On-shell diagrams in $\mathcal{N} = 4$ SYM beyond the planar limit

Total Positivity: a bridge between Representation Theory and Physics

- University of Kent -

based on:

hep-th/1502.02034 - Franco, Galloni, BP, Wen

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12/01/16

GATIS
Gauge Theory as an Integrable System

Overview

- Scattering amplitudes in planar N=4 SYM admit a description in terms of the positive Grassmannian.
  Arkani-Hamed, Cachazo, Cheung, Kaplan / Mason, Skinner

- Loop leading singularities are residues of a Grassmannian integral.

- All-loop integrand determined via the BCFW recursion relation.
  Britto, Cachazo, Feng, Witten / Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka

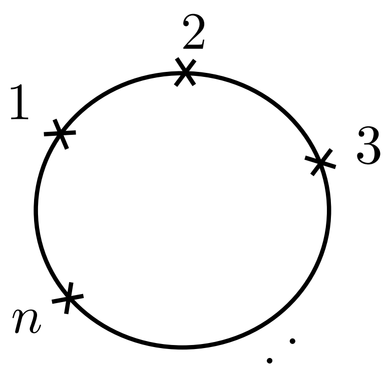
... but how to go beyond planar N=4 SYM?

No ordering = No positivity

$SU(N)$ gauge group

Planar limit: $N \to \infty$, with $\lambda = g^2_{\mathrm{YM}} N$ fixed

$$\mathcal{A}_n = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots t^{a_{\sigma(n)}}) \underbrace{A_n(\sigma(1), \sigma(2), \ldots, \sigma(n))}_{\substack{\text{Partial amplitude} \\ \text{(colour ordered)}}}$$



(Finite N corrections $\propto$ multiple traces)

Planar loop integrand

- Tree-level amplitudes enjoy *Yangian* symmetry

Drummond, Henn, Plefka

[[Yangian = Superconformal + Dual Superconformal]]

Drummond, Henn, Korchemsky, Sokatchev

Loop level:

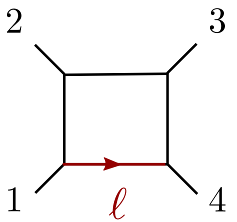
- Yangian symmetry broken due to IR divergences
- Loop integrand

$$\int d^4\ell_1 \ldots d^4\ell_L \times \boxed{\text{Rational function of external and loop momenta}}$$

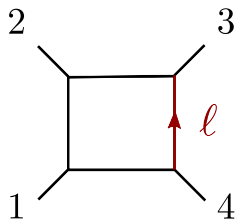
ℓ_i dummy variables, but must be defined consistently among various terms

Planar loop integrand

Ambiguities:

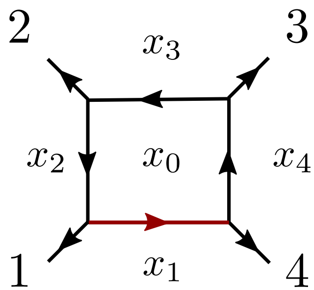


$$\leftrightarrow \quad \int d^4\ell \frac{1}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$



$$\leftrightarrow \quad \int d^4\ell \frac{1}{\ell^2(\ell+p_4)^2(\ell+p_1+p_4)^2(\ell-p_3)^2}$$

- Planar loop integrand well defined: dual variables x_i



$$= \int d^4x_0 \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}$$

$$p_i = x_i - x_{i+1}$$
$$x_{ij} = x_i - x_j$$
$$\ell = x_{01}$$

State of the art in the planar limit

All-loop **integrand** determined by the all-loop recursion relation

Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka

- Dual variables x_i allow different terms in recursion relation to be combined in a non-ambiguous way

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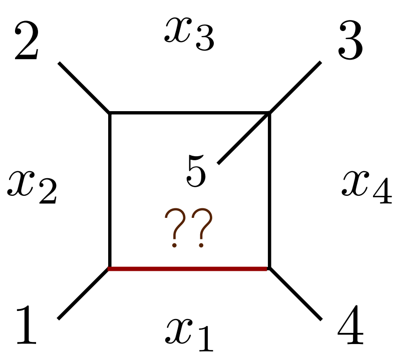


Unavailable for non-planar integrands

Non-planar integrand not well-defined

Can still study non planar **Leading Singularities**

Eden, Landshoff, Olive, Polkinghorne / Britto, Cachazo, Feng

Leading Singularities

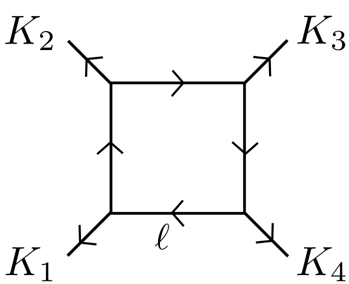
[[Eden, Landshoff, Olive, Polkinghorne / Britto, Cachazo, Feng]]

"Cut" propagators: $$\frac{1}{P^2} \longrightarrow \delta(P^2)$$

Compute residue of the integrand

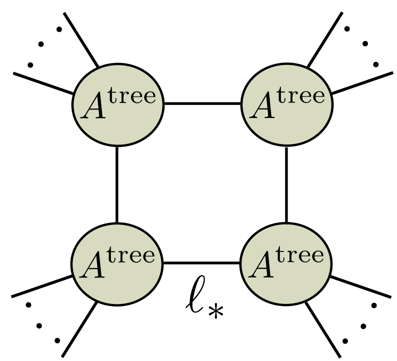
Leading singularities: Maximal number of propagators cut (4xL)

Ex: 1-loop

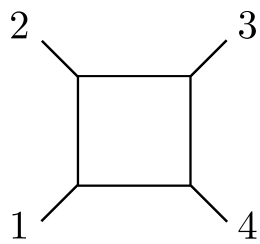


$$= \int d^4\ell \frac{1}{\ell^2(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell+K_4)^2}$$

cut 4 propagators $\longrightarrow$ $$\sum_{\ell_* \text{ solutions}} \mathcal{J} \times$$



Planar



In the planar limit ∃ basis of
dual conformal integrands with
"unit leading singularity"
[[Arkani-Hamed, Bourjaily, Cachazo, Trnka - 2010]]

↓

LS are sufficient to determine
the all-loop integrand!

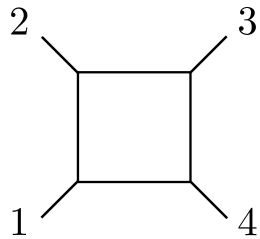
↓

All planar LS are residues of a
(positive) **Grassmannian** integral

↓

Positive Grassmannian
parametrised by
planar on-shell diagrams

Planar



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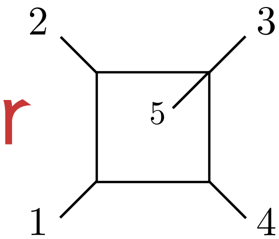
Non-Planar



Non-planar integrand
not well defined

↓

Consider non-planar LS

↓

Residues of a
Grassmannian integral

↓

Parametrised by
non-planar on-shell diagrams

Motivation

- Grassmannian formulation is linked to *on-shell diagrams*
  Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka

Region of the Grassmannian + dlog on-shell form

[[Arkani-Hamed, Bourjaily, Cachazo, Trnka]]

Conjecture: Non-planar amps have only log singularities and no poles at infinity.

Non-planar on-shell diagrams are the natural objects to study

Outline

1. Grassmannian formulation for amplitudes

    Review of planar
    Non-planar corrections

2. On-shell diagrams

    Review of planar
    Generalised face variables
    General boundary measurement
    A new type of singularity
    Equivalence, reductions, and polytopes

3. Conclusions + more possible applications

Notation

SU(N) colour generators

$$\mathcal{A}_n = \mathcal{A}_n(p^i_\mu, \epsilon^i_\mu, t^a_i) \qquad i = 1, \ldots, n$$

Kinematics

Polarisation vectors

Spinor variables

(kinematics and polarisation)

$$p_\mu \to p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu \xrightarrow{\text{massless}} \lambda_\alpha \widetilde{\lambda}_{\dot{\alpha}}$$

Spinor products

$$\langle ab \rangle = \epsilon^{\alpha\beta} \lambda^a_\alpha \lambda^b_\beta$$

$$[ab] = \epsilon^{\dot{\alpha}\dot{\beta}} \widetilde{\lambda}^a_{\dot{\alpha}} \widetilde{\lambda}^b_{\dot{\beta}}$$

Partial amplitude: $A_n(\eta^i, \lambda^i, \widetilde{\lambda}^i)$

(colour ordered)

Auxiliary fermionic variables

(helicity)

$$A = 1, 2, 3, 4 \in SU(4)_{\mathrm{R}}$$

$$\Phi(p, \eta) \quad := \quad g^+(p) + \eta_A \, \lambda^A(p) + \frac{\eta_A \eta_B}{2!} \, \phi^{AB}(p) + \epsilon^{ABCD} \frac{\eta_A \eta_B \eta_C}{3!} \, \bar{\lambda}_D(p) + \eta_1 \eta_2 \eta_3 \eta_4 \, g^-(p)$$

Grassmannian Formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]
[[Mason, Skinner - 2009]]

The geometry of momentum conservation

- External data: $\lambda^a, \widetilde{\lambda}^a,\, \eta^a$ for each particle $a = 1, 2, \ldots, n$
- Organise them in the $\Lambda,\, \widetilde{\Lambda}$ 2-planes in $\mathbb{C}^n$:

$$\Lambda \equiv \begin{pmatrix} \vec{\lambda}_1 \\ \vec{\lambda}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \cdots & \lambda_1^n \\ \lambda_2^1 & \lambda_2^2 & \cdots & \lambda_2^n \end{pmatrix} \qquad \widetilde{\Lambda} \equiv \begin{pmatrix} \vec{\widetilde{\lambda}}_1 \\ \vec{\widetilde{\lambda}}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{\lambda}_1^1 & \widetilde{\lambda}_1^2 & \cdots & \widetilde{\lambda}_1^n \\ \widetilde{\lambda}_2^1 & \widetilde{\lambda}_2^2 & \cdots & \widetilde{\lambda}_2^n \end{pmatrix}$$

and a fermionic 4-plane η:

$$\eta \equiv \begin{pmatrix} \vec{\eta}_1 \\ \vec{\eta}_2 \\ \vec{\eta}_3 \\ \vec{\eta}_4 \end{pmatrix} = \begin{pmatrix} \eta_1^1 & \eta_1^2 & \cdots & \eta_1^n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_4^1 & \eta_4^2 & \cdots & \eta_4^n \end{pmatrix}$$

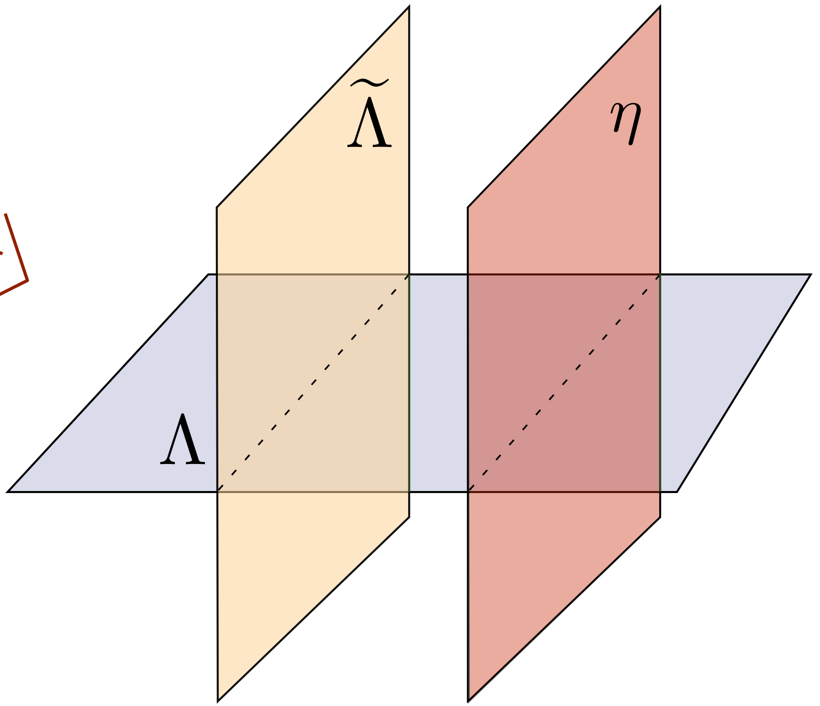
The geometry of momentum conservation

- (Super) momentum conservation:

$$\sum_{a=1}^{n} \lambda^a \widetilde{\lambda}^a = 0 \Rightarrow \Lambda \perp \widetilde{\Lambda}$$

$$\sum_{a=1}^{n} \lambda^a \eta^a = 0 \Rightarrow \Lambda \perp \eta$$

$\Rrightarrow$



Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

DEF: Grassmannian $Gr_{k,n}$ is the space of k-planes in $\mathbb{C}^n$

- Element of $Gr_{k,n}$: choose k n-vectors:

$$C_{\alpha a} = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \dots & C_{kn} \end{pmatrix}$$

- $GL(k)$ gauge redundancy $\longrightarrow$ $\dim(Gr_{k,n}) = nk - k^2$

For us: Scattering of gluons g^+, g^-

n : total number of gluons

k : number of gluons g^-

Recall:

k=0, 1, n-1, n $\longrightarrow$ Amp = 0

k=2 $\longrightarrow$ MHV

k=n-2 $\longrightarrow$ $\overline{\text{MHV}}$

Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

$$C_{\alpha a} = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \dots & C_{kn} \end{pmatrix}$$

- Coordinates in $Gr_{k,n}$ $\longrightarrow$ Maximal minors (Plücker coords.)

$$\Delta_{i_1,i_2,\dots,i_k} = (i_1 i_2 \cdots i_k) = \det \begin{pmatrix} C_{1i_1} & C_{1i_2} & \cdots & C_{1i_k} \\ C_{2i_1} & C_{2i_2} & \cdots & C_{2i_k} \\ \vdots & & & \vdots \\ C_{ki_1} & C_{ki_2} & \cdots & C_{ki_k} \end{pmatrix}$$

Plücker relations:

Ex: $Gr_{2,4} \longrightarrow \Delta_{1,2}\Delta_{3,4} + \Delta_{1,3}\Delta_{4,2} + \Delta_{1,4}\Delta_{2,3} = 0$

Positive Grassmannian $Gr^{+}_{k,n} \longrightarrow \Delta_{i_1,i_2,\dots,i_k} > 0 \begin{cases} \forall\, C_{\alpha a} > 0 \\ i_1 < i_2 < \cdots < i_k \end{cases}$

Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

Planar LS are residues of the following integral over $Gr^{+}_{k,n}$

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\Lambda})\,\delta(C^{\perp}\cdot\Lambda)\,\delta(C\cdot\eta)}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)}$$

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Gauge fix k^2 entries of C

Grassmannian formulation

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Ensure $C \perp \widetilde{\Lambda}$ $C \supset \Lambda$ $C \perp \eta$

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Gauge fix k^2 entries of C

$k \times k$ consecutive minors of C

Ex: $(12\ldots k) = \det \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \vdots & & & \vdots \\ C_{k1} & C_{k2} & \cdots & C_{kk} \end{pmatrix}$

Grassmannian formulation

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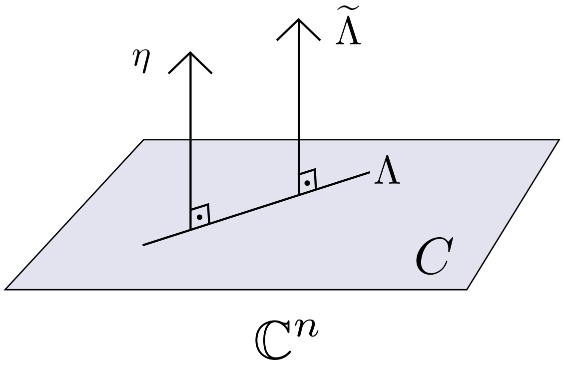
- Non-zero only if the plane C is orthogonal to $\widetilde{\Lambda}$, η and contains Λ

$\Longrightarrow$ Momentum conservation



Note:

For $k = 0, 1$ impossible to have $C \supset \Lambda$

For $k = n-1, n$ impossible to have $C \perp \widetilde{\Lambda}$

$\Longrightarrow$ Amplitudes automatically zero!

Grassmannian formulation

[[Arkani-Hamed, Cachazo, Cheung, Kaplan - 2009]]

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\Lambda})\,\delta(C^{\perp}\cdot\Lambda)\,\delta(C\cdot\eta)}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)}$$

Poles when **consecutive** minors vanish

Non-planar

[[Galloni, Franco, BP, Wen - 2015]]

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\Lambda})\,\delta(C^{\perp}\cdot\Lambda)\,\delta(C\cdot\eta)}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)} \times \mathcal{F}$$

$GL(k)$ invariance: cross ratio of minors

Ex: k=3 $\quad \mathcal{F} = \frac{(123)(245)}{(124)(235)}$

No notion of ordering or positivity in non-planar case $\rightarrow$ $\cancel{Gr^{+}_{k,n}}$

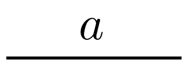
On-shell diagrams

[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

On-shell formulation

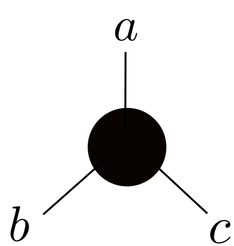
[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

Bi-coloured graphs made of the building blocks:

Edges: 

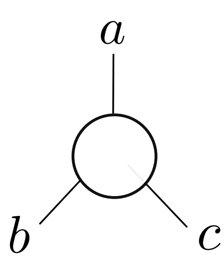
on-shell momentum $p_a = \lambda^a \widetilde{\lambda}^a$

Nodes



MHV amplitude $\leftrightarrow$ $Gr_{2,3}$

$\widetilde{\lambda}^a \propto \widetilde{\lambda}^b \propto \widetilde{\lambda}^c$

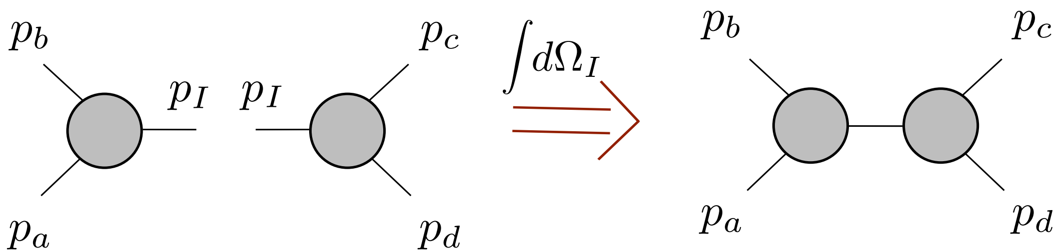


$\overline{\text{MHV}}$ amplitude $\leftrightarrow$ $Gr_{1,3}$

$\lambda^a \propto \lambda^b \propto \lambda^c$

Constructing on-shell diagrams

- To connect two nodes, integrate over on-shell phase space of edge in common:

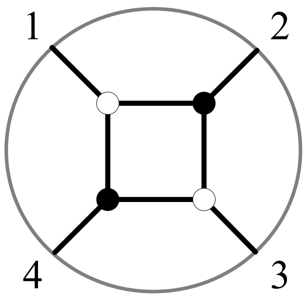


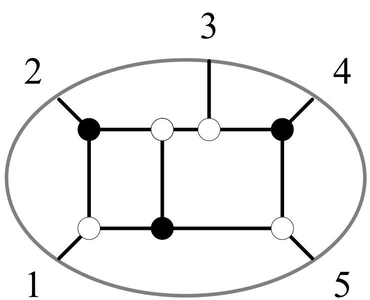
$$d\Omega_I = \frac{d^2\lambda^I d^2\widetilde{\lambda}^I d^4\eta^I}{\underbrace{\text{Vol}(GL(1))_I}_{\text{Little group}}}$$

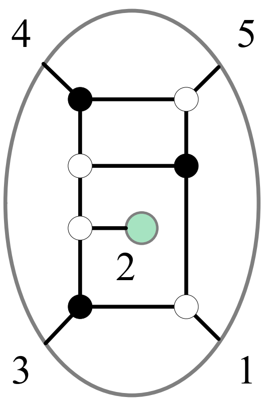
- Can construct more complicated diagrams
- Each node has two degrees of freedom

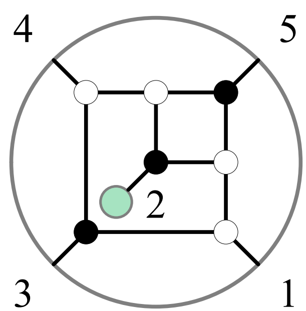
Constructing on-shell diagrams

Examples:



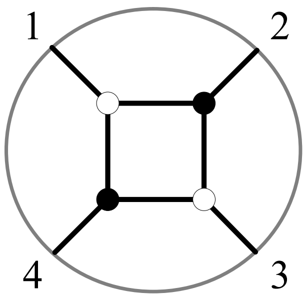


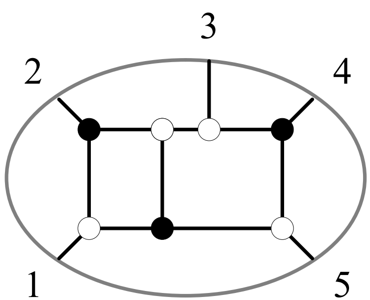




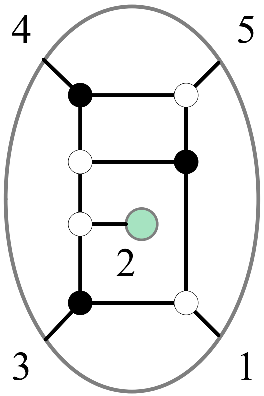
Constructing on-shell diagrams

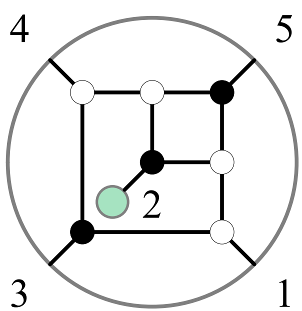
Examples:





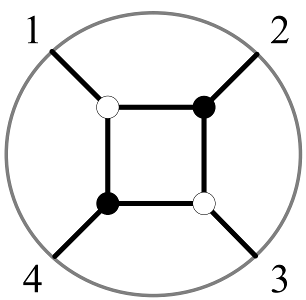
Planar:
Can be embedded on a disk

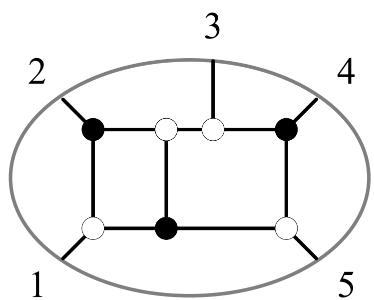




Constructing on-shell diagrams

Examples:





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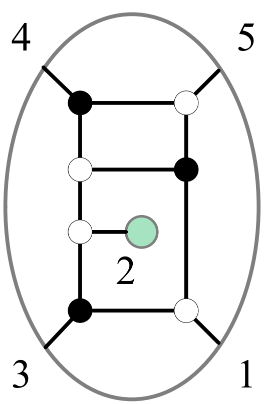
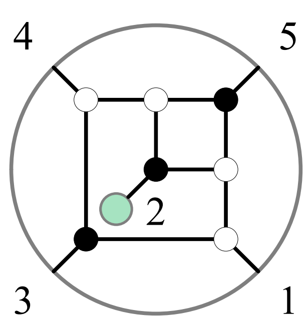


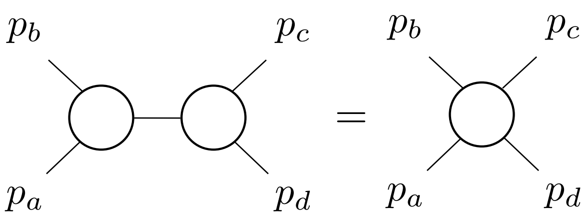


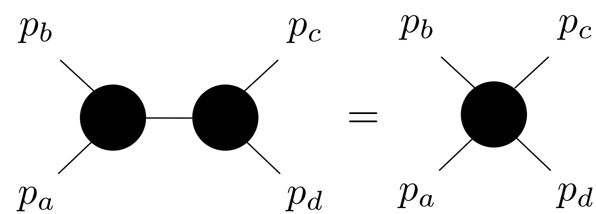
Non planar:
Can be embedded on a surface with multiple boundaries/ higher genus

Equivalence Moves

- On-shell diagrams are *equivalent* if they are related by the following moves:

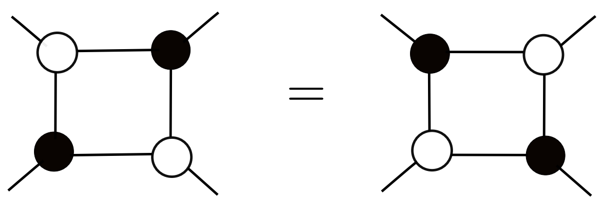
Merger:





$$\lambda^a \propto \lambda^b \propto \lambda^c \propto \lambda^d \qquad\qquad \widetilde{\lambda}^a \propto \widetilde{\lambda}^b \propto \widetilde{\lambda}^c \propto \widetilde{\lambda}^d$$

Square move:



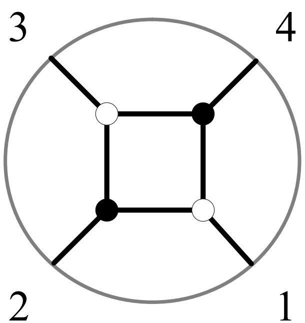
- Note that every on-shell diagram can be made **bipartite**

Fusing Grassmannians

- An on-shell diagram with n_B black nodes, n_W white nodes and n_I internal edges is associated to $Gr_{k,n}$, where:

$$k = 2n_B + n_W - n_I$$

Ex:



$$n_B = 2$$
$$n_W = 2$$
$$n_I = 4$$

$$k = 2 \times 2 + 2 - 4 = 2$$
$$n = 4$$

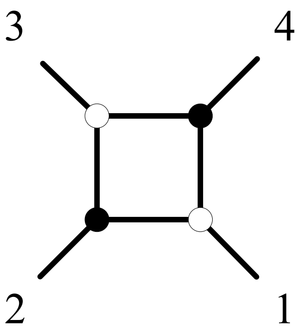
$$\boxed{Gr_{2,4}}$$

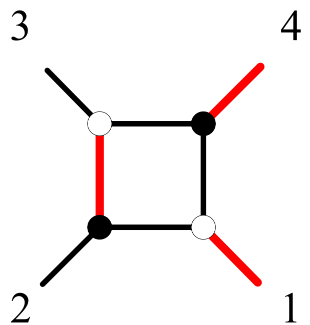
Boundary measurement

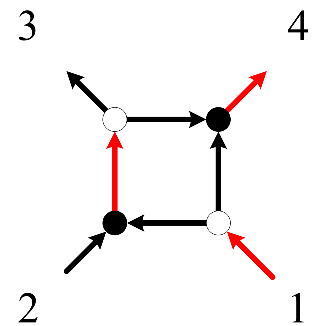
On-shell diagram

$C \in Gr_{k,n}$

Bipartite technology



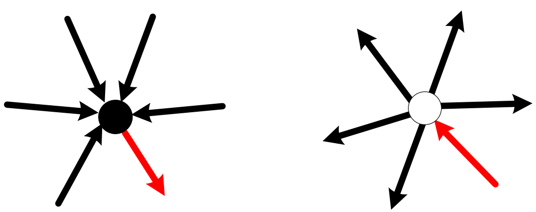




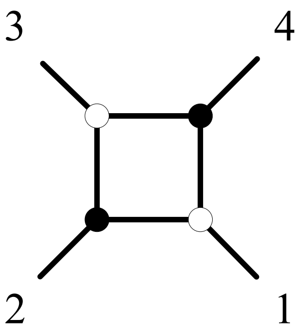
Perfect matching

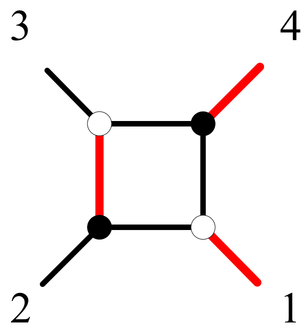
Choice of edges such that every internal node is the endpoint of only one edge

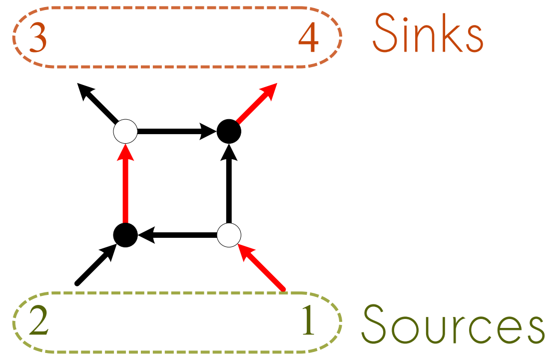
Perfect orientation



Bipartite technology







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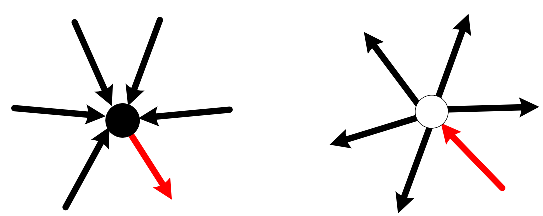
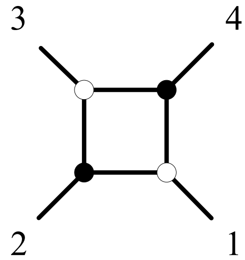
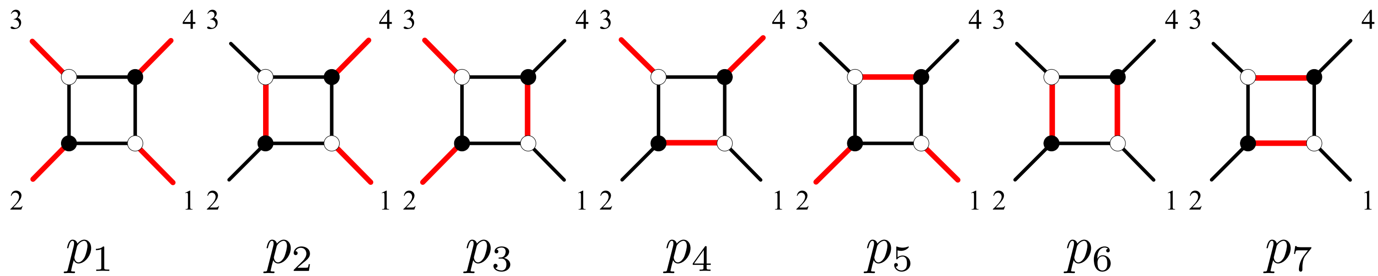


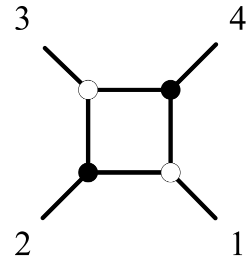
Example



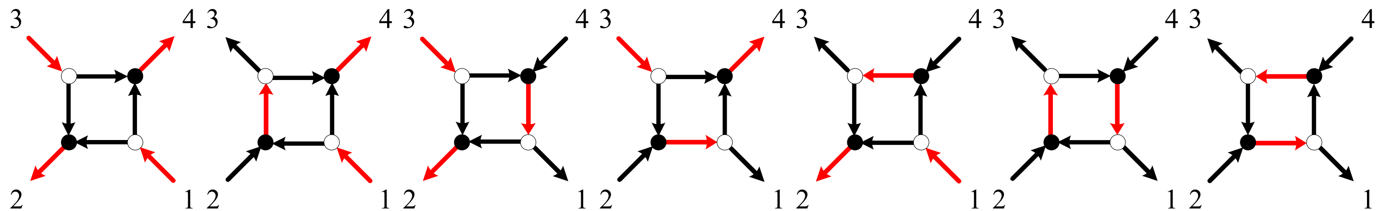
Perfect matchings:



Example

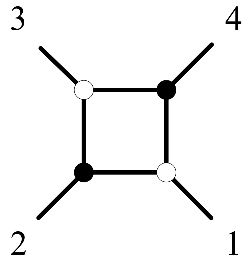


Oriented perfect matchings:



$\widetilde{p}_1$ $\widetilde{p}_2$ $\widetilde{p}_3$ $\widetilde{p}_4$ $\widetilde{p}_5$ $\widetilde{p}_6$ $\widetilde{p}_7$

Example



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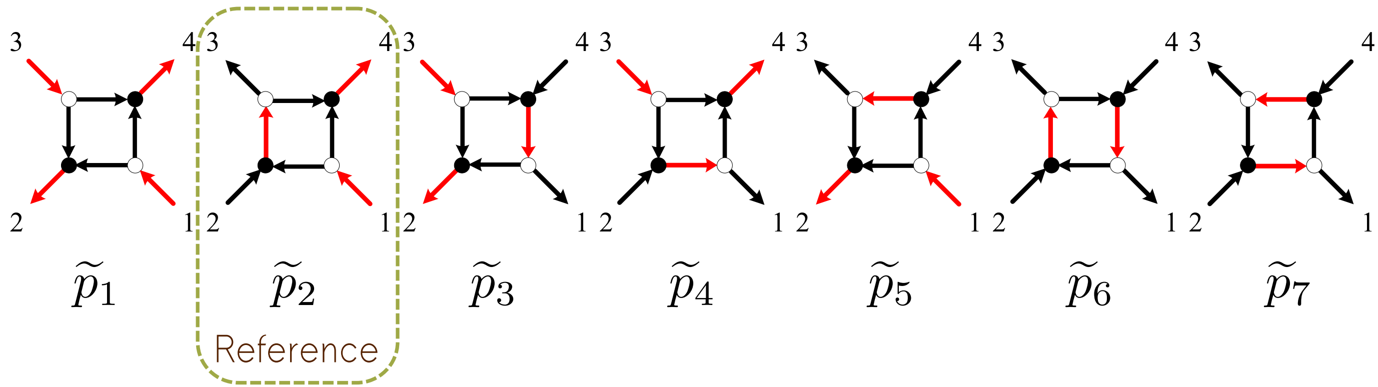
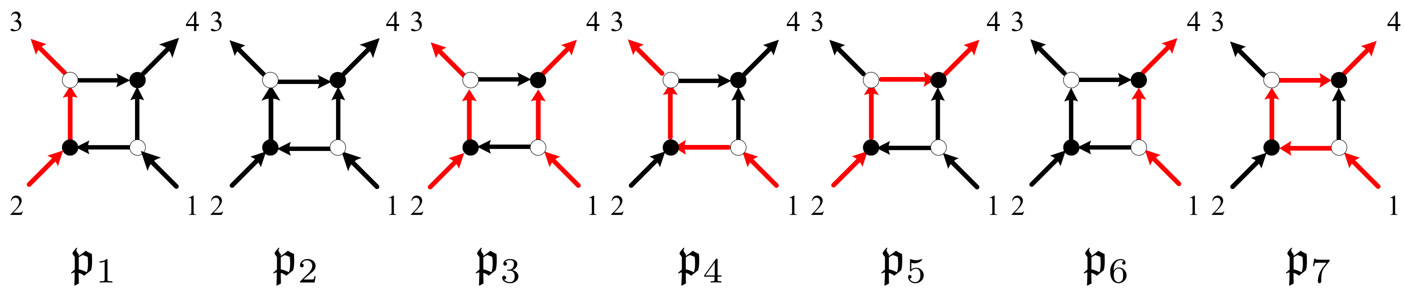
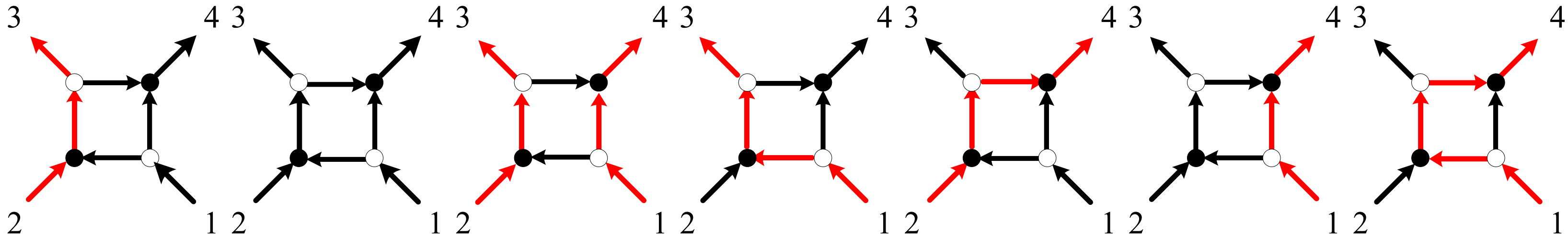


Flows:



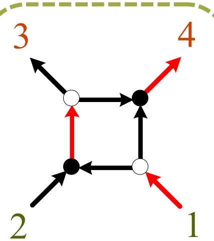
Boundary measurement

Flows:



$\mathfrak{p}_1$ $\mathfrak{p}_2$ $\mathfrak{p}_3$ $\mathfrak{p}_4$ $\mathfrak{p}_5$ $\mathfrak{p}_6$ $\mathfrak{p}_7$

Map between on-shell diagram and element of the Grassmannian



$\widetilde{p}_2$

Reference

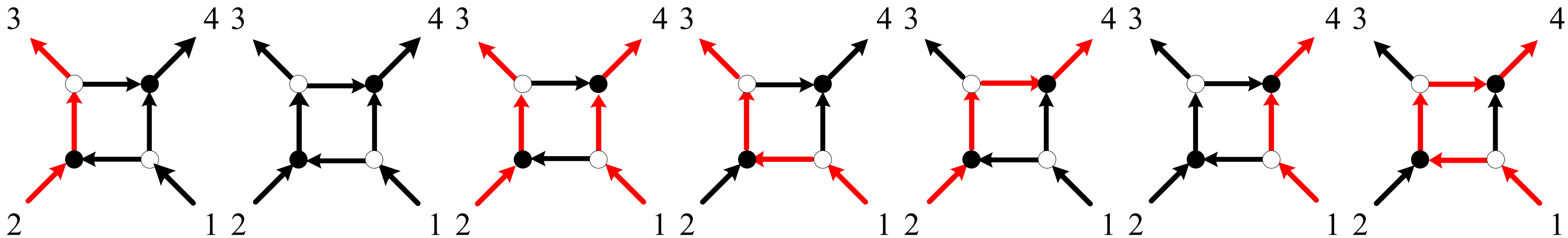
$$C = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \end{pmatrix} \end{array}$$

$$C_{ij} = \sum_{\Gamma\{i \rightsquigarrow j\}} (-1)^{s_\Gamma} \, \mathfrak{p}_{\{i \rightsquigarrow j\}}$$

Flows from i to j

Sign prescription

Boundary measurement



$\mathfrak{p}_1$	$\mathfrak{p}_2$	$\mathfrak{p}_3$	$\mathfrak{p}_4$	$\mathfrak{p}_5$	$\mathfrak{p}_6$	$\mathfrak{p}_7$
$\{1,3\}$	$\{1,2\}$	$\{3,4\}$	$\{2,3\}$	$\{1,4\}$	$\{2,4\}$	$\{2,4\}$

$$C = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & -\mathfrak{p}_4 & -(\mathfrak{p}_6+\mathfrak{p}_7) \\ 0 & 1 & \mathfrak{p}_1 & \mathfrak{p}_5 \end{pmatrix} \end{array}$$

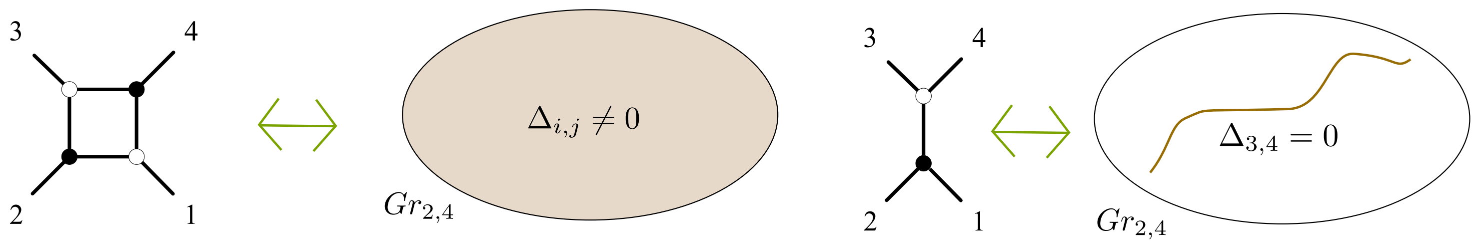
$(-1)^{s_\Gamma}$ → Plücker coordinates are positive in planar case and are a sum of flows with corresponding source set.

Sign prescription

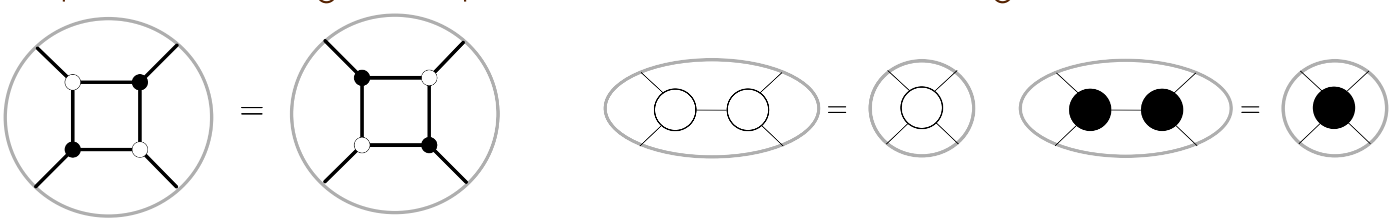
Ex: $\Delta_{1,2} = \mathfrak{p}_2\,,\ \Delta_{2,4} = \mathfrak{p}_6 + \mathfrak{p}_7$

Important

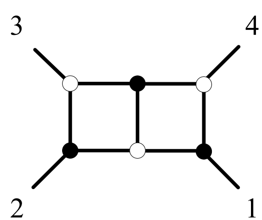
On-shell diagrams parametrise regions of the Grassmannian



Equivalent diagrams parametrise the same region:



Graph is *reducible* if possible to delete edges while preserving region.



Planar: set of non-zero minors preserved

This changes for non-planar graphs!

Otherwise graph is *reduced*.

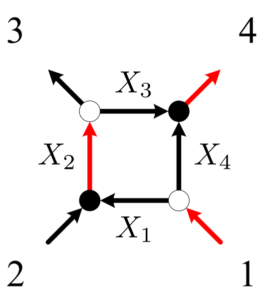
Parametrising on-shell diagrams

Planar:

- On-shell dlog form: variables unfixed by delta-functions mapped to loop integration variables.
- # degrees of freedom of a planar on-shell diagram is $d = F - 1$

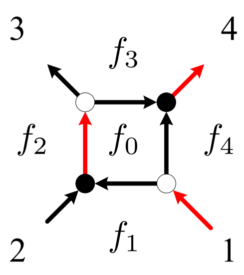
faces

- Bases for expressing flows: **Edges** and **Faces**



Edge variables:

$$\frac{dX_1}{X_1}\frac{dX_2}{X_2}\frac{dX_3}{X_3}\frac{dX_4}{X_4}\delta(C(X)\cdot\widetilde{\Lambda})\,\delta(C(X)^{\perp}\cdot\Lambda)\,\delta(C(X)\cdot\eta)$$



Face variables:

General for non-planar ?

$$\frac{df_1}{f_1}\frac{df_2}{f_2}\frac{df_3}{f_3}\frac{df_4}{f_4}\delta(C(f)\cdot\widetilde{\Lambda})\,\delta(C(f)^{\perp}\cdot\Lambda)\,\delta(C(f)\cdot\eta)$$

Generalised face variables

[[Galloni, Franco, BP, Wen - 2015]]

$$d = \underbrace{(F-1)}_{f_i} + \underbrace{(B-1)}_{b_a} + \underbrace{2g}_{\{\alpha_m,\beta_m\}} \quad = \quad F - \xi$$

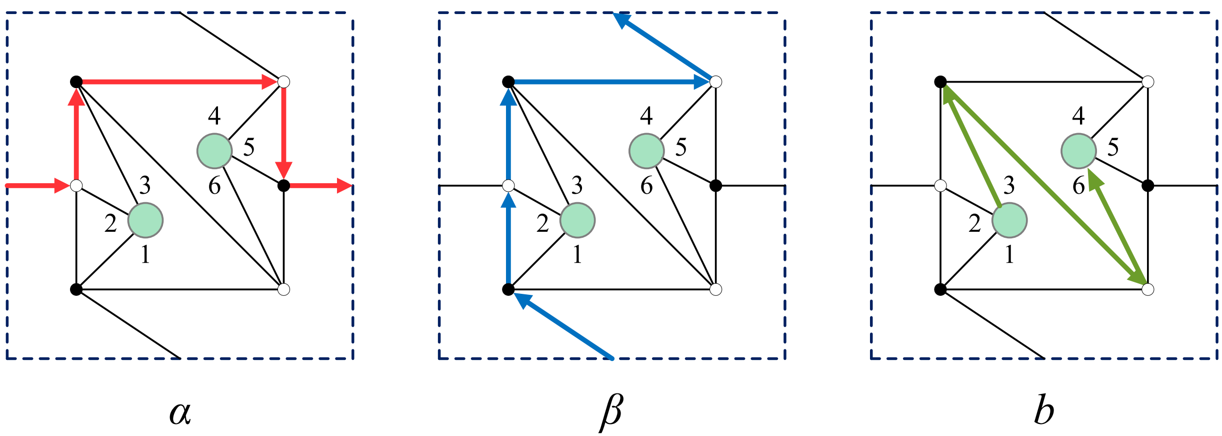
F = # faces
B = # boundaries
g = genus

$f_i,\ i = 1,\dots,F \qquad \prod_{i=1}^{F} f_i = 1$ — Faces

$b_a,\ a = 1,\dots,B-1$ — Paths connecting different boundaries

$\{\alpha_m,\beta_m\},\ m = 1,\dots g$ — Fundamental cycles

Ex: Genus 1



Generalised face variables

[[Galloni, Franco, BP, Wen - 2015]]

$$d = \underbrace{(F-1)}_{f_i} + \underbrace{(B-1)}_{b_a} + \underbrace{2g}_{\{\alpha_m, \beta_m\}} \quad = \quad F - \xi$$

F = # faces
B = # boundaries
g = genus

$f_i,\ i = 1, \ldots, F$	$\prod_{i=1}^{F} f_i = 1$	Faces
$b_a,\ a = 1, \ldots, B-1$		Paths connecting different boundaries
$\{\alpha_m, \beta_m\},\ m = 1, \ldots g$		Fundamental cycles

dlog on-shell form:

$$\boxed{\frac{dX_1}{X_1}\frac{dX_2}{X_2}\cdots\frac{dX_d}{X_d} \quad \leftrightarrow \quad \prod_{i=1}^{F-1}\frac{df_i}{f_i}\prod_{a=1}^{B-1}\frac{db_a}{b_a}\prod_{m=1}^{g}\frac{d\alpha_m}{\alpha_m}\frac{d\beta_m}{\beta_m}}$$

Some properties of planar graphs

[[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka - 2012]]

- If it is impossible to remove an edge of a graph without sending some Plücker coord to zero, the graph is *reduced*.

$\Longrightarrow$ (Positroid stratification of $Gr^{+}_{k,n}$)

- Boundaries of the regions in the positive Grassmannian are always $\Delta_i = 0$

Recall:

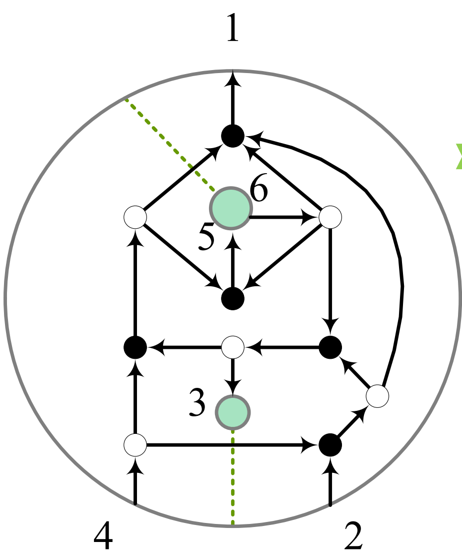
$$\mathcal{L}_{n,k} = \frac{1}{\text{Vol}(GL(k))} \int d^{k\times n} C_{\alpha a} \frac{\delta(C\cdot\widetilde{\Lambda})\,\delta(C^{\perp}\cdot\Lambda)\,\delta(C\cdot\eta)}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots k-1)}$$

Non-planar novelties

[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014, Galloni, Franco, BP, Wen - 2015]]

A non-planar novelty:

- It is possible to remove an edge of a **reduced** graph without sending any Plücker coord to zero!



Recall: Deformation from planar Grassmannian integrand

$$\mathcal{F} = \frac{(346)^2(356)(123)(612)}{(136)(236)[(124)(346)(365)-(456)(234)(136)]}$$

OBS: Method for determining $\mathcal{F}$: generalisation of [[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014]] from k=2 leading singularities to higher k

Reducibility & Equivalence: Non-planar

[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014, Galloni, Franco, BP, Wen - 2015]]

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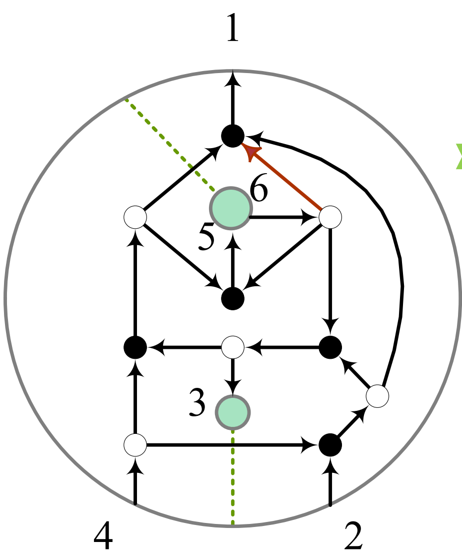

Recall: Deformation from planar Grassmannian integrand

$$\mathcal{F} = \frac{(346)^2(356)(123)(612)}{(136)(236)[(124)(346)(365) - (456)(234)(136)]}$$

Removal of an edge does not set any $\Delta_{i,j,k}$ to zero, but gives rise to the relation

$$\boxed{\Delta_{1,2,4}\Delta_{3,4,6}\Delta_{3,6,5} = \Delta_{4,5,6}\Delta_{2,3,4}\Delta_{1,3,6}}$$

Reducibility & Equivalence: Non-planar

[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014, Galloni, Franco, BP, Wen - 2015]]

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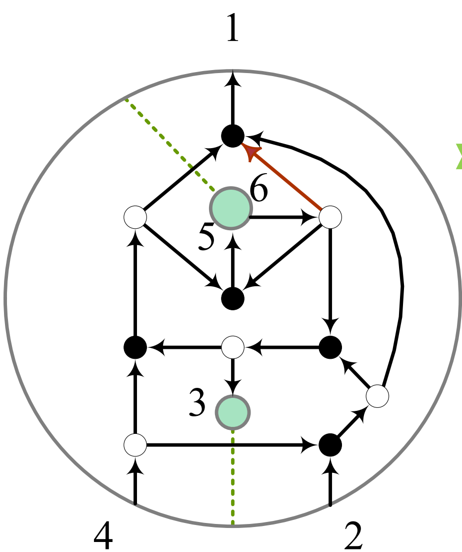
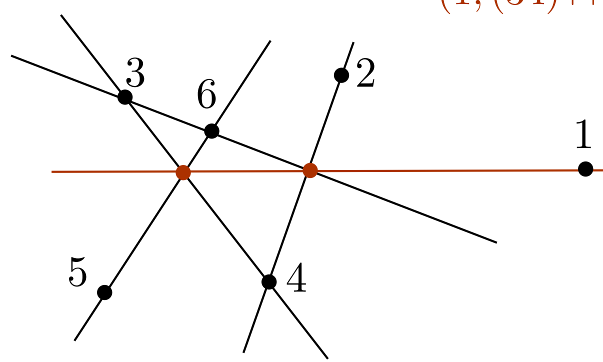


Recall: Deformation from planar Grassmannian integrand

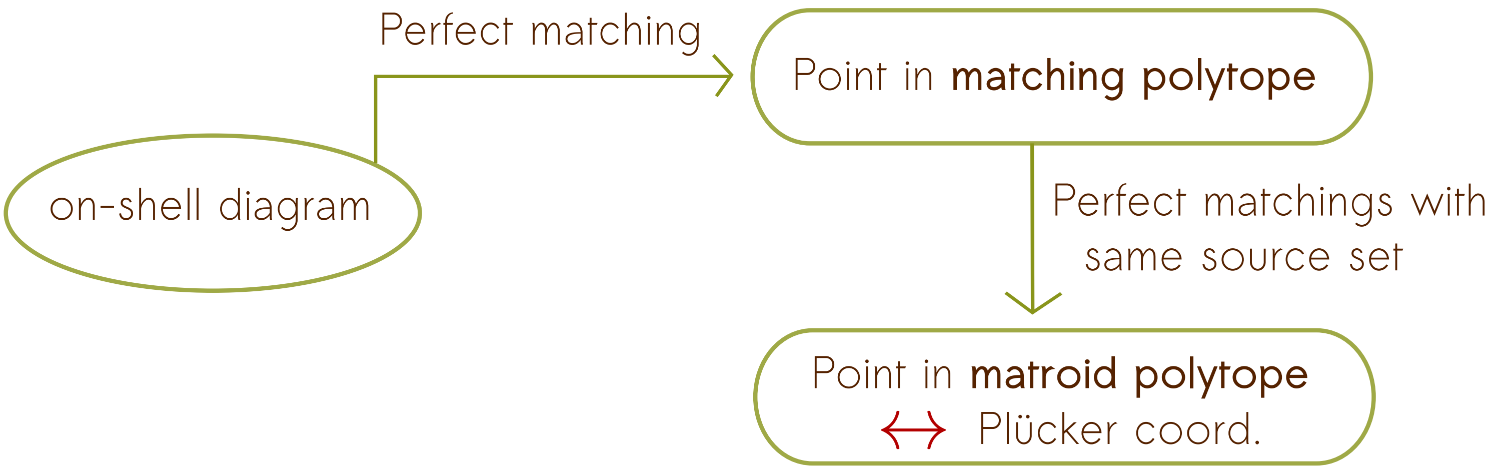
$$\mathcal{F} = \frac{(346)^2(356)(123)(612)}{(136)(236)[(124)(346)(365)-(456)(234)(136)]}$$

$$(1,(34)\cap(56),(24)\cap(36))=0$$



Polytopes

Notions of equivalence/reduction can be rephrased in terms of polytopes:
[[Postnikov, Speyer, Williams - 2008, Franco, Galloni, Mariotti - 2013]]



$\Delta_{i_1,i_2,\ldots,i_k} \leftrightarrow$ Sum of flows with source set $\{i_1,i_2,\ldots,i_k\}$ with coefficients ± 1

[[Gekhtman, Shapiro, Vainshtein - 2013]] → Annulus
[[Franco, Galloni, Mariotti - 2013]] → Arbitrary *B*, genus zero
[[Franco, Galloni, BP, Wen - 2015]] → Any graph

Non-vanishing minors and relations can be seen from the polytopes

Characterisation of on-shell diagrams

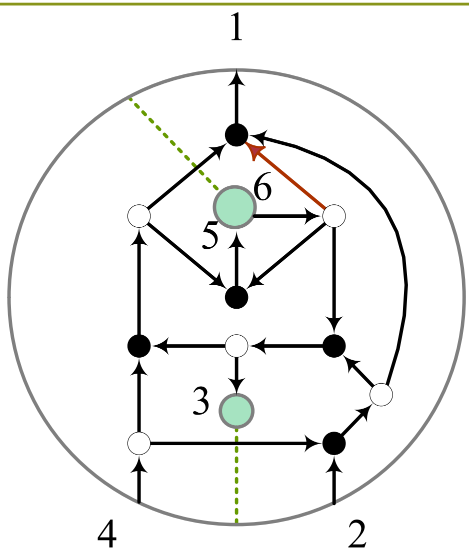
planar / reduced → Cell of the Grassmannian characterised by a zig zag path

Non-planar **without extra constraints** on Plücker coordinates → Cell of the Grassmannian characterised by matroid polytope

- Equivalent graphs have the same matroid polytope
- Reduction of a diagram removes 1 dof while preserving the matroid polytope
- A graph is **reduced** if it is impossible to remove edges while preserving the matroid polytope.

Constraints and polytopes: example

[[Galloni, Franco, BP, Wen - 2015]]



Before removal: 40 perfect matchings

$\Delta_{1,2,4}$	p_7, p_1	$\Delta_{4,5,6}$	p_4
$\Delta_{3,4,6}$	p_2	$\Delta_{2,3,4}$	p_5
$\Delta_{3,5,6}$	p_3	$\Delta_{1,3,6}$	p_6

After removal: 33 perfect matchings

Before and after removal:

$$p_1 p_2 p_3 = p_4 p_5 p_6$$

After removal

p_7 disappears

$$\Delta_{1,2,4}\Delta_{3,4,6}\Delta_{3,6,5} = \Delta_{4,5,6}\Delta_{2,3,4}\Delta_{1,3,6}$$

Concluding remarks & Outlook

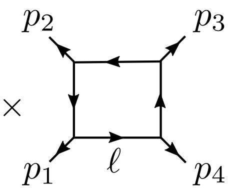
1) Physical interpretation:

Planar: All tree level amplitudes and loop integrands via BCFW recursion relation.

Britto, Cachazo, Feng, Witten / Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka

dlog form of the loop integrand:

$$\mathcal{A}^{L=1} = \mathcal{A}^{L=0} \times$$



$$= \mathcal{A}^{L=0} \times \int d^4\ell \, \frac{(p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

$$\boxed{d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)}$$

Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka

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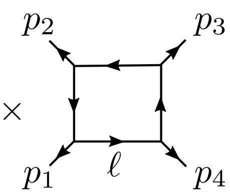

$$= \mathcal{A}^{L=0} \times \int d^4\ell \, \frac{(p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

$$\boxed{d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)}$$

Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka

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Non planar: Leading singularities of the loop integrand

? Non-planar loop integrand

? Non-planar Grassmannian formulation

[[Arkani-Hamed, Bourjaily, Cachazo, Trnka - 2014]]

Conjecture: Non-planar amps have only log singularities and no poles at infinity.

Concluding remarks & Outlook

2) Non-planar diagrams parametrise regions of $Gr_{k,n}$ with hidden relations between Plücker coordinates.
$\hookrightarrow$? Method for finding representative graph given a constraint

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[[Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka - 2014]]

Same not true for Non-MHV, however similar method can be used to find the deformation of the integrand $\mathcal{F}$.

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4) Positive Grassmannian $Gr^{+}_{k,n}$ $\rightarrow$ Amplituhedron
[[Arkani-Hamed, Trnka - 2013]]
? Non-planar generalisation
[[Evidence: Bern, Herrmann, Litsey, Stankowicz, Trnka - 2015]]