Multisymplectic conservation laws for differential and differential-difference equations

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Many well-known partial differential equations can be written as multisymplectic systems. Such systems have a structural conservation law from which scalar conservation laws can be derived. These conservation laws arise as differential consequences of a 1-form 'quasi-conservation law,' which is related to Noether's Theorem. This paper develops the above framework and uses it to introduce a multisymplectic structure for differential-difference equations. The shallow water equations and the Ablowitz-Ladik equations are used to illustrate the general theory. It is found that conservation of potential vorticity is a differential consequence of two conservation laws; this surprising result and its implications are discussed.

Keywords: Multisymplectic systems, partial differential equations, differential-difference equations, conservation laws, potential vorticity

1. Introduction

Hamiltonian systems of ordinary differential equations (ODEs) are endowed with a rich geometrical structure that is associated with the symplectic 2-form. This structure is masked in the Lagrangian formulation, which is nonlocal. Some partial differential equations (PDEs) have one or more Hamiltonian formulations (Olver, 1993) but these, like their Lagrangian counterparts, are nonlocal.

Bridges (1997) observed that many PDEs can be written as *multisymplectic* systems, which are local and have a distinct symplectic structure associated with each independent variable. The multisymplectic formulation is useful for the study of nonlinear waves (Bridges, 1997a), stability and bifurcation analysis (Bridges & Derks, 2001), and the development of numerical methods (Bridges & Reich, 2001).

For any Hamiltonian ODE, the symplectic structure is preserved on trajectories. Similarly, every multisymplectic system has a conservation law that involves each of the symplectic structures. We shall refer to this as the *structural conservation law*, because it is associated with the multisymplectic structure rather than the specific PDE. Other conservation laws have been obtained for some systems by using a restricted version of Noether's Theorem (Bridges, 1997a).

The current paper shows how the general (unrestricted) form of Noether's Theorem is linked to the structural conservation law for multisymplectic systems.

This knowledge provides a rationale for generalizing multisymplectic structures to include differential-difference equations. For brevity, we restrict attention to differential-difference equations with only two independent variables. (Multisymplectic difference equations and their applications to geometric integration will be treated in a more extensive paper.) As examples, we consider the shallow water equations and the Ablowitz-Ladik equation, which each have a multisymplectic formulation with associated conservation laws.

One observation which comes from the analysis has an application in meteorology: conservation of potential vorticity is a differential consequence of two more fundamental conservation laws. This surprising result and its possible implications for numerical weather prediction are discussed briefly.

2. The structural conservation law and its consequences

It is helpful to begin by considering some well-known features of the following simple Hamiltonian system of ODEs:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H(q,p)}{\partial q} \\ \frac{\partial H(q,p)}{\partial p} \end{pmatrix} . \tag{2.1}$$

where $\dot{}$ denotes the derivative with respect to the independent variable, t. The Hamiltonian system (2.1) has the symplectic 2-form

$$\kappa = \mathrm{d}p \wedge \mathrm{d}q.$$

Here we use the standard notation for Cartan's exterior calculus; in particular, d is the exterior derivative, which maps r-forms to $(r+1)$-forms and satisfies the identity $\mathrm{d}^2 = 0$. Consequently the symplectic 2-form is closed, i. e.

$$\mathrm{d}\kappa = 0.$$

Let D_t denote the total derivative with respect to t, which treats the dependent variables and their derivatives as functions of the independent variable:

$$D_t = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} + \dots.$$

The exterior derivative d commutes with D_t; hence, on solutions of (2.1),

$$D_t\kappa = \mathrm{d}\dot{p} \wedge \mathrm{d}q + \mathrm{d}p \wedge \mathrm{d}\dot{q} = -\mathrm{d}\left(\frac{\partial H}{\partial q}\right) \wedge \mathrm{d}q + \mathrm{d}p \wedge \mathrm{d}\left(\frac{\partial H}{\partial p}\right) = 0. \tag{2.2}$$

In other words, the symplectic 2-form is conserved, irrespective of the function $H(q,p)$. This exemplifies the idea of a 'structural conservation law.'

The symplectic 2-form and its time-derivative are locally exact, and

$$\mathrm{d}\{D_t(p\,\mathrm{d}q)\} = D_t\kappa = 0.$$

The Poincaré Lemma implies that, locally, there exists a function $g(q,p)$ satisfying

$$D_t(p\,\mathrm{d}q) = \mathrm{d}g. \tag{2.3}$$

We call such a relationship a '1-form quasi-conservation law'. Explicitly,

$$D_t(p\,\mathrm{d}q) = \dot{p}\,\mathrm{d}q + p\,\mathrm{d}(\dot{q}) = \mathrm{d}\left(p\frac{\partial H}{\partial p} - H\right).$$

Pulling this relationship back to the base space (replacing each 1-form $\mathrm{d}f(q,p)$ by $D_t\{f(q,p)\}\,\mathrm{d}t$) yields

$$D_t(p\,\dot{q}\,\mathrm{d}t) = D_t\left(p\frac{\partial H}{\partial p} - H\right)\mathrm{d}t;$$

on solutions of (2.1) this amounts to

$$D_t(H)\,\mathrm{d}t = 0. \tag{2.4}$$

Therefore $D_t(H) = 0$ on the trajectories in phase space. In the following, these simple observations for the Hamiltonian system (2.1) are extended to multisymplectic systems, with useful consequences.

The general form for a multisymplectic system of PDEs with dependent variables z^i and independent variables m^α is

$$K^\alpha_{ij}(\mathbf{z})z^j_{,\alpha} = \frac{\partial H(\mathbf{z})}{\partial z^i}. \tag{2.5}$$

We use the Einstein summation convention in (2.5) and hereafter; total differentiation with respect to m^α is denoted by the subscript α after a comma. The following constraints are usually imposed on the functions $K^\alpha_{ij}(\mathbf{z})$:

1. the symplectic 2-forms $\kappa^\alpha = \frac{1}{2}K^\alpha_{ij}(\mathbf{z})\,\mathrm{d}z^i \wedge \mathrm{d}z^j$ satisfy $\mathrm{d}\kappa^\alpha = 0$ for each α;
2. $K^\alpha_{ij}(\mathbf{z}) = -K^\alpha_{ji}(\mathbf{z})$.

Note that (2.5) is a generalization of the Hamiltonian ODE system (2.1). The consequence of the first of the above constraints is that (at least) locally there exist functions $L^\alpha_i(\mathbf{z})$ such that

$$\kappa^\alpha = \mathrm{d}(L^\alpha_j\,\mathrm{d}z^j) = \frac{\partial L^\alpha_j}{\partial z^i}\,\mathrm{d}z^i \wedge \mathrm{d}z^j = \frac{1}{2}\left(\frac{\partial L^\alpha_j}{\partial z^i} - \frac{\partial L^\alpha_i}{\partial z^j}\right)\mathrm{d}z^i \wedge \mathrm{d}z^j.$$

Therefore

$$K^\alpha_{ij} = \frac{\partial L^\alpha_j}{\partial z^i} - \frac{\partial L^\alpha_i}{\partial z^j},$$

and so the second constraint is automatically satisfied. In practice, it is usually fairly easy to construct the functions L^α_j by inspection; if necessary, they can be constructed systematically with the aid of a homotopy operator (Olver, 1993).

Bridges (1997b) derived the structural conservation law

$$\kappa^\alpha_{,\alpha} = 0. \tag{2.6}$$

By following the argument that was used above for Hamiltonian ODEs, we derive the 1-*form quasi-conservation law*:

$$\left(L^\alpha_j\,\mathrm{d}z^j\right)_{,\alpha} = \mathrm{d}\left(L^\alpha_j z^j_{,\alpha} - H(\mathbf{z})\right), \tag{2.7}$$

which holds on solutions of the multisymplectic system (2.5).

A set of conservation laws can be found by pulling (2.6) back to the base space (Bridges *et al.*, 2003). To do this, replace each $\mathrm{d}z^i$ by $z^i_{,\alpha}\,\mathrm{d}m^\alpha$ and use the anti-symmetry of the wedge product. If there are n independent variables this yields $\frac{1}{2}n(n-1)$ conservation laws:

$$\left(K^\alpha_{ij} z^i_{,\beta} z^j_{,\gamma}\right)_{,\alpha} = 0, \qquad (\beta < \gamma). \tag{2.8}$$

Just as for the Hamiltonian ODE, we can also pull the quasi-conservation law back to the space of independent variables, obtaining n conservation laws

$$\left(L^\alpha_j z^j_{,\beta} - L^\gamma_j z^j_{,\gamma}\delta^\alpha_\beta + H(\mathbf{z})\delta^\alpha_\beta\right)_{,\alpha} = 0, \tag{2.9}$$

where δ^α_β is the Kronecker delta. The first set of conservation laws (2.8) can be obtained from (2.9) by cross-differentiation, and therefore it is reasonable to regard (2.9) as the more fundamental set. However, in some instances the interpretation of a 2-form law may be more obvious than that of the 1-form laws. To illustrate this, we find the conservation laws (2.8) and (2.9) for the shallow water equations, which are the basis for much of meteorology and oceanography.

The shallow water equations are written in the Lagrangian (particle) description. Let $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^2$ represent the position and velocity of the fluid particles, which are functions of t and the Lagrangian 'label-space' coordinates $\mathbf{m} = (m^1, m^2)$. (Note: we use t rather than m^3 to denote time in this example). The system rotates with constant angular velocity $f/2$. The internal energy of the fluid is $e(\tau)$, where τ is the reciprocal of the fluid depth. Writing τ in terms of new dependent variables $x^i_\alpha = x^i_{,\alpha}$, we obtain

$$\tau = x^1_1 x^2_2 - x^1_2 x^2_1.$$

To write the equations of motion in multisymplectic form, we introduce four further dependent variables

$$w^\alpha_i = -\frac{\partial e(\tau)}{\partial x^i_\alpha}$$

Then the shallow water equations can be written as the following multisymplectic system (Bridges *et al.*, 2003):

$$\begin{aligned}
-u^1_{,t} + f x^2_{,t} - w^1_{1,1} - w^2_{1,2} &= 0, \\
-u^2_{,t} - f x^1_{,t} - w^1_{2,1} - w^2_{2,2} &= 0, \\
x^i_{,t} &= u^i\,, && i = 1,2, \\
x^i_{,\alpha} &= x^i_\alpha\,, && i,\alpha = 1,2, \\
0 &= \frac{\partial e(\tau)}{\partial x^i_\alpha} + w^\alpha_i\,, && i,\alpha = 1,2.
\end{aligned}$$

This system is of the form (2.5), with

$$z^1 = x^1, \quad z^2 = x^2, \quad z^3 = u^1, \quad z^4 = u^2, \quad z^5 = w^1_1, \quad z^6 = w^2_1,$$

$$z^7 = w^1_2, \quad z^8 = w^2_2, \quad z^9 = x^1_1, \quad z^{10} = x^1_2, \quad z^{11} = x^2_1, \quad z^{12} = x^2_2,$$

and

$$H = \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau) + x^i_\alpha w^\alpha_i, \qquad \text{where} \qquad \mathbf{u}\cdot\mathbf{u} = (u^1)^2 + (u^2)^2.$$

The only nonzero components K^α_{ij} with $i < j$ are

$$K^t_{12} = f, \qquad K^t_{13} = K^t_{24} = K^1_{15} = K^1_{27} = K^2_{16} = K^2_{28} = -1.$$

By substituting the results into (2.8) and (2.9), we obtain conservation laws. In the following, the *potential vorticity* for the shallow water system is

$$q = f\tau + x^1_2 u^1_{,1} + x^2_2 u^2_{,1} - x^1_1 u^1_{,2} - x^2_1 u^2_{,2}\,.$$

The $\mathrm{d}m^1 \wedge \mathrm{d}m^2$ component of the 2-form conservation law (2.6) is

$$q_{,t} = 0. \tag{2.10}$$

In other words, potential vorticity is conserved by fluid particles. The $\mathrm{d}t$ component of the (pulled back) 1-form quasi-conservation law describes conservation of energy:

$$\left(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau)\right)_{,t} + (u^i w^\alpha_i)_{,\alpha} = 0. \tag{2.11}$$

The remaining components of the structural conservation law are merely label-space derivatives of the energy conservation law (2.11), so are of no interest. However, the remaining components of the 1-form quasi-conservation law are interesting; they are

$$\left(u^1x^1_1 + u^2x^2_1 + fx^1x^2_1\right)_{,t} + \left(e(\tau) - \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - fx^1u^2 + x^i_1 w^1_i\right)_{,1} + (x^i_1 w^2_i)_{,2} = 0; \tag{2.12}$$

$$\left(u^1x^1_2 + u^2x^2_2 + fx^1x^2_2\right)_{,t} + (x^i_2 w^1_i)_{,1} + \left(e(\tau) - \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - fx^1u^2 + x^i_2 w^2_i\right)_{,2} = 0. \tag{2.13}$$

Conservation of potential vorticity is a differential consequence of (2.12) and (2.13). At present, it is generally agreed that numerical schemes should respect the conservation of potential vorticity. However, we have just seen that fluid particles retain potential vorticity because they are bound by the conservation laws (2.12) and (2.13). Furthermore, potential vorticity is not the only information contained in these conservation laws. This raises an intriguing question: would schemes that preserve (2.12) and (2.13) be more accurate than those which preserve potential vorticity? A major advantage of using constant potential vorticity is that singularities such as fronts can be dealt with. It remains to be seen whether the conservation laws (2.12) and (2.13) are as useful as (2.10); however, they are more fundamental (mathematically speaking) than the conservation of potential vorticity.

3. Conservation laws from symmetries

In a seminal paper on multisymplectic systems, Bridges (1997a) demonstrated that Noether's Theorem could be applied to the multisymplectic system (2.5) subject to some constraints on the symmetry group. Most notably, these constraints included the requirement that $H(\mathbf{z})$ should be invariant under the group action.

In fact, Noether's Theorem applies in full generality to any multisymplectic system (2.5), without the need for additional constraints. To see this, all that is needed is a variational formulation. It is straightforward to check that (2.5) is the Euler-Lagrange equation for the variational problem

$$\delta \int L(\mathbf{z}, \mathbf{z}^{(1)})\, \mathrm{d}\mathbf{m},$$

where

$$L(\mathbf{z}, \mathbf{z}^{(1)}) = L^{\alpha}_{j} z^{j}_{,\alpha} - H(\mathbf{z}). \tag{3.1}$$

(Here $\mathbf{z}^{(1)}$ denotes the set of first derivatives of $\mathbf{z}$.) Note that the right-hand side of the quasi-conservation law (2.7) is the exterior derivative of the Lagrangian, L. By applying Noether's Theorem to this Lagrangian, we can use variational symmetries to derive conservation laws for multisymplectic systems.

Let $[\mathbf{m}, \mathbf{z}]$ denote the set of independent and dependent variables together with derivatives of $\mathbf{z}$ of all orders. The partial differential operator

$$X = Q^{i}[\mathbf{m}, \mathbf{z}] \frac{\partial}{\partial z^{i}} + D_{\alpha}(Q^{i}[\mathbf{m}, \mathbf{z}]) \frac{\partial}{\partial z^{i}_{,\alpha}}$$

generates variational symmetries of the Lagrangian (3.1) if

$$XL = B^{\alpha}_{,\alpha}$$

for some functions $B^{\alpha}[\mathbf{m}, \mathbf{z}]$. Equivalently, XL is in the kernel of the Euler-Lagrange operator (see Olver, 1993). Note that X acts only on the dependent variables and their derivatives; every nontrivial symmetry can be written in this form without loss of generality. Noether's Theorem states that the variational symmetry generator X yields the conservation law

$$\left(L^{\alpha}_{j} Q^{j} - B^{\alpha}\right)_{,\alpha} = 0. \tag{3.2}$$

This result can also be proved directly, by taking the interior product of the generalized vector field X with the 1-form quasi-conservation law (2.7).

As particular application of Noether's Theorem, note that every multisymplectic system (2.5) is invariant under the group of translations in each independent variable m^{β}. The corresponding symmetry generators are

$$X_{\beta} = z^{i}_{,\beta} \frac{\partial}{\partial z^{i}} + z^{i}_{,\alpha\beta} \frac{\partial}{\partial z^{i}_{,\alpha}}.$$

Then $X_{\beta} L = L_{,\beta}$ and so Noether's Theorem yields

$$\left(L^{\alpha}_{j} z^{j}_{,\beta} - L\delta^{\alpha}_{\beta}\right)_{,\alpha} = 0,$$

which are the conservation laws (2.9) that were obtained by pulling the quasi-conservation law back to the base space. Therefore the structural conservation law can be regarded as a differential consequence of the conservation laws corresponding to translational invariance in the independent variables.

4. Multisymplectic differential-difference equations

So far, we have established that every multisymplectic system of PDEs is an Euler-Lagrange equation, where the Lagrangian L is first-order and is linear the derivatives. Furthermore, the interior product of any variational symmetry generator X with the one-form quasi-conservation law gives the conservation law that corresponds to X according to Noether's Theorem. These facts may be used to introduce a multisymplectic structure for differential-difference equations, and to obtain conservation laws. For brevity, we shall adopt the approach of considering only the simplest class of problems, because the generalizations are obvious but notationally cumbersome. Therefore attention is restricted to equations whose dependent variables z_i are functions of two variables only, namely $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

The forward shift operator $S : n \to n+1$ acts on functions of n as follows:

$$S : f(n) \to Sf(n) = f(n+1) = f(Sn),$$

and therefore

$$S\big(f(n)g(n)\big) = \big(Sf(n)\big)\big(Sg(n)\big).$$

Henceforth, we omit the argument n except where it is needed for clarity. The forward difference and backward difference operators are respectively $\Delta^+ = S - \mathrm{id}$ and $\Delta^- = S^{-1}\Delta^+ = \mathrm{id} - S^{-1}$, where id is the identity map. Unlike D_t, these difference operators are not derivations; however, they satisfy the useful identity

$$\Delta^+\left(fS^{-1}g\right) = \left(\Delta^+ f\right)g + f\left(\Delta^- g\right).$$

The adjoint of the forward shift operator is the backward shift: $S^* = S^{-1}$, because

$$\sum_{n=-\infty}^{\infty} \big(Sf(n)\big)g(n) = \sum_{n=-\infty}^{\infty} f(n)\big(S^{-1}g(n)\big),$$

provided that these series converge. Consequently $(\Delta^+)^* = -\Delta^-$.

In the following, we use the notation z_+^j and z_-^j to denote $\Delta^+ z^j$ and $\Delta^- z^j$ respectively. Consider the variational problem

$$\delta \int \sum_{n=-\infty}^{\infty} L\,\mathrm{d}t = 0. \tag{4.1}$$

The simplest type of first-order Lagrangian that is linear in derivatives and differences is of the form

$$L = L_j(\mathbf{z})z_{,t}^j + R_j(\mathbf{z})z_+^j - H(\mathbf{z}), \tag{4.2}$$

where $L_j(\mathbf{z})$, $R_j(\mathbf{z})$, and $H(\mathbf{z})$ are given functions. The Euler-Lagrange equation obtained by varying z^i is

$$\frac{\partial L}{\partial z^i} - D_t\left(\frac{\partial L}{\partial z_{,t}^i}\right) - \Delta^-\left(\frac{\partial L}{\partial z_+^i}\right) = 0.$$

When this is written out explicitly and simplified, it amounts to the following *multisymplectic differential-difference equation*:

$$K_{ij}(\mathbf{z})z_{,t}^j + \frac{\partial R_j(\mathbf{z})}{\partial z^i}\,z_+^j - \Delta^-\big(R_i(\mathbf{z})\big) = \frac{\partial H(\mathbf{z})}{\partial z^i}\,, \tag{4.3}$$

where

$$K_{ij}(\mathbf{z}) = \frac{\partial L_j(\mathbf{z})}{\partial z^i} - \frac{\partial L_i(\mathbf{z})}{\partial z^j}\,.$$

By analogy with the continuous case, the 1-form quasi-conservation law is derived by taking the exterior derivative of L and using (4.3) to obtain

$$D_t\Big(L_j(\mathbf{z})\,\mathrm{d}z^j\Big) + \Delta^+\Big(S^{-1}\big(R_j(\mathbf{z})\big)\,\mathrm{d}z^j\Big) = \mathrm{d}L. \qquad (4.4)$$

Taking the exterior derivative of (4.4) yields the structural conservation law

$$D_t\Big(\tfrac{1}{2}K_{ij}(\mathbf{z})\,\mathrm{d}z^i\wedge\mathrm{d}z^j\Big) + \Delta^+\left\{S^{-1}\left(\frac{\partial R_j(\mathbf{z})}{\partial z^i}\right)\,\mathrm{d}(S^{-1}z^i)\wedge\mathrm{d}z^j\right\} = 0. \qquad (4.5)$$

By construction, the 2-form enclosed in braces in (4.5) is closed, because it is exact. The difference term in the structural conservation law would not have been easy to find by direct extrapolation from the continuous multisymplectic system, because it is not antisymmetric in (i,j). By construction, it gives a continuous multisymplectic system in the limit as the lines of constant n coalesce. Most importantly, because it comes from a variational problem, Noether's Theorem (for differential-difference equations) is retained. The differential operator

$$X = Q^i[t,n,\mathbf{z}]\,\frac{\partial}{\partial z^i} + D_t(Q^i[t,n,\mathbf{z}])\,\frac{\partial}{\partial z^i_{,t}} + \Delta^+(Q^i[t,n,\mathbf{z}])\,\frac{\partial}{\partial\Delta^+(z^i)}$$

generates variational symmetries of (4.1) if

$$XL = B^t_{,t} + \Delta^+B^n$$

for some functions $B^t[t,n,\mathbf{z}]$ and $B^n[t,n,\mathbf{z}]$. Here $[t,n,\mathbf{z}]$ denotes dependence on t, n, $\mathbf{z}$, and shifts and derivatives of $\mathbf{z}$. Taking the interior product of X and the 1-form quasi-conservation law yields (by Noether's Theorem) the conservation law

$$D_t\left(L_j(\mathbf{z})Q^j - B^t\right) + \Delta^+\left(S^{-1}\big(R_j(\mathbf{z})\big)Q^j - B^n\right) = 0. \qquad (4.6)$$

Every multisymplectic system (4.3) is invariant under translations in t, which are variational symmetries, so Noether's Theorem gives the conservation law

$$D_t\Big(H(\mathbf{z}) - R_j(\mathbf{z})z^j_+\Big) + \Delta^+\Big(S^{-1}\big(R_j(\mathbf{z})\big)z^j_{,t}\Big) = 0. \qquad (4.7)$$

Unlike the continuous case, there is no local conservation law corresponding to invariance under translations in the second independent variable, because such translations cannot be made infinitesimally. (For information on nonlocal symmetries of difference equations, see Levi & Winternitz (1993,1996).)

In many continuous multisymplectic systems (including the shallow water equations) all coefficients K^α_{ij} are constants, so each $L^\alpha_j(\mathbf{z})$ is linear. If the functions $R_j(\mathbf{z})$ are linear then (4.3) simplifies, as follows. Suppose that

$$R_j(\mathbf{z}) = M_{ij}z^i,$$

where each M_{ij} is constant. Then the multisymplectic system (4.3) reduces to

$$K_{ij}(\mathbf{z})z^j_{,t} + M_{ij}z^j_+ - M_{ji}z^j_- = \frac{\partial H(\mathbf{z})}{\partial z^i}\,. \qquad (4.8)$$

The structural conservation law for the simplified system (4.8) is

$$D_t\big(\tfrac{1}{2}K_{ij}(\mathbf{z})\,\mathrm{d}z^i\wedge\mathrm{d}z^j\big)+\Delta^+\big(M_{ij}\,\mathrm{d}(S^{-1}z^i)\wedge\mathrm{d}z^j\big)=0, \tag{4.9}$$

and the conservation law (4.7) becomes

$$D_t\left(H(\mathbf{z})-M_{ij}z^iz^j_+\right)+\Delta^+\left(M_{ij}(S^{-1}z^i)z^j_{,t}\right)=0. \tag{4.10}$$

In a pioneering paper, Maeda (1980) introduced a symplectic structure for ordinary difference equations. The discrete part of the simplified multisymplectic system (4.8) is a generalization of Maeda's structure (which corresponds to particular choices of M_{ij}).

As an application of the multisymplectic differential-difference structure, consider the Ablowitz-Ladik equation,

$$iu_t+(1+|u|^2)\left(\Delta^+\Delta^-u\right)+2\,|u|^2u=0, \tag{4.11}$$

which is integrable (Ablowitz & Ladik, 1976). In terms of the variables

$$z^1=\Re\{u\},\qquad z^2=\Im\{u\},\qquad z^3=z^1_+,\qquad z^4=z^2_+,$$

the Ablowitz-Ladik equation amounts to the multisymplectic system (4.8), where

$$H(\mathbf{z})=(z^1)^2+(z^2)^2-\ln\left\{1+(z^1)^2+(z^2)^2\right\}+\tfrac{1}{2}\left\{(z^3)^2+(z^4)^2\right\};$$

the only nonzero coefficients $K_{ij}(\mathbf{z})$ and M_{ij} are

$$K_{12}=-K_{21}=\left\{1+(z^1)^2+(z^2)^2\right\}^{-1},\qquad M_{31}=M_{42}=1.$$

Having identified the multisymplectic formulation of the Ablowitz-Ladik equation, we can immediately write down the conservation law (4.10). In terms of the original complex variable u, (4.10) amounts to

$$D_t\left(\,|u|^2-\ln\left\{1+|u|^2\right\}-\tfrac{1}{2}|\Delta^+u|^2\,\right)+\Delta^+\left(\,\Re\left\{\Delta^-(\overline{u})\,u_{,t}\right\}\,\right)=0. \tag{4.12}$$

To the best of my knowledge, the conservation law (4.12) is new. Other conservation laws can be obtained from (4.6) provided that X generates variational symmetries. Some higher symmetries of the Ablowitz-Ladik equation have recently been computed by Göktaş & Hereman, 1998. For further discussion of local symmetries of differential-difference equations, see Quispel *et al.* (1992).

5. Conclusions

For any multisymplectic system of PDEs, the two-forms κ^α are closed and satisfy the structural conservation law $\kappa^\alpha_{,\alpha}=0$. From this starting-point, we have shown that the structural conservation law is a differential consequence of a quasi-conservation law that is connected with Noether's Theorem. Furthermore, we have also seen that every multisymplectic system of PDEs can be written as the Euler-Lagrange equations for a Lagrangian that is first-order and linear in the derivatives. By reversing the line of argument, we have constructed a multisymplectic structure for a discrete independent variable that shares almost all of the important features of the continuous multisymplectic structure. The generalization to multiple discrete variables is obvious: simply add these variables to the Lagrangian, whilst retaining the property that the Lagrangian is first-order and linear in all differences and derivatives.

For brevity, most of this paper has been devoted to developing the general theory of multisymplectic systems. Nevertheless, for each of the applications that we have considered, the multisymplectic formulation gives new information. Application of the discrete multisymplectic structure to numerical integration will be considered in a separate paper. It remains to be seen whether this structure is as useful as its continuous counterpart for stability analysis.

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