Friezes, difference equations and moduli spaces

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(illustrations shown during the talk given at University of Kent- January 2016)

Frieze of width 4

$$
\begin{array}{ccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
& 2 & & 1 & & 3 & & 2 & & 2 & & 1 & & 4 & & 2 & & 1 & & \cdots \\
\cdots & & 1 & & 2 & & 5 & & 3 & & 1 & & 3 & & 7 & & 1 & & 2 & \\
& 3 & & 1 & & 3 & & 7 & & 1 & & 2 & & 5 & & 3 & & 1 & & \cdots \\
\cdots & & 2 & & 1 & & 4 & & 2 & & 1 & & 3 & & 2 & & 2 & & 1 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Periodicity

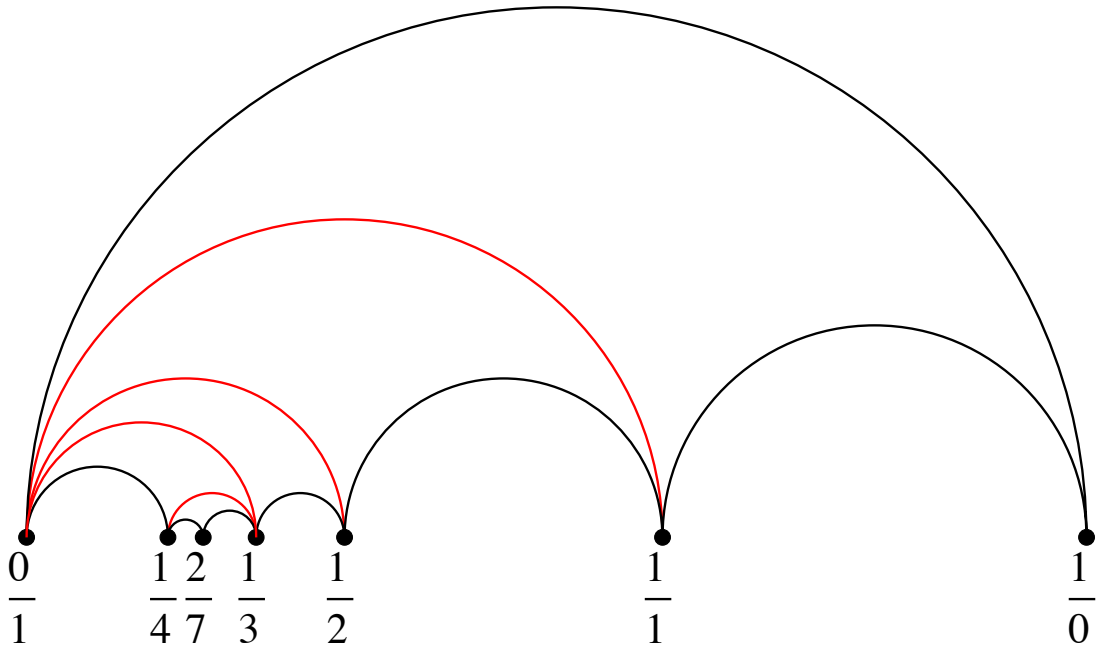
$$
\begin{array}{ccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
& 2 & & 1 & & 3 & & 2 & & 2 & & 1 & & 4 & & 2 & & 1 & & \cdots \\
\cdots & & 1 & & 2 & & 5 & & 3 & & 1 & & 3 & & 7 & & 1 & & 2 & \\
& 3 & & 1 & & 3 & & 7 & & 1 & & 2 & & 5 & & 3 & & 1 & & \cdots \\
\cdots & & 2 & & 1 & & 4 & & 2 & & 1 & & 3 & & 2 & & 2 & & 1 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Glide symmetry

1		1		1		1		1		1		1		1		1		1	
	2		1		3		2		2		1		4		2		1		...
...		1		2		5		3		1		3		7		1		2	
	3		1		3		7		1		2		5		3		1		...
...		2		1		4		2		1		3		2		2		1	
	1		1		1		1		1		1		1		1		1		1

Farey graph

$\cdots$		1		1		1		1		1		1		1		1		1		
	2		1		3		2		2		1		4		2		1		$\cdots$	
$\cdots$		1		2		5		3		1		3		7		1		2		
	3		1		3		7		1		2		5		3		1		$\cdots$	
$\cdots$		2		1		4		2		1		3		2		2		1		
	1		1		1		1		1		1		1		1		1		1	



SL_3-frieze of width 4

1		1		1		1		1		1		1		1					
	7		2		2		2		5		3		2		2				
11		10		2		2		8		5		5		1					
	15		7		1		7		5		6		2		2				
2		10		1		3		3		4		2		2					
	1		1		1		1		1		1		1		1				

SL_3-frieze of width 4 - Projective duality on friezes

1		1		1		1		1		1		1		1					
	7		4		2		2		5		3		2		2				
11		10		2		2		8		5		5		1					
	15		7		1		7		5		6		2		2				
2		10		1		3		3		4		2		2					
	1		1		1		1		1		1		1		1				

SL_3-frieze of width 4 - Projective duality on friezes

1		1		1		1		1		1		1		1		
	7		4		2		2		5		3		2		2	
11		10		2		2		8		5		5		1		
	15		7		1		7		5		6		2		2	
2		10		1		3		3		4		2		2		
	1		1		1		1		1		1		1		1	

SL_3-frieze of width 4 - Projective duality on friezes

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	7	4	2	2	2	2	2	2	5	10	3	1	2	3	2
11	5	10	6	2	2	2	2	8	15	5	7	5	1	1	7
8	15	5	7	5	1	1	7	11	5	10	6	2	2	2	2
2	5	10	3	1	2	3	2	3	7	4	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Gale duality on friezes and equations

Theorem ([MGOST])

Let F be an SL_{k+1}-frieze of width w associated with the equation

$$V_i = a_i^1 V_{i-1} - a_i^2 V_{i-2} + \cdots - (-1)^k a_i^k V_{i-k} + (-1)^k V_{i-k-1}.$$

The coefficients of the equation forms an SL_{w+1}-frieze of width k:

$$\begin{array}{ccccccccccccccccccc}
\ldots & & 1 & & & & 1 & & & & 1 & & & & 1 & & & & 1 \\
& & & & \ldots & & & & a_n^1 & & & & a_1^1 & & & & a_2^1 & & \ldots & & & & a_n^1 & & \\
& & & & & & a_n^2 & & & & a_1^2 & & & & a_2^2 & & & & & & a_n^2 & & & & \\
\ldots & & & & \ddots & & & & \ddots & & & & \ddots & & & & & & \ddots & & & & \ldots & & \\
& & a_n^k & & & & a_1^k & & & & a_2^k & & & & \ldots & & & & a_n^k & & & & \ldots & & \\
1 & & & & 1 & & & & 1 & & & & 1 & & & & 1 & & & & \ldots & &
\end{array}$$

This defines the Gale dual of F.