London Taught Course Centre - Mechanics Exercises - 12.11.08

(Note: Unless otherwise stated, the Einstein summation convention is assumed throughout.)

1. Geodesic equations on a Riemannian manifold: free motion in curved space
Given a Riemannian manifold M with metric g, consider the action $S = \int_{t_0}^{t_1} L\, dt$ with the purely kinetic Lagrangian

$$L = \frac{1}{2} g(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k, \tag{1}$$

where $g = (\, g_{jk}(\mathbf{x})\,)$ is the metric given in terms of local coordinates x^j for a point $\mathbf{x} \in M$ with tangent vector $\dot{\mathbf{x}}$; the dot denotes differentiation with respect to the parameter t ("time") along a curve.

(i) Applying the principle of least action, $\delta S = 0$, calculate the Euler-Lagrange equations for this Lagrangian, and show that they can be written in the form

$$\ddot{x}^j + \Gamma^j_{kl} \dot{x}^k \dot{x}^l = 0, \tag{2}$$

where Γ^j_{kl} (the Christoffel symbols) should be found in terms of g and its derivatives.

(ii) Perform a Legendre transformation and hence reformulate the geodesic equations as an Hamiltonian system on T^*M.

Remarks: In general, a curve γ in M with tangent vector T^j is a geodesic if $T^k \nabla_k T^j = f\, T^j$, for some function f along γ, where ∇_j denotes covariant derivative (the Levi-Civita connection associated with g). If γ is parametrized by $x^j(\lambda)$ in local coordinates, so $T^j = dx^j/d\lambda$, then the equation for geodesics is given by

$$\frac{d^2 x^j}{d\lambda^2} + \Gamma^j_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = f(\lambda) \frac{dx^j}{d\lambda}. \tag{3}$$

Additional exercise (optional): Given (3), show that it is possible to parametrize γ by a parameter $t = t(\lambda)$ (called an affine parameter) such that (2) holds.

In general relativity, space-time is a pseudo-Riemannian manifold (the metric has indefinite signature) and the geodesics correspond to the trajectories of test particles in the gravitational field defined by the metric. (One should assume that the test particles are of small mass, because the presence of mass alters the gravitational field, and hence the metric.) The null geodesics, such that $g(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = 0$, correspond to the paths of photons.

2. Geodesics on SO(3) and the Euler top
For the Lie group SO(3) of rotations in three-dimensional space,

$$\mathrm{SO}(3) = \{ R \in M_3(\mathbb{R}) \,|\, RR^T = 1 \},$$

a left-invariant Riemannian metric is defined by

$$g(A, B) = -\mathrm{tr}\,(R^{-1} A \Sigma R^{-1} B),$$

where $A, B \in T_R\mathrm{SO}(3)$ are tangent vectors at $R \in \mathrm{SO}(3)$ and Σ is an arbitrary positive definite symmetric matrix. Geodesics on SO(3) with respect to this metric are calculated from the kinetic energy Lagrangian

$$L = \frac{1}{2} g(\dot{R}, \dot{R})$$

defined on the tangent bundle $T\mathrm{SO}(3)$.

(i) Show that, without loss of generality, after a suitable transformation $R \to R\tilde{R}$ for another rotation $\tilde{R}$, the above Lagrangian can be rewritten as

$$L = -\frac{1}{2}\mathrm{tr}\,(\Omega\Lambda\Omega), \tag{4}$$

where $\Omega = R^{-1}\dot{R} \in \mathfrak{so}(3)$ (i.e. Ω is a skew-symmetric matrix) and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, is the diagonal matrix of eigenvalues of Σ.

(ii) Taking variations of the Lagrangian with respect to R, derive the formula

$$\delta L = \langle \delta\Omega, \Pi \rangle \,,$$

where

$$\Pi = \frac{\partial L}{\partial \Omega} = \Lambda\Omega + \Omega\Lambda \in \mathfrak{so}(3)$$

and $\langle \, , \, \rangle$ is the Killing form (inner product) on $\mathfrak{so}(3)$, defined by $\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{so}(3)$. Show also that

$$\delta\Omega = \dot{\Xi} + [\Omega, \Xi]$$

where $\Xi = R^{-1}\delta R$ is an $\mathfrak{so}(3)$-valued variation.

(iii) Use the principle of least action ($\delta S = 0$) for the action $S = \int_{t_0}^{t_1} L\,dt$, with variations δR vanishing at the endpoints, to show that the Euler-Lagrange equations for the rigid body Lagrangian above can be written as

$$\dot{\Pi} = [\Pi, \Omega] \tag{5}$$

Hint: Use the ad-invariance of the Killing form, that is

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \,.$$

(iv) The Lie algebra $\mathfrak{so}(3)$ with commutator $[\, , \,]$ is isomorphic to $\mathbb{R}^3$ with the cross product $\times$, so that

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

is identified with the angular velocity vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$. Peform the Legendre transformation

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \boldsymbol{\omega}} = I\,\boldsymbol{\omega}, \qquad H(\boldsymbol{\pi}) = \boldsymbol{\omega} \cdot \boldsymbol{\pi} - L(\boldsymbol{\omega}),$$

where the diagonalized intertia tensor $I = \mathrm{diag}(I_1, I_2, I_3)$ is given in terms of the eigenvalues λ_j. Hence show that Euler's equations for the rigid body take the Hamiltonian form

$$\frac{d\boldsymbol{\pi}}{dt} = \boldsymbol{\pi} \times \nabla H = \{\boldsymbol{\pi}, H\} \tag{6}$$

with the $\mathfrak{so}(3)$ Lie-Poisson bracket

$$\{\pi_j, \pi_k\} = -\epsilon_{jkl}\pi_l$$

on the reduced phase space $\mathbb{R}^3$ for the angular momentum $\boldsymbol{\pi}$.

(v) Show that the total angular momentum $C = |\boldsymbol{\pi}|^2$ is a Casimir for the Lie-Poisson bracket, i.e. $\{C, F\} = 0$ for all smooth functions F on $\mathbb{R}^3$. By considering the geometry of the level sets $H = $const, $C = $const, or otherwise, show that (almost) all of the orbits of the system (6) are periodic. Given $\boldsymbol{\pi}(t)$, how is the rotation matrix $R(t)$ obtained? *Hint:* (Otherwise.) Integrate the equations of motion (6) explicitly using Jacobi elliptic functions.

Remarks: With minor modifications, the Lagrangian description and the Lie-Poisson formulation of geodesic flows on other finite-dimensional Lie groups is entirely analogous to the above. However, in general such flows are not necessarily completely integrable, as is the case here.

3. Free motion on matrices and rational Calogero-Moser systems

Let $\mathcal{H} = -i\mathfrak{u}(n)$ denote the vector space of self-adjoint (Hermitian) $n \times n$ matrices, that is

$$\mathcal{H} = \{A \in M_n(\mathbb{C}) \,|\, A = A^\dagger\},$$

which (up to factors of i) is isomorphic to the Lie algebra $\mathfrak{u}(n)$ (anti-Hermitian matrices). The Lie group $\mathrm{U}(n)$ of unitary matrices acts on $\mathcal{H}$ by conjugation,

$$A \mapsto Ad(U) \cdot A := UAU^\dagger, \qquad U \in \mathrm{U}(n).$$

The cotangent bundle $T^*\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ is a symplectic manifold with the symplectic form $\omega = \mathrm{tr}\, dB \wedge dA = dB_{jk} \wedge dA_{kj}$.

(i) What is the dimension of $\mathcal{H}$?

(ii) Write down Hamilton's equations for the Hamiltonian

$$H = \frac{1}{2}\mathrm{tr}\, B^2 = \frac{1}{2} B_{jk} B_{kj},$$

show that they correspond to free motion ($\ddot{A} = 0$) on $\mathcal{H}$, and hence write down the solution of these equations for $A(t)$, $B(t)$. Show also that the action of $\mathrm{U}(n)$ on $\mathcal{H}$ extends to an action on $T^*\mathcal{H}$ that preserves ω and H.

(iii) A moment map $\varphi : T^*\mathcal{H} \to \mathfrak{u}(n)^* \cong \mathfrak{u}(n)$ is defined by

$$\varphi : \ (A, B) \mapsto i\, C := [A, B].$$

Verify that $\mathrm{tr}\, C = 0$ and C is preserved by the Hamiltonian flow in (ii) above. For a unitary matrix $U = U(t)$, define $Q = UAU^\dagger$, $L = UBU^\dagger$, $\tilde{C} = UCU^\dagger$, and show that the equations

$$i\, \tilde{C} = [Q, L], \qquad \dot{Q} = L + [M, Q], \qquad \dot{L} = [M, L] \tag{7}$$

all hold, where $M = \dot{U}U^\dagger$. The third equation in (7) is called the *Lax equation*.

(iv) Suppose that C is a rank one perturbation of the identity given by

$$C = 1 - \mathbf{e}\mathbf{e}^\dagger = 1 - \mathbf{e} \otimes \overline{\mathbf{e}},$$

where $\mathbf{e} = (1, 1, \ldots, 1)^T$. Let U be chosen to diagonalize A, so that $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$, and assume that $U\mathbf{e} = \mathbf{e}$ also holds. Then show that the Lax matrix L has components

$$L_{jk} = p_j \delta_{jk} - i(1 - \delta_{jk})(q_j - q_k)^{-1}$$

(no summation), where $p_j = \dot{q}_j$. Hence show that in the coordinates (q_j, p_j), the Hamiltonian reduces to

$$H = \frac{1}{2}\sum_j p_j^2 + \sum_{j<k}(q_j - q_k)^{-2},$$

which is the rational Calogero-Moser Hamiltonian. Assuming that these are canonically conjugate coordinates and momenta, calculate the equations of motion for the rational Calogero-Moser system.

(v) Show that the n quantities

$$I_j = \operatorname{tr} L^j, \qquad j = 1, \ldots, n$$

(with $H = 2I_2$) are conserved under the Calogero-Moser equations of motion, and hence (assuming that they Poisson commute with one another) prove that this is a completely integrable system in the sense of Liouville. Use the Lax equation to show that the anti-Hermitian matrix M has components

$$M_{jk} = i\delta_{jk}\Big(\mu + \sum_{l\neq j}(q_j - q_l)^{-2}\Big) - i(1 - \delta_{jk})(q_j - q_k)^{-2}$$

(with no summation on j, k), for some $\mu \in \mathbb{R}$.
Additional exercises (optional): Given that $U(t)$ diagonalizes $A(t)$, justify the assumption that this unitary matrix can be chosen so that $U(t)\,\mathbf{e} = \mathbf{e}$ (and hence $\tilde{C} = C$) for all t. Prove that the variables $(\mathbf{q}, \mathbf{p}) = (q_j, p_j)_{j=1,\ldots,n}$ are canonically conjugate, and show that the functions $I_j = I_j(\mathbf{q}, \mathbf{p})$ are in involution with respect to the canonical Poisson bracket.

Remarks: The rational Calogero-Moser system above belongs to a rather large family of integrable models. One generalization is to write down the Hamiltonian in the form

$$H = \frac{1}{2}\sum_j p_j^2 + \sum_{j<k} f(q_j - q_k), \tag{8}$$

and then it turns out that the system is still integrable if one takes $f(x) = \wp(x)$, where $\wp$ is the Weierstrass elliptic function, which satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

with constants g_2, g_3 (known as the invariants). The rational case $f(x) = 1/x^2$ appears when $g_2 = g_3 = 0$, and the trigonometric/hyperbolic cases $(1/\sin^2 x\,/\,1/\sinh^2 x)$ arise when the discriminant $g_2^3 - 27g_3^2 = 0$ (with these latter cases often being referred to as Calogero-Sutherland models, especially in the quantum setting).

The above construction (based on coadjoint orbits of Lie groups) is called the orbit method, and it can be generalized to the case of other Lie groups and algebras. Observe that the potential in (8) is invariant under the symmetric group S_n (permuting the particle positions $q_1, \ldots, q_n$), which is the Weyl group corresponding to the Lie algebra $\mathfrak{u}(n)$ (or the A_{n-1} root system). Analogous potentials can be written down for other simple Lie algebras and the associated root systems.

A further generalization is to consider Hamiltonians of the form

$$H_\pm = \sum_j e^{\pm p_j} \prod_{k\neq j} f(q_j - q_k),$$

which leads to the family of integrable Ruijsenaars-Schneider models. These again come in rational/trigonometric/elliptic flavours, and generalize to arbitrary root systems. They are associated with a realization of the two-dimensional Poincaré algebra in terms of Poisson brackets, so they are sometimes referred to as "relativistic" versions of ordinary Calogero-Moser systems, to which they reduce in a certain limit.

Finally, it is worth mentioning that the quantum versions of all of these systems also have many remarkable properties, and both the classical and quantum systems are related to contemporary research problems in mathematical physics and representation theory (supersymmetric Yang-Mills theory, topological field theory, Knizhnik-Zamolodchikov equations to name but a few). In fact, Calogero originally found the quantum version of these systems in 1969 (*before* the classical case) by requiring a factorizable ground state wave function.

Dr Andy Hone, November 2008.