LTCC Geometry and Physics: Exercise Sheet 1

1. On S^n define the coordinate neighbourhoods

$$\begin{aligned} U_{i+} &= \{(x_0, x_1, \ldots, x_n) \in S^n : x_i > 0\}\,, \\ U_{i-} &= \{(x_0, x_1, \ldots, x_n) \in S^n : x_i < 0\}\,. \end{aligned}$$

   Define the coordinate maps $\phi_{i\pm} \to \mathbb{R}^n$ via

$$\phi_{i\pm} = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

   For $n = 2$ calculate $\phi_{y-} \circ \phi_{x+}^{-1}$.

2. If f is a function on an open set $U \subset \mathbb{R}^n$, show that $d(df) = 0$.

3. Let $F\colon M = \mathbb{R}^2 \to N = \mathbb{R}^2$ be defined by $F(x_1, x_2) = (x_2 - x_1^3, x_1)$ (or, if you prefer, by saying $y_1 = x_2 - x_1^3$, etc.). If $\omega = y_1 dy_1 \wedge dy_2$, compute $F^*\omega$ (which should be a 2-form on M expressed using $dx_1 \wedge dx_2$—differentiating something may help).

4. Recall the pullback formula for metrics:

$$g^M_{\mu\nu}(x) = g^N_{\alpha\beta}(F(x))\frac{\partial F^\alpha}{\partial x^\mu}\frac{\partial F^\beta}{\partial x^\nu}.$$

   Let $F\colon M = \mathbb{R}^2 \to N = \mathbb{R}^2$ be defined by $F(r, \phi) = (r\cos\phi, r\sin\phi)$. Compute the induced metric on M. Also calculate the induced map of the sphere $S^2 \subset \mathbb{R}^3$ for both polar coordinates and stereographic coordinates.

5. Recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

   is a basis of $(T_pM)^{\mathbb{C}}$ (I omit indices z^μ on the variables). Show that $dz = dx + idy$ and $d\overline{z} = dx - idy$ make a dual basis.

6. Show how the Hopf bundle can be derived by considering $z_1, z_2 \in \mathbb{C}$, with $|z_1|^2 + |z_2|^2 = 1$, as the total space, and $[z_1, z_2]$ as homogeneous coordinates of the base space $\mathbb{C}P^1 \cong S^2$. Here homogeneous coordinates are defined via the equivalence relation

$$[z_1, z_2] = [\lambda z_1, \lambda z_2] \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

   Find two suitable coordinate charts for S^2 and write down the corresponding local trivialisations. Give the projections. What are the transition functions?

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