ON A VARIATIONAL COMPLEX FOR DIFFERENCE EQUATIONS

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Abstract. A variational complex for difference equations is described, yielding a characterization of difference Euler Lagrange equations.

1. Variational methods for difference systems

The variational complex enables several fundamental problems to be resolved. For differential equations, this complex has been shown to be exact, and the resulting homotopy operators can be used to construct conservation laws [**O**]. At present, it is not clear how to discretize differential equations so as to preserve conservation laws. This problem has been studied by using Hamiltonian or Lagrangian structures to devise a discrete analogue of Noether's theorem [**WM, M**]. However, to deal with arbitrary difference equations, a discrete analogue of the variational complex is needed.

There is no *a priori* translation of the continuous variational complex to the discrete case. The main reasons are that there is no de Rham complex on $\mathbb{Z}^p$, and neither S nor $S - \mathrm{id}$ are derivations (obey a Leibnitz rule). The variational complex we construct in this article was designed for the discrete Euler-Lagrange operator that is obtained when a discrete variation of a discrete action is calculated in the natural way. The proof of the main result, that the complex is formally locally exact, is fully detailed in [**HM**], together with a discrete version of Noether's theorem, a discussion of the continuum limit of our complex, and more examples.

It has been known for some time that a discrete analogue of the variational complex should exist which is independent of any implicit

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continuum limit [**K**], and we start by describing the calculations for a simple example. In §2 we describe the analogues of the horizontal and functional vertical complexes that will be spliced together to form our full variational complex, and an analogue of integration by parts. We follow the line of argument for the continuous case, as well as the notation, of [**O**] where appropriate. In §3 we splice the complexes together, and prove the resulting complex is exact at the splicing point.We conclude with open problems.

1.1. A simple example. We consider a simple example on a one-dimensional lattice. Suppose a discrete Lagrangian density is given by

$$L_n = u_n^2 u_{n+2}$$

and that the *action* is the discrete analogue of the integral, namely the sum of L_n over the lattice. Thus we take the action to be

$$W = \sum_n L_n = \sum_n u_n^2 u_{n+2}.$$

Then the variation of the action is given by

$$\begin{aligned} \Delta W &= \sum_{n,m} \frac{\partial L_n}{\partial u_m} \Delta u_m \\ &= \sum_n \left\{ 2u_n u_{n+2} \Delta u_n + u_n^2 \Delta u_{n+2} \right\} \\ &= \sum_n \left\{ 2u_n u_{n+2} + u_{n-2}^2 \right\} \Delta u_n \end{aligned}$$

by a change of dummy variable in the second summand. For the action to be zero for all variations Δu_n, we conclude that

$$E(L_n) = 2u_n u_{n+2} + u_{n-2}^2 = 0, \qquad \text{for all } n.$$

In general, if S is the shift operator sending u_n to u_{n+1}, we have that

$$\Delta W = \sum_{n,m} \frac{\partial L_n}{\partial u_m} \Delta u_m = \sum_{n,m} \left(S^{-m} \frac{\partial L_n}{\partial u_{n+m}} \right) \Delta u_n \equiv 0$$

for arbitrary Δu_n, if and only if

$$E(L_n) = \sum_m S^{-m} \frac{\partial L_n}{\partial u_{n+m}} = 0$$

for all n. We take n to be the generic index, and write "$E(L_n) = 0$ for all n" as $E(L) = 0$. Then $E(L) = 0$ is a difference equation, just as the Euler-Lagrange equation of a continuous Lagrangian is a differential equation.

It is simple to see that the Euler operator is zero on total differences, the analogue of total divergences. That is, $E(Sf - f) \equiv 0$ for all f.

Next, we consider the analogue for the Helmholtz condition on our simple example. Thus, we seek to show that the linearization of $E(L)$ is self-adjoint. Set $P = E(L) = 2u_n u_{n+2} + u_{n-2}^2$. Define

$$\mathbf{D}_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} P(u + \epsilon Q) = 2(u_{n+2}Q_n + u_n Q_{n+2} + u_{n-2}Q_{n-2})$$

We take the inner product on the Hilbert space ℓ_2 of square summable sequences, and observe that

$$\begin{aligned} \mathbf{D}_P(Q) \cdot R &= 2\sum_n (u_{n+2}Q_n + u_n Q_{n+2} + u_{n-2}Q_{n-2})R_n \\ &= 2\sum_n (u_{n+2}R_n + u_{n-2}R_{n-2} + u_n R_{n+2})Q_n \\ &= \mathbf{D}_P(R) \cdot Q \end{aligned}$$

where we have used a change of dummy variable in the last two summands. Thus, the operator $\mathbf{D}_P$ is formally self-adjoint.

We write the Helmholtz condition as the map $\mathcal{H}$ given by

$$\mathcal{H}(P) = \mathbf{D}_P - \mathbf{D}_P^*.$$

Then in fact we have a sequence of maps, $\cdots S - \mathrm{id}$, E, $\mathcal{H} \cdots$ satisfying the condition that the composition of two adjacent maps is the zero map, that is, $\cdots \mathrm{im}(S - \mathrm{id}) \subseteq \ker(E)$, $\mathrm{im}(E) \subseteq \ker(\mathcal{H}) \cdots$ The fact that the complex we describe is *locally exact* means that these inclusions are in fact equalities, when the domains of the functions involved are "star-shaped". Further, in [**HM**], homotopy operators are given which solve the problem of finding a pre-image.

2. The Horizontal & Vertical Complexes

We consider discrete equations on a p-dimensional lattice. The independent discrete variables are denoted $n_1, \ldots n_p$. The dependent variables $u^1, \ldots u^q$ are assumed to be smooth and to take values in $\mathbb{C}$.

Definition 2.1. A smooth function depending on $\mathbf{n}$, $u^\alpha_{\mathbf{n}}$ and finitely many iterates of $u^\alpha_{\mathbf{n}}$ is written as $P[u]$. The algebra of such functions is denoted $\mathcal{A}$ while the algebra consisting of ℓ-tuples of such functions is denoted $\mathcal{A}^\ell$.

Definition 2.2. We define the shift map in the n_k direction to be

$$S_k u^\alpha_{\mathbf{n}} = u^\alpha_{(n_1,\ldots,n_k+1,\ldots,n_p)} = u^\alpha_{\mathbf{n}+1_k} \tag{2.1}$$

and write the composite of shifts using multi-index notation as

$$S^K = S_1^{K_1} \dots S_p^{K_p} \tag{2.2}$$

so that $u^\alpha_{\mathbf{n}+K} = S^K u^\alpha_{\mathbf{n}}$. The shift map acting on $\mathcal{A}$ is given by

$$S_k P(\mathbf{n}, \dots, u^\alpha_{\mathbf{n}+K}, \dots) = P(\mathbf{n}+1_k, \dots, u^\alpha_{\mathbf{n}+K+1_k}, \dots) \tag{2.3}$$

2.1. The Horizontal Complex. Let $\mathsf{Ex}(p)$ be the exterior algebra on p symbols $\Delta_1, \dots \Delta_p$, so that

$$\Delta_i^2 = 0, \qquad \Delta_i\Delta_j = -\Delta_i\Delta_j.$$

Definition 2.3. We define the algebra of horizontal forms to be

$$\mathsf{Ex} = \bigcup_{\mathbf{n}\in\mathbb{Z}^p} \mathsf{Ex}(p)$$

with coefficients in $\mathcal{A}$ and pointwise multiplication and addition.

A typical element of Ex takes the form

$$\omega = \sum_{1\le j\le p} P_{i_1 i_2 \dots i_j}[u]\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_j} \tag{2.4}$$

where $P_{i_1 i_2 \dots i_j}[u] \in \mathcal{A}$.

Definition 2.4. The *shift map* acting on elements of Ex is defined by (2.3), $S_k(\Delta_i) = \Delta_i$ and $S_k(\eta\omega) = S(\eta)S(\omega)$, so that S_k on the typical element (2.4) is

$$S_k\omega = \sum_{1\le j\le p} S_k(P_{i_1 i_2 \dots i_j}[u])\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_j}.$$

There is a natural grading of Ex. We say $\omega \in \mathsf{Ex}^j$, $j = 1\dots p$ if

$$\omega = \sum_{i_1\neq\dots\neq i_j} P_{i_1 i_2 \dots i_j}[u]\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_j}.$$

Further, $\mathsf{Ex}^0 = \mathcal{A}$.

Definition 2.5. The *total difference* map $\boldsymbol{\Delta} : \mathsf{Ex}^j \longrightarrow \mathsf{Ex}^{j+1}$ is

$$\boldsymbol{\Delta}(\omega) = \sum_{k=1}^{p} \Delta_k \cdot (S_k - id)\omega \tag{2.5}$$

for $j = 0, \dots, p-1$.

On the typical element (2.4), we have

$$\boldsymbol{\Delta}(\omega) = \sum_{1\le j\le p}\sum_{k=1}^{p}(S_k - id)(P_{i_1\dots i_j}[u])\Delta_k\Delta_{i_1}\dots\Delta_{i_j}.$$

EXAMPLE 2.6. If $\omega = n_1 u_{n_1 n_2} u_{n_1+1,n_2} \Delta_2$ then

$$\boldsymbol{\Delta}(\omega) = [(n_1+1)u_{n_1+1,n_2}u_{n_1+2,n_2} - n_1 u_{n_1 n_2} u_{n_1+1,n_2}]\,\Delta_1\Delta_2.$$

Since the maps $(S_k - id)$ commute pairwise, the proof that $\boldsymbol{\Delta}^2 = 0$ is standard.

THEOREM 2.7. *The horizontal complex*

$$0 \longrightarrow \mathbb{C} \xrightarrow{\ i\ } \mathbf{Ex}^0 \xrightarrow{\ \boldsymbol{\Delta}\ } \cdots \xrightarrow{\ \boldsymbol{\Delta}\ } \mathbf{Ex}^p \tag{2.6}$$

where i is the inclusion map, is exact; that is,

$$ker\boldsymbol{\Delta}\big|_{\mathbf{Ex}^j} = im\boldsymbol{\Delta}\big|_{\mathbf{Ex}^{j-1}}$$

for $j = 1, \dots p-1$ and $ker\boldsymbol{\Delta}\big|_{\mathbf{Ex}^0} = \mathbb{C}$.

The proof of this result, which is long and technical, is given in [**HM**]. Note that $\boldsymbol{\Delta}$ is *not* a derivation, that is, $\boldsymbol{\Delta}(w\eta) \neq (\boldsymbol{\Delta}w)\eta \pm w(\boldsymbol{\Delta}\eta)$. Hence $\boldsymbol{\Delta}$ does not respect the natural product structure on $\mathbf{Ex}$. Further, there is no obvious sense in which the space $\mathbf{Ex}^1\big|_{\mathbf{n}}$ can be considered as the dual space to some tangential object. The geometric meaning of this exterior horizontal complex, if there is one, remains to be investigated.

2.2. Vertical forms.

DEFINITION 2.8. A vertical k form is a finite sum

$$\widehat{w} = \sum_{\alpha,\mathbf{n}^1\dots\mathbf{n}^k} P^{\boldsymbol{\alpha}}_{\mathbf{n}^1\dots\mathbf{n}^k}[u]\,du^{\alpha_1}_{\mathbf{n}^1}\wedge\dots\wedge du^{\alpha_k}_{\mathbf{n}^k}$$

where the coefficients $P^{\boldsymbol{\alpha}}_{\mathbf{n}^1\dots\mathbf{n}^k}[u] \in \mathcal{A}$. We define the differential $\widehat{d}$ to be

$$\widehat{d}(\widehat{w}) = \sum_{\beta,\mathbf{n}^j}\ \sum_{\alpha,\mathbf{n}^1\dots\mathbf{n}^k} \frac{\partial}{\partial u^{\beta}_{\mathbf{n}^j}} P^{\boldsymbol{\alpha}}_{\mathbf{n}^1\dots\mathbf{n}^k}[u]\,du^{\beta}_{\mathbf{n}^j}\wedge du^{\alpha_1}_{\mathbf{n}^1}\wedge\dots\wedge du^{\alpha_k}_{\mathbf{n}^k}.$$

EXAMPLE 2.9. For $\widehat{w} = n u_n du_{n+1} - u^2_{n+2} du_{n+2}$,
$\widehat{d}\widehat{w} = n du_n du_{n+1}$.

Since any given vertical form can depend on only finitely many of the iterates, it is readily seen that the $\widehat{\Lambda}$ complex with differential $\widehat{d}$ is an extension of the well-known de Rham complex, with independent variables $u^{\boldsymbol{\alpha}}_{\mathbf{J}}$; the n_i play the role of parameters. Indeed, $\widehat{d}$ is bilinear, is a derivation,

$$\widehat{d}(\widehat{w}\wedge\widehat{\eta}) = d\widehat{w}\wedge\widehat{\eta} + (-1)^k\widehat{w}\wedge\widehat{d}\,\widehat{\eta}$$

and satisfies $\widehat{d}^{\,2} = 0$. The Poincaré lemma for the continuous vertical complex extends immediately to yield the following result.

THEOREM 2.10. *The vertical complex*

$$\widehat{\Lambda}^0 \xrightarrow{\widehat{d}} \widehat{\Lambda}^1 \xrightarrow{\widehat{d}} \widehat{\Lambda}^2 \xrightarrow{\widehat{d}} \dots \tag{2.7}$$

is exact.

DEFINITION 2.11. The shift map on vertical forms is defined to be linear, to commute with $\widehat{d}$, and to satisfy $S(\widehat{\omega} \wedge \widehat{\eta}) = S(\widehat{\omega}) \wedge S(\widehat{\eta})$ together with the standard action on the coefficients given in (2.3).

Hence

$$S_j \mathrm{d}u^\alpha_{\mathbf{n}+K} = \mathrm{d}u^\alpha_{\mathbf{n}+K+I_j} \quad \text{and} \quad S^K \mathrm{d}u^\alpha_{\mathbf{n}} = \mathrm{d}u^\alpha_{\mathbf{n}+K}. \tag{2.8}$$

2.3. Functional forms. The right hand side of the discrete variational complex is a quotient of the vertical complex described above under an equivalence relation. In the continuous case, two functions are equivalent if they differ by a total divergence, while here we say that two functions of the iterates are equivalent if they differ by a total difference. The equivalence classes are said to be *functionals*.

DEFINITION 2.12. We define an equivalence class on $\mathcal{A}$ by,

$$f \sim g \quad \Longleftrightarrow \quad f - g = \sum_{k=1}^{p} (S_k - id) h_k$$

for some $(h_1, \dots, h_p) \in \mathcal{A}^p$. The set of *functionals* $\mathcal{F}$ is defined to be the set of equivalence classes,

$$\mathcal{F} = \mathcal{A} / \sim .$$

We denote the equivalence class of $f \in \mathcal{A}$ by $\sum f$. The notation reflects the (formal) identity that $\sum_K f_K = 0$ if f is a total difference. Note that $\mathcal{F}$ is not an algebra, that is, products of functionals are not functionals.

An equivalence relation on $\widehat{\Lambda}^k$ of vertical k-forms can be defined similarly,

$$\widehat{w} \sim \widehat{w}^1 \qquad \Longleftrightarrow \qquad \widehat{w} = \widehat{w}^1 + \sum_{i=1}^{p} (S_i - id)\widehat{\eta}_i \qquad \widehat{\eta}_i \in \widehat{\Lambda}^k$$

for some $\widehat{\eta}_i$, $i = 1, \dots, p$, where S_i acts on $\widehat{\eta}_i$ according to the definition (2.11). Again, we denote the equivalence class of $\widehat{w}$ by $\sum \widehat{w}$. The equivalence classes are called *functional forms*, and the set of equivalence classes $\widehat{\Lambda}^k / \sim$ is denoted by Λ^k_*

2.4. Analogue of integration by parts. In the continuous variational complex, much use is made of the product (Leibnitz) rule of differential calculus, and the consequent integration by parts, at every stage in the proof of exactness of the continuous variational complex. In the discrete case studied here, neither the shift maps S_i nor the difference maps $S_i - id$ obey the Leibnitz rule.

First note that given two sequences $\{f_n\}$, $\{g_n\}$, we have the identity

$$\sum_{-\infty}^{\infty}(Sf)_n\; g_n = \sum_{-\infty}^{\infty} f_{n+1}\; g_n = \sum_{-\infty}^{\infty} f_n\; g_{n-1} = \sum_{-\infty}^{\infty} f_n(S^{-1}g)_n$$

by a change of dummy variable. Further, given f, $g \in \mathcal{A}$, and letting S denote S_j for some j,

$$(Sf)g - f(S^{-1}g) = (S - id)(fS^{-1}g)$$

Hence, we have

$$\textstyle\sum(Sf)g = \sum f(S^{-1}g) \tag{2.9}$$

using both the definition and the natural interpretation. Equation (2.9) is our analogue of "integration by parts".

2.5. The vertical functional complex.

Definition 2.13. If $w = \sum \widehat{w}$ is a functional k-form corresponding to the vertical k-form $\widehat{w}$, define the *variational derivative* to be

$$\delta w = \textstyle\sum \widehat{d}\widehat{w}.$$

It is simple to show that $\widehat{d}S_j = \widehat{d}$, and therefore that δ is well-defined. Further, since $\widehat{d}^2 = 0$, it follows immediately that $\delta^2 = 0$.

Theorem 2.14. *The vertical functional complex*

$$0 \longrightarrow \Lambda^0_* \xrightarrow{\delta} \Lambda^1_* \xrightarrow{\delta} \cdots \tag{2.10}$$

is exact.

3. The discrete variational complex

Definition 3.1. The **Euler-Lagrange operator** acting on an element of $\mathcal{A}$, corresponding to a dependent variable u^α is

$$E_\alpha(f) = \sum_K S^{-K}\partial_{u^\alpha_{\mathbf{n}+K}}(f).$$

The Euler-Lagrange operator acting on an element of $\mathbf{Ex}^p$ is

$$E(f\Delta_1\cdots\Delta_p) = \sum_{\alpha=1}^{q} E_\alpha(f)\,\mathrm{d}u^\alpha.$$

We let $\pi : \widehat{\Lambda} \longrightarrow \Lambda_*$ denote the projection which takes a vertical form to its equivalence class, $\pi(\widehat{\omega}) = \sum \widehat{\omega}$. Then the **discrete variational complex** is, writing $\pi \circ E$ as E

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathsf{Ex}^0 \xrightarrow{\boldsymbol{\Delta}} \mathsf{Ex}^1 \longrightarrow \cdots \xrightarrow{\boldsymbol{\Delta}} \mathsf{Ex}^p \xrightarrow{E} \Lambda^1_* \xrightarrow{\delta} \Lambda^2_* \xrightarrow{\delta} \cdots \tag{3.1}$$

Consider Figure 1, with $E(f)du$ written for $\sum_\alpha E_\alpha(f)\mathrm{d}u^\alpha_{\mathbf{n}}$.

$$\begin{array}{ccccccc} \xrightarrow{\boldsymbol{\Delta}} \mathsf{Ex}^{p-1} \xrightarrow{\boldsymbol{\Delta}} & \mathsf{Ex}^p & \xrightarrow{E} & \widehat{\wedge}^1 & & \\ & \pi\downarrow & & \downarrow\pi & & \\ 0 \longrightarrow & \wedge^0_* & \xrightarrow{\delta} & \wedge^1_* & \xrightarrow{\delta} \wedge^2_* \longrightarrow \end{array}$$

$$\begin{array}{ccc} f\ \Delta_1 \cdots \Delta_p & \xrightarrow{E} & E(f)du \\ \pi\downarrow & & \downarrow\pi \\ \sum f & \xrightarrow{\delta} & \sum E(f)du \end{array}$$

FIGURE 1. Splicing the horizontal and vertical functional complexes

LEMMA 3.2. $\delta \circ \pi = \pi \circ E$.

PROOF. Using the discrete 'integration by parts' formula (2.9)

$$\begin{aligned} \pi \circ E(f) &= \sum \sum\nolimits_\alpha E_\alpha(f) du^\alpha_{\mathbf{n}} \\ &= \sum \left(\sum\nolimits_{\alpha,K} (S^{-K} \partial_{u^\alpha_{\mathbf{n}+K}} f) du^\alpha_{\mathbf{n}} \right) \\ &= \sum \left(\sum\nolimits_{\alpha,K} (\partial_{u^\alpha_{\mathbf{n}+K}} f)(S^K du^\alpha_{\mathbf{n}}) \right) \\ &= \sum \left(\sum\nolimits_{\alpha,K} \partial_{u^\alpha_{\mathbf{n}+K}} f du^\alpha_{\mathbf{n}+K} \right) \\ &= \sum \widehat{d} f \\ &= \delta \circ \pi(f) \end{aligned}$$

where S^K is defined in (2.2) and we have used the action of the shift maps on vertical forms given in (2.8). □

THEOREM 3.3. *The variational complex is exact at* Ex^p *and at* Λ^1_*. *Specifically,*
(A) *The Euler Lagrange operator has for its kernel precisely those functions in* $\mathcal{A}$ *that are total differences, that is,*

$$\sum E(f)du = 0 \Leftrightarrow \sum f = 0$$

(B) *The variational derivative* $\delta\big|_{\Lambda^1_*}$ *has for its kernel precisely those expressions which are Euler-Lagrange equations, that is,*

$$\delta(\sum\sum_\alpha f_\alpha \mathrm{d}u^\alpha) = 0 \Leftrightarrow f_\alpha = E_\alpha(\mathcal{L}) \text{ for some } \mathcal{L}$$

PROOF. To show Part A, note that $\sum E(f)du = 0 \Longleftrightarrow \pi\circ E(f) = 0 \Longleftrightarrow \delta\circ\pi(f) = 0 \Longleftrightarrow \pi(f) = 0 \Longleftrightarrow \sum f = 0$ or f is a total difference, where we have used the preceeding Lemma and the exactness of (2.10).

To show Part B, if $\omega^* \in \Lambda^1_*$ is such that $\delta\omega^* = 0$, then by exactness of the vertical functional complex, there exists $\eta^* \in \Lambda^0_*$ such that $\delta\eta^* = \omega^*$. But $\eta^* = \sum \eta = \pi(\eta)$ for some $\eta \in \widehat{\Lambda}^0$. Hence $\omega^* = \delta\pi(\eta) = \pi E(\eta)$, showing that ω^* is in the image of πE as required. Since $\eta \in \mathbf{Ex}^p$ it is of the form $\mathcal{L}\Delta_1\cdots\Delta_p$, with $\mathcal{L}\in\mathcal{A}$. The desired Lagrangian is $\mathcal{L}$. □

So, what do these results mean for the study of systems of partial difference equations? Given $f_1 = 0$, $f_2 = 0$, ... , $f_q = 0$ in the dependent variables u^α, $\alpha = 1,\ldots,q$, we write down an element of Λ^*_1, namely

$$\mathbf{f} = f_1 du^1_{\mathbf{n}} + f_2 du^2_{\mathbf{n}} + \cdots + f_q du^q_{\mathbf{n}}.$$

This involves deciding which equation belongs to which dependent variable, that is, for which j is $f_i = E_j(\mathcal{L})$ for some (as yet unspecified) $\mathcal{L}$. Assigning the wrong f_i to each $du^j_{\mathbf{n}}$ will result in the discrete Helmholtz condition failing, i.e. $\delta f \neq 0$, and the fact that the system is an Euler-Lagrange system may be missed. Worse is if the system is only *equivalent* to an Euler-Lagrange system, for example, if $E_1(\mathcal{L}) = g_1$, $E_2(\mathcal{L}) = g_2$, and $f_1 = g_1 + g_2$, $f_2 = g_1 + Sg_2$. Even for the continuous case, the general equivalence problem, of detecting when a system is equivalent to an Euler-Lagrange system, is open. (See [**O**] §5.4)

Given that $\delta\mathbf{f} = 0$, we may use the homotopy operator for the vertical functional complex to find $\mathcal{L}$. We show the results for a single ordinary difference equation. The general result may be found in [**HM**].

EXAMPLE 3.4. A special case of the first discrete Painlevé equation is

$$P \equiv u_{n+1} + u_n + u_{n-1} + \frac{\alpha n + \beta}{1 + u_n} + \mu = 0 \tag{3.2}$$

([**GNR**]; we have translated u_n to $1 + u_n$ to simplify our calculations here). This equation satisfies $\mathbf{D}_P = \mathbf{D}^*_P$ and hence a Lagrangian exists for this equation. Calculating the homotopy yields

$$\begin{aligned} L &= \int_0^1 P[\lambda u]u_n\,\mathrm{d}\lambda \\ &= \int_0^1 \left[\mu u_n + \lambda(u_{n+1} + u_n + u_{n-1})u_n + \frac{\alpha n + \beta}{1 + \lambda u_n}u_n\right]\mathrm{d}\lambda \\ &= \mu u_n + \tfrac{1}{2}(u_{n+1} + u_n + u_{n-1})u_n + (\alpha n + \beta)\log(1 + u_n) \end{aligned}$$

It is straightforward to check that $E(L) = P$.

4. Open problems

- The geometric meaning of the horizontal complex.
- The relationship to other versions of a variational complex for difference equations, designed to have specific continuum limits including boundary behaviour ([**KP**], Chapter 8).
- Comparisons with discretizations of variational principles, to preserve symplecticity [**WM**] or "integrability" [**V**].
- Formulation of a discrete variational complex for Lagrangians which are invariant under a group action (cf. [**AP**]).

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