Applications of Differential Geometry to Mathematical Physics

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Outline

- Manifolds
- Fibre bundles
- Vector bundles, principal bundles
- Sections in fibre bundles
- Metric, Connection
- General Relativity, Yang-Mills theory
The 2-sphere $S^2$

- $S^2 = \left\{ (x_1, x_2, x_3) : \sum_{i=1}^{3} x_i^2 = 1 \right\}$.

- Polar coordinates:
  
  \begin{align*}
  x_1 &= \cos \phi \sin \theta, \\
  x_2 &= \sin \phi \sin \theta, \\
  x_3 &= \cos \theta.
  \end{align*}

- Problem: We can’t label $S^2$ with single coord system such that
  
  1. Nearby points have nearby coords.
  2. Every point has unique coords.

- Stereographic Projection

  \begin{align*}
  X_1 &= \frac{x_1}{1 - x_3}, \\
  X_2 &= \frac{x_2}{1 - x_3}.
  \end{align*}
Def: $M$ is an $m$-dimensional (differentiable) manifold if

- $M$ is a topological space.
- $M$ comes with family of charts $\{(U_i, \phi_i)\}$ known as atlas.
- $\{U_i\}$ is family of open sets covering $M$: $\bigcup_i U_i = M$.
- $\phi_i$ is homeomorphism from $U_i$ onto open subset $U'_i$ of $\mathbb{R}^m$.
- Given $U_i \cap U_j \neq \emptyset$, then the map

$$\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is $C^\infty$. $\psi_{ij}$ are called crossover maps.
\[
\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)
\]
Example: $S^2$

- Projection from North pole:
  \[
  X_1 = \frac{x_1}{1 - x_3}, \quad X_2 = \frac{x_2}{1 - x_3}.
  \]

- $U_1 = S^2 \setminus N$, $U'_1 = \mathbb{R}^2$:
  \[
  \phi_1 : U_1 \rightarrow \mathbb{R}^2 : (x_1, x_2, x_3) \mapsto (X_1, X_2)
  \]

- Projection from South pole:
  \[
  Y_1 = \frac{x_1}{1 + x_3}, \quad Y_2 = \frac{x_2}{1 + x_3}.
  \]

- $U_2 = S^2 \setminus S$, $U'_2 = \mathbb{R}^2$:
  \[
  \phi_2 : U_2 \rightarrow \mathbb{R}^2 : (x_1, x_2, x_3) \mapsto (Y_1, Y_2)
  \]

Crossover map $\psi_{21} = \phi_2 \circ \phi_1^{-1}$:

\[
\psi_{21} : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^2 : (X_1, X_2) \mapsto (Y_1, Y_2) = \left( \frac{x_1}{x_1^2 + x_2^2}, \frac{-x_2}{x_1^2 + x_2^2} \right).
\]

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Example: $S^3$

- Stereographic coords for $S^3$ work the same way as for $S^2$, e.g.
  \[ X_i = \frac{x_i}{1 - x_4}, \]
  where $i = 1, 2, 3$ for the projection from the “North pole”.

- Note $S^3$ can be identified with $SU(2)$, i.e. complex $2 \times 2$ matrices which satisfy

  \[ U \ U^\dagger = U^\dagger \ U = 1 \quad \text{and} \quad \det U = 1. \quad \text{(1)} \]

  Setting

  \[ U = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \]

  satisfies all the conditions in (1) provided

  \[ |z_1|^2 + |z_2|^2 = 1, \]

  which is the equation for $S^3$. 

The trivial bundle

- Often manifolds can be build up from smaller manifolds.
- An important example is the Cartesian product: $E = B \times F$ (known as trivial bundle)
- Fibre bundles are manifolds which look like Cartesian products, locally, but not globally.
- This concept is very useful for physics. Non-trivial fibre bundles occur for example in general relativity, but also due to boundary conditions “at infinity”.

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Def: A fibre bundle \((E, \pi, M, F, G)\) consists of

- A manifold \(E\) called total space, a manifold \(M\) called base space and a manifold \(F\) called fibre (or typical fibre).
- A surjection \(\pi : E \to M\) called the projection. The inverse image of a point \(p \in M\) is called the fibre at \(p\), namely \(\pi^{-1}(p) = F_p \cong F\).
- A Lie group \(G\) called structure group which acts on \(F\) on the left.
- A set of open coverings \(\{U_i\}\) of \(M\) with diffeomorphism \(\phi_i : U_i \times F \to \pi^{-1}(U_i)\), such that \(\pi \circ \phi_i(p, f) = p\). The map is called the local trivialization, since \(\phi_i^{-1}\) maps \(\pi^{-1}(U_i)\) to \(U_i \times F\).
- Transition functions \(t_{ij} : U_i \cap U_j \to G\), such that \(\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)\). Fix \(p\) then \(t_{ij} = \phi_i^{-1} \circ \phi_j\).
The importance of the transition functions

- Consistency conditions (ensure $t_{ij} \in G$)
  
  $$
  t_{ii}(p) = e \quad p \in U_i \\
  t_{ij}(p) = t_{ji}^{-1}(p) \quad p \in U_i \cap U_j \\
  t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p) \quad p \in U_i \cap U_j \cap U_k
  $$

- If all transition functions are the identity map $e$, then the fibre bundle is called the trivial bundle, $E = M \times F$.

- The transition functions of two local trivializations \{\phi_i\} and \{\tilde{\phi}_i\} for fixed \{U_i\} are related via
  
  $$
  \tilde{t}_{ij}(p) = g_i^{-1}(p) \cdot t_{ij}(p) \cdot g_j(p).
  $$

  where for fixed $p$, we define $g_i : F \to F : g_i = \phi_i^{-1} \circ \tilde{\phi}_i$.

- For the trivial bundle, $t_{ij}(p) = g_i^{-1}(p) \cdot g_j(p)$. 

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Tangent vectors

- Given a curve \( c : (-\epsilon, \epsilon) \to M \) and a function \( f : M \to \mathbb{R} \), we define the tangent vector \( X[f] \) at \( c(0) \) as directional derivative of \( f(c(t)) \) along \( c(t) \) at \( t = 0 \), namely

\[
X[f] = \left. \frac{df(c(t))}{dt} \right|_{t=0}.
\]

- In local coords, this becomes

\[
\left. \frac{\partial f}{\partial x^\mu} \frac{dx^\mu(c(t))}{dt} \right|_{t=0},
\]

hence

\[
X[f] = X^\mu \left( \frac{\partial f}{\partial x^\mu} \right).
\]

- To be more mathematical, the tangent vectors are defined via equivalence classes of curves.
More about Tangent vectors

- Vectors are independent of the choice of coordinates, hence
  \[ X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu}. \]

- The components of \( X^\mu \) and \( \tilde{X}^\mu \) are related via
  \[ \tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}. \]

- It is very useful to define the pairing
  \[ \left\langle dx^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu. \]

- This leads us to one-forms \( \omega = \omega_\mu dx^\mu \), also independent of choice of coordinates. Now, we have
  \[ \omega = \omega_\mu dx^\mu = \tilde{\omega}_\nu dy^\nu \implies \tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}. \]

- This can be generalized further to tensors \( T^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_r} \).
The Tangent bundle

- At each point $p \in M$ all the tangent vectors form an $n$ dimensional vector space $T_p M$, so tangent vectors can be added and multiplied by real numbers:

$$\alpha X_1 + \beta X_2 \in T_p M \quad \text{for} \quad \alpha, \beta \in \mathbb{R}, \quad X_1, X_2 \in T_p M.$$ 

- A basis of $T_p M$ is given by $\partial/\partial x^\mu$, $(1 \leq \mu \leq n)$, hence $\dim M = \dim T_p M$.

- The union of all tangent spaces forms the tangent bundle

$$TM = \bigcup_{p \in M} T_p M.$$ 

- $TM$ is a $2n$ dimensional manifold with base space $M$ and fibre $\mathbb{R}^n$. It is an example of a vector bundle.
The Tangent bundle $TS^2$

- We use the two stereographic projections as our charts.
- The coords $(X_1, X_2) \in U'_1$ and $(Y_1, Y_2) \in U'_2$ are related via

  \[
  Y_1 = \frac{X_1}{X_1^2 + X_2^2}, \quad Y_2 = \frac{-X_2}{X_1^2 + X_2^2}.
  \]

- Given $u \in TS^2$ with $\pi(u) = p \in U_1 \cap U_2$, then the local trivializations $\phi_1$ and $\phi_2$ satisfy $\phi_1^{-1}(u) = (p, V_1^\mu)$ and $\phi_2^{-1}(u) = (p, V_2^\mu)$. The local trivialization is

  \[
  t_{12} = \frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} = \frac{1}{(X_1^2 + X_2^2)^2} \begin{pmatrix}
  X_2^2 - X_1^2 & -2X_1X_2 \\
  -2X_1X_2 & X_1^2 - X_2^2
  \end{pmatrix}.
  \]

- Check: $t_{21}(p) = t_{12}^{-1}(p)$. 

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Example: $U(1)$ bundle over $S^2$

- Consider a fibre bundle with fibre $U(1)$ and base space $S^2$.
- Let $\{U_N, U_S\}$ be an open covering of $S^2$ where

  $$
  U_N = \{(\theta, \phi) : 0 \leq \theta < \pi/2 + \epsilon, 0 \leq \phi < 2\pi\}
  $$

  $$
  U_S = \{(\theta, \phi) : \pi/2 - \epsilon < \theta \leq \pi, 0 \leq \phi < 2\pi\}
  $$

- The intersection $U_N \cap U_S$ is a strip which is basically the equator. Local trivializations are

  $$
  \phi_N^{-1}(u) = (p, e^{i\alpha_N}), \quad \phi_S^{-1}(u) = (p, e^{i\alpha_S})
  $$

  where $p = \pi(u)$.

- Possible transition functions are $t_{NS} = e^{in\phi}$, where $n \in \mathbb{Z}$.

- The fibre coords in $U_N \cap U_S$ are related via

  $$
  e^{i\alpha_N} = e^{in\phi} e^{i\alpha_S}.
  $$

- If $n = 0$ this is the trivial bundle $P_0 = S^2 \times S^1$. For $n \neq 0$ the $U(1)$ bundle $P_n$ is twisted.
$P_n$ is an example of a **principal bundle** because the fibre is the same as the structure group $G = U(1)$.

In physics, $P_n$ is interpreted as a magnetic monopole of charge $n$.

Given $S^3 = \{ x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}$ we can define the Hopf map: $\pi : S^3 \to S^2$ by

\[
\begin{align*}
\xi_1 &= 2(x_1x_3 + x_2x_4) \\
\xi_2 &= 2(x_2x_3 - x_1x_4) \\
\xi_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2.
\end{align*}
\]

which implies $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$.

It turns out that with this choice of coords $S^3$ can be identified with $P_1$, a nontrivial $U(1)$ bundle over $S^2$, known as the **Hopf bundle**.
**Def:** Let \((E, M, \pi)\) be a fibre bundle. A section \(s : M \to E\) is a smooth map which satisfies \(\pi \circ s = id_M\). Here, \(s|_p\) is an element of the fibre \(F_p = \pi^{-1}(p)\). The space of section is denoted by \(\Gamma(E)\).

- A local section is defined on \(U \subset M\), only.
- Note that not all fibre bundles admit global sections!
- Example: The wave function \(\psi(x, t)\) in quantum mechanics can be thought of as a section of a complex line bundle \(E = \mathbb{R}^{3,1} \times \mathbb{C}\).
- Vector fields associate a tangent vector to each point in \(M\). They can be thought of as sections of \(TM\).
Vector bundles always have at least one section, the null section $s_0$ with
\[ \phi_i^{-1}(s_0(p)) = (p, 0) \]
in any local trivialization.

A principal bundle $E$ only admits a global section if it is trivial: $E = M \times F$.

A section in a principal bundle can be used to construct the trivialization of the bundle which uses that we can define a right action which is independent of the local trivialization:
\[ ua = \phi(p, g_i a), \quad a \in G \]
Given a principal fibre bundle $P(M, G, \pi)$ and a $k$-dimensional vector space $V$, and let $\rho$ be a $k$ dimensional representation of $G$ then the associated vector bundle $E = P \times_\rho V$ is defined by identifying the points

$$(u, v) \quad \text{and} \quad (ug, \rho(g)^{-1}v) \in P \times V$$

where $u \in P$, $g \in G$, and $v \in V$.

The projection $\pi_E : E \to M$ is defined by $\pi_E(u, v) = \pi(u)$, which is well defined because

$$\pi_E(ug, \rho(g)^{-1}v) = \pi(ug) = \pi(u) = \pi_E(u, v)$$

The transition functions of $E$ are given by $\rho(t_{ij}(p))$ where $t_{ij}(p)$ are the transition functions of $P$.

Conversely, a vector bundle naturally induces a principal bundle associated with it.
Manifolds can carry further structure, for example a *metric*.

A metric $g$ is a $(0,2)$ tensor which satisfies at each point $p \in M$:

1. $g_p(U, V) = g_p(V, U)$
2. $g_p(U, U) \geq 0$, with equality only when $U = 0$.

where $U, V \in T_p M$.

The metric $g$ provides an inner product for each tangent space $T_p M$.

Notation:

$$g = g_{\mu \nu} dx^\mu dx^\nu.$$

If $M$ is a submanifold of $N$ with metric $g_N$ and $f : M \to N$ is the embedding map, then the induced metric $g_M$ is

$$g_{M\mu \nu}(x) = g_{N\alpha \beta}(f(x)) \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu}.$$
Consider the “derivative” of a vector field \( V = V^\mu \frac{\partial}{\partial x^\mu} \) w.r.t. \( x^\nu \):
\[
\frac{\partial V^\mu}{\partial x^\nu} = \lim_{\Delta x \to 0} \frac{V^\mu(\ldots, x^\nu + \Delta x^\nu, \ldots) - V^\mu(\ldots, x^\nu, \ldots)}{\Delta x^\nu}
\]

This doesn’t work as the first vector is defined at \( x + \Delta x \) and the second at \( x \).

We need to transport the vector \( V^\mu \) from \( x \) to \( x + \Delta x \) “without change”. This is known as parallel transport.

This is achieved by specifying a connection \( \Gamma^\mu_{\nu\lambda} \), namely the parallel transported vector \( \tilde{V}^\mu \) is given by
\[
\tilde{V}^\mu(x + \Delta x) = V^\mu(x) - V^\lambda(x)\Gamma^\mu_{\nu\lambda}(x)\Delta x^\nu.
\]

The covariant derivative of \( V \) w.r.t. \( x^\nu \) is
\[
\lim_{\Delta x^\nu \to 0} \frac{V^\mu(x + \Delta x) - \tilde{V}^\mu(x + \Delta x)}{\Delta x^\nu} = \frac{\partial V^\mu}{\partial x^\nu} + V^\lambda \Gamma^\mu_{\nu\lambda}.
\]
We demand that the metric $g$ is covariantly constant.

This means, if two vectors $X$ and $Y$ are parallel transported along any curve, then the inner product $g(X, Y)$ remains constant.

The condition
\[ \nabla_V (g(X, Y)) = 0, \]
gives us the Levi-Civita connection.

The Levi-Civita connection can be written as
\[ \Gamma^\kappa_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left( \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right). \]
The Levi-Civita connection doesn’t transform like a tensor. However, from it, we can build the curvature tensor:

\[ R^{\kappa}_{\lambda \mu \nu} = \partial_{\mu} \Gamma^{\kappa}_{\nu \lambda} - \partial_{\nu} \Gamma^{\kappa}_{\mu \lambda} + \Gamma^{\eta}_{\nu \lambda} \Gamma^{\kappa}_{\mu \eta} - \Gamma^{\eta}_{\mu \lambda} \Gamma^{\kappa}_{\nu \eta}. \]

Important contractions of the curvature tensor are the Ricci tensor \( \text{Ric} \):

\[ \text{Ric}_{\mu \nu} = R^{\lambda}_{\mu \lambda \nu}. \]

and the scalar curvature \( \mathcal{R} \):

\[ \mathcal{R} = g^{\mu \nu} \text{Ric}_{\mu \nu}. \]

Now, we have the ingredients for Einstein’s Equations of General Relativity, namely

\[ \text{Ric}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \mathcal{R} = 8\pi G T_{\mu \nu}, \]

where \( G \) is the gravitational constant and \( T_{\mu \nu} \) is the energy momentum tensor which describes the distribution of matter.
An example of Yang-Mills theory is given by the following Lagrangian density,

\[ \mathcal{L} = \frac{1}{8} \text{Tr} \left( F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{2} (D_\mu \Phi)^\dagger D^\mu \Phi - U(\Phi^\dagger \Phi). \]  

(2)

where

\[ D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \]

Here \( \Phi \) is a two component complex scalar field.

\( A_\mu \) is called a gauge field and is \( su(2) \)-valued, i.e. \( A_\mu \) are anti-hermitian \( 2 \times 2 \) matrices.

\( F_{\mu\nu} \) is known as the field strength (also \( su(2) \)-valued)

This Lagrangian is Lorentz invariant.
Lagrangian (2) is also invariant under local gauge transformations:

- Let $g \in SU(2)$ be a space-time dependent gauge transformation with
  \[ \Phi \mapsto g\Phi, \quad \text{and} \quad A_\mu \mapsto gA_\mu g^{-1} - \partial_\mu gg^{-1}. \]

- The covariant derivative $D_\mu \Phi$ transforms as
  \[ D_\mu \Phi \mapsto \partial_\mu (g\Phi) + (gA_\mu g^{-1} - \partial_\mu gg^{-1}) g\Phi = gD_\mu \Phi. \]

- Hence $\Phi^\dagger \Phi \mapsto (g\Phi)^\dagger g\Phi = \Phi^\dagger g^\dagger g\Phi = \Phi^\dagger \Phi$, and similarly for $(D_\mu \Phi)^\dagger D^\mu \Phi$.

- Finally, $F_{\mu\nu} \mapsto gF_{\mu\nu} g^{-1}$, so
  \[ \text{Tr} (F_{\mu\nu} F^{\mu\nu}) \]
  is also invariant.
In a more mathematical language:

- The gauge field $A_\mu$ corresponds to the connection of the principal $SU(2)$ bundle.
- The field strength $F_{\mu\nu}$ corresponds to the curvature of the principal $SU(2)$ bundle.
- The complex scalar field $\Phi$ is a section of the associated $\mathbb{C}^2$ vector bundle.
- The action of $g \in SU(2)$ on $\Phi$ and $A_\mu$ is precisely what we expect for an associated fibre bundle.
- Surprisingly, mathematicians and physicist derived the same result very much independently!