_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\
\tilde{m}_{21} & \tilde{m}_{22} & \tilde{m}_{23} \\
\tilde{m}_{31} & & \\
\end{array}
$$

$\tilde{m}_{ij} := [???] \cdot [135]^{-1}$ obey quantum matrix rules  

(eg. $\tilde{m}_{11} = [356] \cdot [135]^{-1}$)
Postnikov graph
Speculation

There is a vertex disjoint set of paths from \(\{1, 3\}\) to \(\{2, 4\}\) so [245] is not in the prime.

There is no vertex disjoint set of paths from \(\{1, 3\}\) to \(\{4, 6\}\) so [456] is in the prime.

- We hope to prove this conjecture by using the path methods that Casteels developed in the quantum matrices setting.
The $i$-order: $i \leq_i i + 1 \leq_i \ldots \leq_i n \leq_i 1 \leq_i \ldots \leq_i i - 1$

The four orderings on $O_q(G(2,4))$

- The quantum grassmannian is a quantum algebra with a straightenning law with respect to each of the $n$ orderings
Fix an invariant prime $P$ in $\mathcal{O}_q(G(k, n))$

- For each $i$-order there is a unique quantum minor $[I_i]$ such that $[I_i] \notin P$ but $[J] \in P$ for each $J \nleq I_i$

Let $\Pi_i(P)$ denote $\{[J] \mid J \nleq I_i\}$. Then

$$\Pi(P) := \bigcup_{i=1}^{n} \Pi_i(P) \subseteq P$$

**Conjecture:** $\Pi(P)$ is the set of quantum minors belonging to $P$, and $P$ is generated as an ideal by $\Pi(P)$

- We hope to prove this conjecture by using the path methods that Casteels developed in the quantum matrices setting
Continuing with the notation on the previous slide:

- The quantum minors $I_1, I_2, \ldots, I_n$ form a **Grassmann necklace**, $\text{Neck}(P)$

- Given a Cauchon diagram for an invariant prime $P$, we can construct $\text{Neck}(P)$

- If $P' \subseteq P$ then $\text{Neck}(P') \leq \text{Neck}(P)$

**Conjecture** The converse is true
• In $\mathcal{O}_q(G(2,4))$ consider the Grassmann necklace

$$(I_1, I_2, I_3, I_4) = (12, 12, 14, 14)$$
• **Grassmann necklace:** $(I_1, I_2, I_3, I_4) = (12, 12, 14, 14)$

- The $\mathcal{H}$-prime $P$ with this necklace is $P = \langle [13], [23], [24], [34] \rangle$

- Note that $\mathcal{O}_q(G(2, 4))/P \cong \mathbb{C}[12, [14]]$ is a quantum plane, so $P$ is prime.