LTCC Geometry and Physics: Exercise Sheet 3

1. Let M, J be an almost complex manifold and g be a hermitian metric. Let X be a vector of length 1: $g(X, X) = 1$. Show that X, JX is an orthonormal pair—that is, $g(JX, JX) = 1$ and $g(X, JX) = 0$.

2. If M is a complex manifold, define an almost complex structure on a patch with coordinates $z^j = x^j + iy^j$ by $J(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial y^j}$ and $J(\frac{\partial}{\partial y^j}) = -\frac{\partial}{\partial x^j}$. Show that this rule is unchanged by holomorphic change of coordinates so that it defines an almost complex structure on the whole of M. (Use the Cauchy–Riemann equations, as usual.)

3. Let X and Y be the stereographic coordinates via projection from the south pole. Invert this projection to show that points on $S^2 \subset \mathbb{R}^3$ are given by

$$\mathbf{n} = \frac{1}{1 + X^2 + Y^2} \begin{pmatrix} 2X \\ 2Y \\ 1 - X^2 - Y^2 \end{pmatrix}.$$

   Let

$$\mathbf{n}_X = \frac{\partial \mathbf{n}}{\partial X} \quad \text{and} \quad \mathbf{n}_Y = \frac{\partial \mathbf{n}}{\partial Y}.$$

   Show that $\mathbf{n}_X$ and $\mathbf{n}_Y$ are orthogonal to each other and to $\mathbf{n}$ and hence span the tangent space T_pM for fixed p. Show that $(\mathbf{n} \times .)$ gives an almost complex structure on T_pM.

4. (i) Calculate the Euler-Lagrange equation for the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + 4(\cos\varphi - 1),$$

   and write down the kinetic and potential energy.

   (ii) Calculate the Euler-Lagrange equation for

$$\mathcal{L} = -\frac{1}{2}\varphi_t\varphi_x + \frac{1}{2}\varphi_{xx}^2 - \varphi_x^3.$$

   Show that this can be written as a single equation for π, the conjugate momentum variable. Finally, show that this is in Hamiltonian form

$$\pi_t = -D_x \frac{\delta H}{\delta \pi},$$

   where $-D_x$ is the Hamiltonian operator and the Hamiltonian is

$$H = \int (\pi\varphi_t - \mathcal{L})\, dx.$$

5. Let

$$\exp(i\varphi/2) = \tau_+/\tau_-$$

for some functions $\tau_\pm$ (Hirota's tau-functions). Verify that if $\tau_\pm$ satisfy the pair of equations

$$\tau_\pm\tau_{\pm,tt} - \tau_{\pm,t}^2 - \tau_\pm\tau_{\pm,xx} + \tau_{\pm,x}^2 = \tau_\pm^2 - \tau_\mp^2$$

then φ is a solution of the sine-Gordon equation

$$\varphi_{tt} - \varphi_{xx} = 4\sin\varphi.$$

Verify that the equations for $\tau_\pm$ (the Hirota bilinear equations) have a solution given by

$$\tau_\pm = 1 \pm \exp(ax + bt + c),$$

for constants a, b, c satisfying a suitable constraint. Show that if $\tau_- = \overline{\tau}_+$ then φ is real, and explain how to choose the constants a, b, c to ensure that this is the case (for real x, t). Sketch the corresponding solution of the sine-Gordon equation.

6. Given the matrices

$$U = \begin{pmatrix} i\varphi_{x_+}/2 & \lambda \\ \lambda & -i\varphi_{x_+}/2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \lambda^{-1}e^{i\varphi} \\ \lambda^{-1}e^{-i\varphi} & 0 \end{pmatrix},$$

let $\partial_{x_+} - U$, $\partial_{x_-} - V$ define a connection on a vector bundle of rank 2 over $\mathbb{R}^2$, with (x_+, x_-) being coordinates on the base space. What are the fibres of the vector bundle? Calculate the condition on $\varphi(x_+, x_-)$ for the connection to be flat.

Dr Steffen Krusch & Prof. Andy Hone, February 2013.