CLASSIFICATION OF DISCRETE SYMMETRIES OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A simple method for determining all discrete point symmetries of a given differential equation has been developed recently. The method uses constant matrices that represent inequivalent automorphisms of the Lie algebra spanned by the Lie point symmetry generators. It may be difficult to obtain these matrices if there are three or more independent generators, because the matrix elements are determined by a large system of algebraic equations. This paper contains a classification of the automorphisms that can occur in the calculation of discrete symmetries of scalar ordinary differential equations, up to equivalence under real point transformations. (The results are also applicable to many partial differential equations.) Where these automorphisms can be realized as point transformations, we list all inequivalent realizations. By using this classification as a look-up table, readers can calculate the discrete point symmetries of a given ordinary differential equation with very little effort.

1. Introduction

Discrete symmetries of differential equations have many applications. They are used in the bifurcation analysis of nonlinear systems, in quantum field theory, and in the numerical solution of boundary value problems [1, 2, 3]. Discrete symmetries may also enable one to construct new solutions from a known solution, either directly or *via* an auto-Bäcklund transformation. Some discrete symmetries correspond to important geometric properties of the differential equation. For example, the Chazy equation has a circle of singularities [4]; its discrete symmetries include inversion in this circle, with the consequence that two apparently different symmetry reductions produce the same reduced ordinary differential equation (ODE).

The chief obstacle to obtaining discrete symmetries is that the symmetry condition amounts to a highly-coupled system of nonlinear partial differential equations (PDEs). In general, this system is intractable; we know of only one nonlinear ODE for which the symmetry condition has been solved directly [5]. An alternative approach is to use an ansatz [6]. This has the advantage that the calculations are tractable; the drawback is that one cannot be sure of finding all discrete symmetries in a given class.

It is usually easy to find all one-parameter Lie groups of point symmetries of a given differential equation. This is done by solving the linearized symmetry condition to obtain the Lie algebra of point symmetry generators. For a simple introduction to this technique, see [7] or [8]; more detailed accounts are given in [9] and [10].

Every real ODE of the form

$$y^{(n)} = \omega\left(x, y, y', \ldots, y^{(n-1)}\right), \qquad n \geq 2, \tag{1.1}$$

has a finite-dimensional Lie algebra $\mathcal{L}$ of point symmetry generators, with a basis

$$X_i = \xi_i(x,y)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1, \ldots, R. \tag{1.2}$$

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The generators in $\mathcal{L}$ can be exponentiated to obtain an R-parameter local Lie group action on the (x, y) plane. All such local Lie group actions that have no fixed points are classified, up to equivalence under real point transformations, in [11]. The corresponding Lie algebras of generators are listed by Olver in Tables 1, 3, and 6 of [12]; these tables contain all possible Lie algebras for real scalar ODEs (1.1).

Hydon [13] introduced a simple method for obtaining all discrete point symmetries of a given ODE (1.1) with a non-zero Lie algebra (1.2). Every symmetry induces an automorphism of $\mathcal{L}$, which is a linear transformation of the basis generators X_i that preserves the commutator relations

$$[X_i, X_j] = c_{ij}^k X_k.$$

In essence, Hydon's method classifies all possible automorphisms of $\mathcal{L}$, factoring out those that are equivalent under the action of any symmetry in the Lie group generated by $\mathcal{L}$. Then it is possible to obtain the most general realization of the inequivalent automorphisms as point transformations. Finally, by substituting these transformations into the symmetry condition, the user obtains a complete list of inequivalent discrete symmetries of the given ODE. The method is outlined in detail (with a worked example) in §2.

The most difficult part of the calculation is the determination of all inequivalent real automorphisms. Our aim in writing this paper is to save the reader the work of doing this. To this end, we have calculated these automorphisms for the Lie algebras that are listed in Olver's tables; the results are listed in a look-up table (see §3). The table includes the most general class of inequivalent point transformations of the plane (for application to scalar ODEs). The results in §3 are nearly (but not quite) exhaustive. A few of the Lie algebras listed in Olver's tables belong to infinite families. The look-up table includes the first two Lie algebras in each such family, which are the ones that are most likely to occur in applications. In §4, we comment on the higher-dimensional Lie algebras in these families.

With the exception of the above infinite families, our classification is complete. To save space, the results are stated without proof. They can be checked using the method described in §2. This requires considerable patience and a reliable computer algebra system; we have used Maple [17] interactively.

Our classification of inequivalent automorphisms is not only applicable to scalar ODEs; it can also be used where the symmetry generators of a scalar PDE or system of ODEs constitute one of the Lie algebras in Olver's tables. Then the realizations of these automorphisms need not act as transformations on a plane, but they can be calculated as shown in [14]. This is illustrated in §5.

As the purpose of this paper is to provide a classification, we have not included examples of the many applications of discrete symmetries. Instead we refer the reader to [7, 13, 14, 15, 16], where some substantial examples are described.

2. The classification method

This section summarizes the ideas behind the classification of the inequivalent automorphisms of the Lie algebra. Suppose that a diffeomorphism

$$\Gamma : (x, y) \mapsto \big(\hat{x}(x, y),\, \hat{y}(x, y)\big)$$

is a symmetry of (1.1). For every one-parameter Lie group of symmetries, $e^{\epsilon X_i}$, there is an associated Lie group of symmetries, $\Gamma e^{\epsilon X_i} \Gamma^{-1}$, whose infinitesimal generator is

$$\hat{X}_i = \Gamma X_i \Gamma^{-1}.$$

In particular, if X_i is a generator in the basis (1.2) then

$$\hat{X}_i = \xi_i(\hat{x}, \hat{y}) \partial_{\hat{x}} + \eta_i(\hat{x}, \hat{y}) \partial_{\hat{y}}.$$

The generators $\hat{X}_i$, $i = 1, \ldots, R$, belong to $\mathcal{L}$, which is the Lie algebra of all point symmetry generators. Furthermore, they are linearly independent, so they constitute a basis for $\mathcal{L}$. Therefore there exists a nonsingular matrix $B = (b_i^l)$ such that

$$X_i = b_i^l \hat{X}_l, \qquad i = 1, \ldots, R. \tag{2.1}$$

(The usual summation convention is adopted.) By applying (2.1) to the unknown functions $\hat{x}(x, y)$ and $\hat{y}(x, y)$ we obtain a system of $2R$ *determining equations*:

$$X_i \hat{x} = b_i^l \xi_l(\hat{x}, \hat{y}), \qquad X_i \hat{y} = b_i^l \eta_l(\hat{x}, \hat{y}). \tag{2.2}$$

The first-order quasilinear PDEs (2.2) can be solved by the method of characteristics or (if $R \geq 3$) by algebraic means. The general solution of the determining equations depends upon the unknown constants b_i^l and (possibly) some unknown constants or functions of integration. Every point symmetry of the ODE (1.1) is included in the general solution, which may also include point transformations that are not symmetries. At this stage, it is necessary to substitute the general solution $(\hat{x}, \hat{y})$ into the symmetry condition,

$$\hat{y}^{(n)} = \omega\left(\hat{x}, \hat{y}, \ldots, \hat{y}^{(n-1)}\right) \qquad \text{when (1.1) holds.} \tag{2.3}$$

The point symmetries of (1.1) are those solutions of the determining equations that also satisfy (2.3).

So far, we have not considered the matrix B in detail. For most Lie algebras, this matrix is strongly constrained, which greatly simplifies the determining equations (2.2). Suppose that in the basis (1.2) the commutator relations are

$$[X_i, X_j] = c_{ij}^k X_k. \tag{2.4}$$

As each $\hat{X}_i$ is obtained from X_i merely by replacing (x, y) with $(\hat{x}, \hat{y})$, the new basis has the commutator relations

$$[\hat{X}_i, \hat{X}_j] = c_{ij}^k \hat{X}_k, \tag{2.5}$$

with the *same* structure constants c_{ij}^k as in (2.4). Hence each symmetry Γ induces an automorphism of the Lie algebra, which is defined by

$$\Gamma : X_i \mapsto b_i^l X_l, \qquad i = 1, \ldots, R; \qquad \det(B) \neq 0. \tag{2.6}$$

By substituting (2.1) into (2.4) and taking (2.5) into account, we obtain the following system of nonlinear constraints on the elements of B:

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, \qquad 1 \leq i < j \leq R, \quad 1 \leq n \leq R. \tag{2.7}$$

If $\mathcal{L}$ is abelian the structure constants are all zero, so there are no constraints. However, most Lie algebras acting on the plane are nonabelian, and the problem of finding all symmetries can be simplified by first finding all matrices B that satisfy the constraints. This usually requires computer algebra, because there can be up to $\frac{1}{2}R^2(R-1)$ constraints.

Our aim is to obtain an inequivalent set of discrete symmetries, so we need to factor out the one-parameter Lie groups generated by each X_j in turn. Let

$$A(j, \epsilon) = \exp\left\{\epsilon C(j)\right\}, \quad \text{where} \quad \left(C(j)\right)_i^k = c_{ij}^k. \tag{2.8}$$

If $\Gamma = e^{\epsilon X_j}$ for some ϵ then $B = A(j, \epsilon)$; further details are given in [7]. Multiplication in the symmetry group corresponds to multiplication of the matrices B that represent the associated automorphisms. Hence, for each j such that $C(j)$ is non-zero, the Lie symmetries generated by X_j can be factored out as follows. First replace B by either $BA(j, \epsilon)$ or $A(j, \epsilon)B$, then choose ϵ to be a value that simplifies at least one entry in the new matrix. The aim is to create zeros in the matrix B, in order to simplify the determining equations and nonlinear constraints. Each matrix

$A(j,\epsilon)$ should be used at most once. However, if $C(j)=0$ for some j then $A(j,\epsilon)$ is the identity matrix (for all ϵ). This occurs when X_j is in the *centre* of $\mathcal{L}$, which means that it commutes with every generator. These symmetries induce a trivial automorphism of $\mathcal{L}$; they can only be factored out once the determining equations have been solved.

To illustrate the above ideas, consider the ODE

$$y''' = y'' - (y'')^2, \tag{2.9}$$

whose Lie point symmetries are generated by

$$X_1 = \partial_x\,, \qquad X_2 = x\partial_y\,, \qquad X_3 = \partial_y\,. \tag{2.10}$$

Although (2.9) is easily solved, it is useful for understanding how to calculate inequivalent discrete symmetries. The Lie algebra is nonabelian and has a nontrivial centre, so most aspects of the method can be seen in this simple example. The only nonzero structure constants are

$$c_{12}^3 = 1, \qquad c_{21}^3 = -1, \tag{2.11}$$

so X_3 is a basis for the centre of the Lie algebra. The matrices corresponding to the automorphisms generated by X_1 and X_2 are

$$A(1,\epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\epsilon \\ 0 & 0 & 1 \end{bmatrix}, \qquad A(2,\epsilon) = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The constraints (2.7) with $n=1$ amount to

$$c_{ij}^k b_k^1 = 0, \qquad 1 \le i < j \le 3.$$

These yield only one constraint, which is obtained by setting $(i,j)=(1,2)$, namely

$$b_3^1 = 0.$$

Similarly, the $n=2$ constraints amount to

$$b_3^2 = 0,$$

so $b_3^3 \neq 0$ (because B is nonsingular). To simplify B further, premultiply it by $A(1, b_2^3/b_3^3)$ to replace b_2^3 by zero. Then premultiply B by $A(2, -b_1^3/b_3^3)$ to replace b_1^3 by zero, so that now

$$B = \begin{bmatrix} b_1^1 & b_1^2 & 0 \\ b_2^1 & b_2^2 & 0 \\ 0 & 0 & b_3^3 \end{bmatrix}. \tag{2.12}$$

We have not yet used the nonlinear constraints with $n=3$; the above simplifications have reduced these constraints to the single equation

$$b_1^1 b_2^2 - b_1^2 b_2^1 = b_3^3. \tag{2.13}$$

The matrices (2.12) satisfying (2.13) represent the inequivalent automorphisms of the abstract 3-dimensional Lie algebra whose only nonzero structure constants are (2.11). To find out which of these automorphisms can be realized as real point transformations of the plane, we must solve the determining equations,

$$\begin{bmatrix} \hat{x}_x & \hat{y}_x \\ x\hat{x}_y & x\hat{y}_y \\ \hat{x}_y & \hat{y}_y \end{bmatrix} = B \begin{bmatrix} 1 & 0 \\ 0 & \hat{x} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2\hat{x} \\ b_2^1 & b_2^2\hat{x} \\ 0 & b_3^3 \end{bmatrix}.$$

Taking (2.13) into account, the general solution of the determining equations is

$$\hat{x} = b_1^1 x, \qquad \hat{y} = \tfrac{1}{2} b_1^1 b_1^2 x^2 + b_1^1 b_2^2 y + c, \qquad b_1^1 b_2^2 \neq 0, \tag{2.14}$$

where c is a constant of integration. Note that the determining equations require that $b_2^1 = 0$, so not every automorphism can be realized as a point transformation of the plane. Now we factor out the one-parameter Lie group generated by X_3, setting $c = 0$ for simplicity. Finally, we must check the symmetry condition (2.3) for the ODE (2.9) to see which of the inequivalent point transformations (2.14) are symmetries. From (2.14),

$$\hat{y}'' = \frac{b_1^2}{b_1^1} + \frac{b_2^2}{b_1^1} y'',$$

$$\hat{y}''' = \frac{b_2^2}{(b_1^1)^2} y'''.$$

Substituting these results into the symmetry condition (2.3), we find that either

$$(\hat{x}, \hat{y}) = (x, y),$$

or

$$(\hat{x}, \hat{y}) = \left(-x,\ \tfrac{1}{2}x^2 - y\right).$$

So the group of inequivalent discrete symmetries of the ODE (2.9) is generated by

$$\Gamma_1 : (x, y) \mapsto \left(-x,\ \tfrac{1}{2}x^2 - y\right).$$

This group is isomorphic to $\mathbb{Z}_2$, because Γ_1^2 is the identity transformation.

3. The classification

In this section, we present the inequivalent real automorphisms of the Lie algebras of point symmetry generators for ODEs. We also present their realizations as inequivalent point transformations of the real plane. Commonly, the inequivalent automorphisms and inequivalent point transformations form finite groups. Where this occurs, we write down a presentation of the finite group in terms of a minimal set of generators. For infinite groups, we state a set of generators for the whole group.

For most Lie algebras, we have used Olver's choice of basis in Tables 1, 3, and 6 of [12]. Exceptions occur where there exists a basis with fewer nonzero structure constants than Olver's basis. Some Lie groups have several realizations as point transformation groups that cannot be mapped to one another by a real point transformation. (The most extreme example is $\mathfrak{sl}(2)$, which has four such realizations.) Where this occurs, we first state the nontrivial commutators and inequivalent automorphisms of the underlying Lie algebra, which are independent of the realizations. Then we write down the discrete point transformations for each realization in turn.

To use our results, begin by calculating the Lie algebra of Lie point symmetry generators for the given ODE. If your chosen basis does not coincide with any of those listed that are of the same dimension, seek a real point transformation that puts the basis into one of the listed forms. It is usually easy to obtain a suitable transformation by simplifying the commutators as far as possible. The given ODE should be written in terms of the new variables if a transformation is used. Then look up the group of inequivalent realizations as point transformations. The generators of this group are of the form

$$\Gamma_i : (x, y) \mapsto \big(f(x, y),\ g(x, y)\big)$$

for some functions f and g. Substitute

$$\hat{x} = f(x, y), \qquad \hat{y} = g(x, y),$$

into the symmetry condition (2.3) to find out which of these are symmetries.

For example, to find the inequivalent discrete symmetries of (2.9) quickly, begin by calculating the Lie algebra (2.10). This Lie algebra is **3d** in our classification, which states that the inequivalent realizations of automorphisms as point transformations are of the form

$$\Gamma_1 : (x,y) \mapsto \left(b_1^1 x,\ \tfrac{1}{2} b_1^1 b_1^2 x^2 + b_1^1 b_2^2 y\right), \qquad b_1^1 b_2^2 \neq 0.$$

The discrete symmetries are then obtained by substituting (2.14) into the symmetry condition as shown in §2.

The following notations are used throughout the classification. Arbitrary constants and functions are denoted by c_i and F_i respectively. The most general nontrivial point transformation for which B is the identity is denoted by Γ_0. Where any finite group occurs, the generators $\bar{\Gamma}_i$ satisfy the standard presentation for that group, which is listed in the Appendix. In referring to Olver's tables, we state the integer k and the functions η_i used by Olver where this is necessary.

One-dimensional Lie algebras

1a Lie algebra: $\mathbb{R}$

Basis (Olver's no. 3.1, $k = 1$):

$$X_1 = \partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : X_1 \mapsto b_1^1 X_1, \qquad b_1^1 \neq 0.$$

Inequivalent realizations:

$$\Gamma_0 : (x,y) \mapsto \left(F_1(x),\ y + F_2(x)\right), \qquad F_1'(x) \neq 0, \quad F_2(0) = 0;$$

$$\Gamma_1 : (x,y) \mapsto \left(x,\ b_1^1 y\right), \qquad b_1^1 \neq 0.$$

Two-dimensional Lie algebras

2a Lie algebra: $\mathbb{R}^2$

Basis A (Olver's no. 3.1, $k = 2$):

$$X_1 = \partial_y, \qquad X_2 = x\partial_y.$$

Basis B (Olver's no. 1.5, $k = 1$, $\eta_1'(x) = 0$):

$$X_1 = \partial_x, \qquad X_2 = \partial_y.$$

Nontrivial commutators: The generators commute.

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2) \mapsto \left(b_1^1 X_1 + b_1^2 X_2,\ b_2^1 X_1 + b_2^2 X_2\right), \qquad b_1^1 b_2^2 - b_1^2 b_2^1 \neq 0.$$

Inequivalent realizations:

Basis A:

$$\Gamma_0 : (x,y) \mapsto \left(x,\ y + F_1(x)\right), \qquad F_1(0) = F_1'(0) = 0;$$

$$\Gamma_1 : (x,y) \mapsto \Big(\frac{b_1^1 x - b_2^1}{b_2^2 - b_1^2 x},\ \frac{b_1^1 b_2^2 - b_1^2 b_2^1}{b_2^2 - b_1^2 x}\, y\Big), \qquad b_1^1 b_2^2 - b_1^2 b_2^1 \neq 0.$$

Basis B:

$$\Gamma_1 : (x,y) \mapsto \big(b_1^1 x + b_2^1 y,\, b_1^2 x + b_2^2 y\big), \qquad b_1^1 b_2^2 - b_1^2 b_2^1 \neq 0.$$

2b Lie algebra: $\mathfrak{a}(1) = \mathbb{R} \ltimes \mathbb{R}$

Basis A (Olver's no. 3.2, $k=1$):

$$X_1 = \partial_y, \qquad X_2 = y\partial_y.$$

Basis B (Olver's no. 1.5, $k=1$, $\eta_1'(x) \neq 0$):

$$X_1 = e^{-x}\partial_y, \qquad X_2 = \partial_x.$$

Nontrivial commutators:

$$[X_1, X_2] = X_1.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2) \mapsto (-X_1,\, X_2).$$

The group of inequivalent automorphisms is $\mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1$.

Inequivalent realizations:

Basis A:

$$\Gamma_0 : (x,y) \mapsto (F_1(x),\, y), \qquad F_1'(x) \neq 0;$$

$$\Gamma_1 : (x,y) \mapsto (x,\, -y).$$

Basis B:

$$\Gamma_0 : (x,y) \mapsto \big(x + c_1,\, c_2 + e^{-c_1} y\big);$$

$$\Gamma_1 : (x,y) \mapsto (x,\, -y).$$

Three-dimensional Lie algebras

3a Lie algebra: $\mathfrak{a}(1) \oplus \mathbb{R}$

Basis A (Olver's no. 1.6, $k=1$; also 1.7, $k=1, \alpha=0$):

$$X_1 = \partial_x, \qquad X_2 = x\partial_x, \qquad X_3 = \partial_y.$$

Basis B (Olver's no. 1.5; $k=2$, $\eta_1(x) = e^{-x}$, $\eta_2(x) = 1$):

$$X_1 = e^{-x}\partial_y, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_1.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(-X_1,\, X_2 + b^3_2 X_3,\, b^3_3 X_3 \big), \qquad b^3_3 \neq 0.$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, b^3_3 y \big), \qquad b^3_3 \neq 0.$$

Basis B:

$$\Gamma_1 : (x, y) \mapsto \big(x + c_1,\, -e^{-c_1} y + b^3_2 x \big).$$

3b Lie algebra: $\mathfrak{a}(1) \ltimes \mathbb{R}$

Basis A (Olver's no. 1.7, $k = 1, \alpha = 1$):

$$X_1 = -x\partial_x - y\partial_y, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y.$$

Basis B (Olver's no. 3.2; $k = 2$):

$$X_1 = -y\partial_y, \qquad X_2 = \partial_y, \qquad X_3 = x\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(X_1,\, b^2_2 X_2 + b^3_2 X_3,\, b^2_3 X_2 + b^3_3 X_3 \big), \qquad b^2_2 b^3_3 - b^3_2 b^2_3 = -1.$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(b^2_2 x + b^2_3 y,\, b^3_2 x + b^3_3 y \big), \qquad b^2_2 b^3_3 - b^3_2 b^2_3 = -1.$$

Basis B:

$$\Gamma_1 : (x, y) \mapsto \Big(\frac{b^2_2 x - b^2_3}{b^3_3 - b^3_2 x},\, -\frac{y}{b^3_3 - b^3_2 x} \Big), \qquad b^2_2 b^3_3 - b^3_2 b^2_3 = -1.$$

3c Lie algebra: $\mathfrak{a}(1) \ltimes \mathbb{R}$

Basis A (Olver's no. 1.7, $k = 1,\ \alpha \notin \{0, 1\}$):

$$X_1 = -x\partial_x - \alpha y\partial_y, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y.$$

Basis B (Olver's no. 1.5; $k = 1,\ \eta_1(x) = e^x,\ \eta_2(x) = e^{\alpha x},\ \alpha \notin \{0, 1\}$):

$$X_1 = \partial_x, \qquad X_2 = e^x\partial_y, \qquad X_3 = e^{\alpha x}\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = \alpha X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(X_1,\, -X_2,\, b^3_3 X_3 \big), \qquad b^3_3 \neq 0.$$

Also, if $\alpha = -1$ only,

$$\Gamma_2 : (X_1, X_2, X_3) \mapsto \big(-X_1,\, X_3,\, X_2 \big).$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, b_3^3 y \big), \qquad b_3^3 \neq 0.$$

Also, if $\alpha = -1$ only,

$$\Gamma_2 : (x, y) \mapsto \big(y,\, x \big).$$

Basis B:

$$\Gamma_1 : (x, y) \mapsto \big(x + c_1,\, -e^{-c_1} y + c_2 \big).$$

Also, if $\alpha = -1$ only,

$$\Gamma_2 : (x, y) \mapsto \big(-x,\, y \big).$$

3d Lie algebra: $\mathbb{R} \ltimes \mathbb{R}^2$

Basis (Olver's no. 1.5; $k = 2$, $\eta_1(x) = x$, $\eta_2(x) = 1$):

$$X_1 = \partial_x, \qquad X_2 = x\partial_y, \qquad X_3 = \partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(b_1^1 X_1 + b_1^2 X_2,\, b_2^1 X_1 + b_2^2 X_2,\, (b_1^1 b_2^2 - b_1^2 b_2^1) X_3 \big),$$

where $b_1^1 b_2^2 - b_1^2 b_2^1 \neq 0$.

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(b_1^1 x,\, \tfrac{1}{2} b_1^1 b_1^2 x^2 + b_1^1 b_2^2 y \big), \qquad b_1^1 b_2^2 \neq 0.$$

3e Lie algebra: $\mathbb{R} \ltimes \mathbb{R}^2$

Basis A (Olver's no. 1.8, $k = 1$):

$$X_1 = -x\partial_x - (x + y)\partial_y, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y.$$

Basis B (Olver's no. 1.5; $k = 1$, $\eta_1(x) = e^x$, $\eta_2(x) = xe^x$):

$$X_1 = \partial_x, \qquad X_2 = e^x\partial_y, \qquad X_3 = xe^x\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = X_2 + X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(X_1,\, b_3^3 X_2,\, b_3^3 X_3 \big), \qquad b_3^3 \neq 0.$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(b_3^3 x,\, b_3^3 y \big), \qquad b_3^3 \neq 0.$$

Basis B:

$$\Gamma_0 : (x, y) \mapsto \big(x,\, y + c_1\big);$$

$$\Gamma_1 : (x, y) \mapsto \big(x,\, b_3^3 y\big), \qquad b_3^3 \neq 0.$$

3f **Lie algebra**: $\mathbb{R} \ltimes \mathbb{R}^2$

Basis A (Olver's no. 6.1, $\alpha > 0$):

$$X_1 = -(\alpha x + y)\partial_x + (x - \alpha y)\partial_y, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y.$$

Basis B (Olver's no. 1.5; $k = 1,\ \eta_1(x) = e^{\alpha x}\cos(x),\ \eta_2(x) = e^{\alpha x}\sin(x),\ \alpha > 0$):

$$X_1 = \partial_x, \qquad X_2 = e^{\alpha x}\cos(x)\partial_y, \qquad X_3 = e^{\alpha x}\sin(x)\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = \alpha X_2 - X_3, \qquad [X_1, X_3] = X_2 + \alpha X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(X_1,\, e^t X_2,\, e^t X_3\big), \qquad t \in [0, 2\pi\alpha).$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(e^t x,\, e^t y\big), \qquad t \in [0, 2\pi\alpha).$$

Basis B:

$$\Gamma_0 : (x, y) \mapsto \big(x + n\pi,\, (-1)^n e^{n\pi\alpha} y + c_1\big), \qquad n \in \mathbb{Z};$$

$$\Gamma_1 : (x, y) \mapsto \big(x,\, e^t y\big), \qquad t \in [0, 2\pi\alpha).$$

3g **Lie algebra**: $\mathbb{R} \ltimes \mathbb{R}^2$

Basis A (Olver's no. 6.1, $\alpha = 0$):

$$X_1 = -y\partial_x + x\partial_y, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y.$$

Basis B (Olver's no. 1.5; $k = 1,\ \eta_1(x) = \cos(x),\ \eta_2(x) = \sin(x)$):

$$X_1 = \partial_x, \qquad X_2 = \cos(x)\partial_y, \qquad X_3 = \sin(x)\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = X_2.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(X_1,\, e^t X_2,\, e^t X_3\big), \qquad t \in \mathbb{R};$$

$$\Gamma_2 : (X_1, X_2, X_3) \mapsto \big(-X_1,\, -X_2,\, X_3\big).$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(e^t x,\, e^t y\big), \qquad t \in \mathbb{R};$$

$$\Gamma_2 : (x, y) \mapsto \big(-x,\, y\big).$$

Basis B:

$$\Gamma_0 : (x, y) \mapsto \big(x + n\pi,\, (-1)^n y + c_1\big), \qquad n \in \mathbb{Z};$$

$$\Gamma_1 : (x, y) \mapsto \big(x,\, e^t y\big), \qquad t \in \mathbb{R};$$

$$\Gamma_2 : (x, y) \mapsto \big(-x,\, -y\big).$$

3h Lie algebra: $\mathfrak{sl}(2)$

Basis A (Olver's no. 1.1):

$$X_1 = \partial_x, \qquad X_2 = x\partial_x - y\partial_y, \qquad X_3 = x^2\partial_x - 2xy\partial_y.$$

Basis B (Olver's no. 1.2):

$$X_1 = \partial_x, \qquad X_2 = x\partial_x - y\partial_y, \qquad X_3 = x^2\partial_x - (2xy + 1)\partial_y.$$

Basis C (Olver's no. 6.3):

$$X_1 = \partial_x, \qquad X_2 = x\partial_x + y\partial_y, \qquad X_3 = (x^2 - y^2)\partial_x + 2xy\partial_y.$$

Basis D (Olver's no. 3.3):

$$X_1 = \partial_y, \qquad X_2 = y\partial_y, \qquad X_3 = y^2\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3) \mapsto \big(-X_1,\, X_2,\, -X_3\big);$$

$$\Gamma_2 : (X_1, X_2, X_3) \mapsto \big(X_3,\, -X_2,\, X_1\big).$$

The group of inequivalent automorphisms is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Inequivalent realizations:

Basis A:

$$\Gamma_0 : (x, y) \mapsto \big(x,\, c_1 y\big);$$

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_2 : (x, y) \mapsto \left(-\frac{1}{x}\,,\ x^2 y\right).$$

Basis B:

$$\Gamma_0 : (x, y) \mapsto \left(x + \frac{1}{y}\,,\ -y\right);$$

$$\Gamma_1 : (x, y) \mapsto \big(-x, -y\big);$$

$$\Gamma_2 : (x,y) \mapsto \Big(-\frac{1}{x}\,,\; x^2y + x \Big)\,.$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2,\ \bar{\Gamma}_3 = \Gamma_0$.

Basis C:

$$\Gamma_0 : (x,y) \mapsto \big(x,\, -y\big);$$

$$\Gamma_1 : (x,y) \mapsto \big(-x, y\big);$$

$$\Gamma_2 : (x,y) \mapsto \Big(-\frac{x}{x^2+y^2}\,,\; \frac{y}{x^2+y^2} \Big)\,.$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2,\ \bar{\Gamma}_3 = \Gamma_0$.

Basis D:

$$\Gamma_0 : (x,y) \mapsto \big(F_1(x),\, y\big), \qquad F_1'(x) \neq 0;$$

$$\Gamma_1 : (x,y) \mapsto \big(x, -y\big);$$

$$\Gamma_2 : (x,y) \mapsto \Big(x\,,\; -\frac{1}{y}\Big)\,.$$

3i Lie algebra: $\mathfrak{so}(3)$

Basis (Olver's no. 6.3):

$X_1 = y\partial_x - x\partial_y, \quad X_2 = \frac{1}{2}(1+x^2-y^2)\partial_x + xy\partial_y, \quad X_3 = xy\partial_x + \frac{1}{2}(1-x^2+y^2)\partial_y.$

Inequivalent automorphisms:

All automorphisms are equivalent to the identity.

Inequivalent realizations:

$$\Gamma_0 : (x,y) \mapsto \Big(-\frac{x}{x^2+y^2}\,,\; -\frac{y}{x^2+y^2} \Big)\,.$$

This group of realizations is $\mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_0$.

Four-dimensional Lie algebras

4a Lie algebra: $\mathfrak{a}(1) \oplus \mathfrak{a}(1)$

Basis A (Olver's no. 1.9, $k = 1$):

$$X_1 = \partial_x, \qquad X_2 = x\partial_x, \qquad X_3 = \partial_y, \qquad X_4 = y\partial_y.$$

Basis B (Olver's no. 1.6; $k = 2,\ \eta_1(x) = e^x,\ \eta_2(x) = e^{\alpha x},\ \alpha \neq 1$):

$$X_1 = e^{\alpha x}\partial_y, \quad X_2 = \frac{1}{1-\alpha}\Big(\partial_x + y\partial_y\Big), \quad X_3 = e^x\partial_y, \quad X_4 = \frac{1}{\alpha-1}\Big(\partial_x + \alpha y\partial_y\Big).$$

Nontrivial commutators:

$$[X_1, X_2] = X_1, \qquad [X_3, X_4] = X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \big(X_3,\, X_4,\, -X_1,\, X_2\big);$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4) \mapsto \big(X_1,\, X_2,\, -X_3,\, X_4\big).$$

The group of inequivalent automorphisms is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(-y,\, x\big);$$

$$\Gamma_2 : (x, y) \mapsto \big(x,\, -y\big).$$

This group of realizations is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Basis B:

$$\Gamma_1\Gamma_2 : (x, y) \mapsto \big(-x,\, -e^{-(\alpha+1)x}y\big);$$

$$\Gamma_1^2 : (x, y) \mapsto \big(x,\, -y\big).$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1\Gamma_2,\ \bar{\Gamma}_2 = \Gamma_1^2$.

4b Lie algebra: $\mathfrak{a}(1) \ltimes \mathbb{R}^2$

Basis (Olver's no. 1.7, $k = 2,\ \alpha \neq 2$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x + \alpha y\partial_y, \quad X_3 = \partial_y, \quad X_4 = x\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \big(-X_1,\, X_2,\, b_3^3X_3,\, -b_3^3X_4\big), \qquad b_3^3 \neq 0.$$

Also, if $\alpha = 0$,

$$\Gamma_2 : (X_1, X_2, X_3, X_4) \mapsto \big(X_4,\, -X_2 + b_2^3X_3,\, X_3,\, -X_1\big).$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, b_3^3y\big), \qquad b_3^3 \neq 0.$$

Note that Γ_2 is not realizable.

4c Lie algebra: $\mathfrak{a}(1) \ltimes \mathbb{R}^2$

Basis (Olver's no. 1.7, $k = 2,\ \alpha = 2$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x + 2y\partial_y, \quad X_3 = \partial_y, \quad X_4 = x\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \big(b_1^1X_1 + b_1^4X_4,\, X_2,\, -X_3,\, b_4^1X_1 + b_4^4X_4\big),$$

where $b_1^1b_4^4 - b_1^4b_4^1 = -1$.

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(b_1^1x,\, \tfrac{1}{2}b_1^1b_1^4x^2 - y\big), \qquad b_1^1 \neq 0.$$

4d Lie algebra: $\mathbb{R}^2 \ltimes \mathbb{R}^2$

Basis A (Olver's no. 6.4):

$$X_1 = y\partial_x - x\partial_y, \quad X_2 = x\partial_x + y\partial_y, \quad X_3 = \partial_y, \quad X_4 = \partial_x.$$

Basis B (Olver's no. 1.6; $k = 2$, $\eta_1(x) = \cos x$, $\eta_2(x) = \sin x$):

$$X_1 = \partial_x, \quad X_2 = y\partial_y, \quad X_3 = \cos x\partial_y, \quad X_4 = \sin x\partial_y.$$

Nontrivial commutators:

$$[X_1, X_3] = -X_4, \quad [X_1, X_4] = X_3, \quad [X_2, X_3] = -X_3, \quad [X_2, X_4] = -X_4.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \big(-X_1,\, X_2,\, X_3,\, -X_4\big).$$

The group of inequivalent automorphisms is $\mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1$.

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big).$$

This group of realizations is $\mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1$.

Basis B:

$$\Gamma_0 : (x, y) \mapsto \big(x + n\pi,\, (-1)^n y\big), \qquad n \in \mathbb{Z};$$

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big).$$

4e Lie algebra: $\mathbb{R}^2 \ltimes \mathbb{R}^2$

Basis (Olver's no. 1.6, $k = 2$, $\eta_1(x) = 1$, $\eta_2(x) = x$):

$$X_1 = \partial_x, \quad X_2 = y\partial_y, \quad X_3 = \partial_y, \quad X_4 = x\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \Big(b_1^1 X_1,\, X_2,\, -X_3,\, -\frac{1}{b_1^1} X_4\Big), \qquad b_1^1 \neq 0.$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(b_1^1 x,\, -y\big), \qquad b_1^1 \neq 0.$$

4f Lie algebra: $\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^2)$

Basis (Olver's no. 1.8, $k = 2$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x + (x^2 + 2y)\partial_y, \quad X_3 = \partial_y, \quad X_4 = x\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \big(b_1^1 X_1,\, X_2,\, (b_1^1)^2 X_3,\, b_1^1 X_4\big), \qquad b_1^1 \neq 0.$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(b_1^1 x,\, (b_1^1)^2 y\big), \qquad b_1^1 \neq 0.$$

4g Lie algebra: $\mathfrak{sl}(2) \oplus \mathbb{R}$

Basis A (Olver's no. 1.3):

$$X_1 = \partial_x, \quad X_2 = x\partial_x - \tfrac{1}{2} y\partial_y, \quad X_3 = x^2\partial_x - xy\partial_y, \quad X_4 = y\partial_y.$$

Basis B (Olver's no. 1.10; $k = 1$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x, \quad X_3 = x^2\partial_x, \quad X_4 = \partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4) \mapsto \big(-X_1,\, X_2,\, -X_3,\, b_4^4 X_4\big), \qquad b_4^4 \neq 0;$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4) \mapsto \big(X_3,\, -X_2,\, X_1,\, X_4\big).$$

Inequivalent realizations:

Basis A:

$$\Gamma_0 : (x, y) \mapsto \big(x,\, -y\big);$$

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_2 : (x, y) \mapsto \left(-\frac{1}{x},\, xy\right).$$

This group of realizations is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_2$, $\bar{\Gamma}_2 = \Gamma_1$.

Basis B:

$$\Gamma_1(b_4^4) : (x, y) \mapsto \big(-x,\, b_4^4 y\big), \qquad b_4^4 \neq 0;$$

$$\Gamma_2 : (x, y) \mapsto \left(-\frac{1}{x},\, y\right).$$

Five-dimensional Lie algebras

5a Lie algebra: $\mathfrak{sa}(2) = \mathfrak{sl}(2) \ltimes \mathbb{R}^2$

Basis A (Olver's no. 6.5):

$$X_1 = y\partial_x, \quad X_2 = \tfrac{1}{2}(x\partial_x - y\partial_y), \quad X_3 = -x\partial_y, \quad X_4 = \partial_x, \quad X_5 = \partial_y.$$

Basis B (Olver's no. 1.10; $k = 2$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x + \tfrac{1}{2} y\partial_y, \quad X_3 = x^2\partial_x + xy\partial_y, \quad X_4 = \partial_y, \quad X_5 = -x\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3, \qquad [X_1, X_5] = -X_4,$$

$$[X_2, X_4] = -\tfrac{1}{2}X_4, \qquad [X_2, X_5] = \tfrac{1}{2}X_5, \qquad [X_3, X_4] = X_5.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_1,\, X_2,\, X_3,\, e^t X_4,\, e^t X_5\big), \qquad t \in \mathbb{R};$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(-X_1,\, X_2,\, -X_3,\, X_4,\, -X_5\big);$$

$$\Gamma_3 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_3,\, -X_2,\, X_1,\, X_5,\, -X_4\big).$$

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(e^t x,\, e^t y\big), \qquad t \in \mathbb{R};$$

$$\Gamma_2 : (x, y) \mapsto \big(x,\, -y\big);$$

$$\Gamma_3 : (x, y) \mapsto \big(-y,\, x\big).$$

Basis B:

$$\Gamma_1 : (x, y) \mapsto \big(x,\, e^t y\big), \qquad t \in \mathbb{R};$$

$$\Gamma_2 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_3 : (x, y) \mapsto \left(-\frac{1}{x},\, \frac{y}{x}\right).$$

5b Lie algebra: $\mathfrak{sl}(2) \oplus \mathfrak{a}(1)$

Basis (Olver's no. 1.11, $k = 1$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x, \quad X_3 = x^2\partial_x, \quad X_4 = \partial_y, \quad X_5 = y\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(-X_1,\, X_2,\, -X_3,\, X_4,\, X_5\big);$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_3,\, -X_2,\, X_1,\, X_4,\, X_5\big);$$

$$\Gamma_3 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_1,\, X_2,\, X_3,\, -X_4,\, X_5\big).$$

The group of inequivalent automorphisms is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_2 : (x, y) \mapsto \left(-\frac{1}{x},\, y\right);$$

$$\Gamma_3 : (x, y) \mapsto \big(x,\, -y\big).$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

5c Lie algebra: $(\mathfrak{a}(1) \oplus \mathbb{R}) \ltimes \mathbb{R}^2$

Basis (Olver's no. 1.9, $k=2$):

$$X_1 = \partial_x, \quad X_2 = x\partial_x, \quad X_3 = y\partial_y, \quad X_4 = \partial_y, \quad X_5 = x\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_5,\, -X_2,\, X_2 + X_3,\, X_4,\, -X_1\big);$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_1,\, X_2,\, X_3,\, -X_4,\, -X_5\big).$$

The group of inequivalent automorphisms is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Inequivalent realizations:

$$\Gamma_1^2 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_2 : (x, y) \mapsto \big(x,\, -y\big).$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1^2,\ \bar{\Gamma}_2 = \Gamma_2$.

Six-dimensional Lie algebras

6a Lie algebra: $\mathfrak{a}(2) = (\mathfrak{sl}(2) \oplus \mathbb{R}) \ltimes \mathbb{R}^2$

Basis A (Olver's no. 6.6):

$$X_1 = y\partial_x, \qquad X_2 = \tfrac{1}{2}(x\partial_x - y\partial_y), \qquad X_3 = -x\partial_y,$$

$$X_4 = \partial_x, \qquad X_5 = \partial_y, \qquad X_6 = x\partial_x + y\partial_y.$$

Basis B (Olver's no. 1.11; $k=2$):

$$X_1 = \partial_x, \qquad X_2 = x\partial_x + \tfrac{1}{2}y\partial_y, \qquad X_3 = x^2\partial_x + xy\partial_y,$$

$$X_4 = \partial_y, \qquad X_5 = -x\partial_y, \qquad X_6 = y\partial_y.$$

Nontrivial commutators:

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3,$$

$$[X_1, X_5] = -X_4, \qquad [X_2, X_4] = -\tfrac{1}{2}X_4, \qquad [X_2, X_5] = \tfrac{1}{2}X_5,$$

$$[X_3, X_4] = X_5, \qquad [X_4, X_6] = X_4, \qquad [X_5, X_6] = X_5.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto \big(-X_1,\, X_2,\, -X_3,\, X_4,\, -X_5,\, X_6\big);$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto \big(X_3,\, -X_2,\, X_1,\, X_5,\, -X_4,\, X_6\big).$$

The group of inequivalent automorphisms is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_2,\ \bar{\Gamma}_2 = \Gamma_1$.

Inequivalent realizations:

Basis A:

$$\Gamma_1 : (x, y) \mapsto \big(x,\, -y\big);$$

$$\Gamma_2 : (x, y) \mapsto \big(-y,\, x\big).$$

This group of realizations is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_2,\ \bar{\Gamma}_2 = \Gamma_1$.

Basis B:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_2 : (x, y) \mapsto \left(-\frac{1}{x},\ \frac{y}{x}\right).$$

This group of realizations is $D(4)$, with $\bar{\Gamma}_1 = \Gamma_2,\ \bar{\Gamma}_2 = \Gamma_1$.

6b Lie algebra: $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

Basis (Olver's no. 1.4):

$$X_1 = \partial_x, \quad X_2 = x\partial_x, \quad X_3 = x^2\partial_x, \quad X_4 = \partial_y, \quad X_5 = y\partial_y, \quad X_6 = y^2\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto \big(X_4,\, X_5,\, X_6,\, -X_3,\, -X_2,\, -X_1\big);$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto \big(-X_4,\, X_5,\, -X_6,\, X_3,\, -X_2,\, X_1\big);$$

$$\Gamma_3 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto \big(X_4,\, X_5,\, X_6,\, X_1,\, X_2,\, X_3\big).$$

The group of inequivalent automorphisms is $\mathrm{dih}(\mathbb{Z}_4 \otimes \mathbb{Z}_4)$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \left(\frac{1}{y},\ x\right);$$

$$\Gamma_2 : (x, y) \mapsto \left(-\frac{1}{y},\ -x\right);$$

$$\Gamma_3 : (x, y) \mapsto \big(y,\, x\big).$$

This group of realizations is $\mathrm{dih}(\mathbb{Z}_4 \otimes \mathbb{Z}_4)$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

6c Lie algebra: $\mathfrak{so}(3,1)$

Basis (Olver's no. 6.7):

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + y\partial_y, \quad X_4 = y\partial_x - x\partial_y,$$

$$X_5 = (x^2 - y^2)\partial_x + 2xy\partial_y, \qquad X_6 = 2xy\partial_x + (y^2 - x^2)\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_1,\, -X_2,\, X_3,\, -X_4,\, X_5,\, -X_6);$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_5,\, -X_6,\, -X_3,\, -X_4,\, X_1,\, -X_2).$$

The group of inequivalent automorphisms is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Inequivalent realizations:

$$\Gamma_1 : (x,y) \mapsto \big(x,\, -y\big);$$

$$\Gamma_2 : (x,y) \mapsto \Big(-\frac{x}{x^2+y^2}\,,\ \frac{y}{x^2+y^2}\Big)\,.$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Eight-dimensional Lie algebras

8a Lie algebra: $\mathfrak{sl}(3)$

Basis (Olver's no. 6.8):

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x, \quad X_4 = y\partial_x, \quad X_5 = x\partial_y,$$

$$X_6 = y\partial_y, \qquad X_7 = x^2\partial_x + xy\partial_y, \qquad X_8 = xy\partial_x + y^2\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8) \mapsto (X_5,X_2,-X_3,X_7,-X_1,X_3+X_6,-X_4,X_8);$$

$$\Gamma_2 : (X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8) \mapsto (X_8,X_7,-X_6,-X_4,-X_5,-X_3,X_2,X_1);$$

$$\Gamma_3 : (X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8) \mapsto (X_7,X_8,-X_3,-X_5,-X_4,-X_6,X_1,X_2).$$

The group of inequivalent automorphisms is $S(4) \otimes \mathbb{Z}_2$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

Inequivalent realizations:

$$\Gamma_1\Gamma_3 : (x,y) \mapsto \Big(\frac{x}{y}\,,\ -\frac{1}{y}\Big)\,;$$

$$\Gamma_2\Gamma_3 : (x,y) \mapsto \big(y,\, x\big).$$

This group of realizations is $S(4)$, with $\bar{\Gamma}_1 = \Gamma_1\Gamma_3,\ \bar{\Gamma}_2 = \Gamma_2\Gamma_3$.

4. Lie algebras with ideals of arbitrary dimension

Nine classes of vector fields on the real plane yield Lie algebras of arbitrary dimension. The general form of all Lie algebras in any one class is the same, apart from an ideal of arbitrary dimension k that is isomorphic to $\mathbb{R}^k$. In five classes, this ideal has a basis

$$X_i = x^{i-1}\partial_y, \qquad i = 1,\ldots,k. \tag{4.1}$$

For these classes, we have already dealt with the cases $k = 1$ and $k = 2$ in §3. If $k \geq 3$, the following inequivalent automorphisms and realizations always occur. There are no other automorphisms or realizations when $k = 3$ or $k = 4$, and we conjecture that the same is true for all $k \geq 5$. (At present, we know of no way of proving this conjecture.)

Lie algebra: $\mathfrak{a}(1) \ltimes \mathbb{R}^k$

Basis (Olver's no. 1.7, $k \geq 3$):

$$X_i = x^{i-1}\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = \partial_x, \quad X_{k+2} = x\partial_x + \alpha y\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_i, X_{k+1}, X_{k+2}) \mapsto \big((-1)^{i-1}b_1^1 X_i,\ -X_{k+1},\ X_{k+2}\big), \qquad b_1^1 \neq 0.$$

Also, if $\alpha = k$,

$$\Gamma_2 : (X_i, X_{k+1}, X_{k+2}) \mapsto \big(X_i,\ X_{k+1} + b_{k+1}^k X_k,\ X_{k+2}\big),$$

or, if $\alpha = 0$,

$$\Gamma_2' : (X_i, X_{k+1}, X_{k+2}) \mapsto \big(X_i,\ X_{k+1},\ X_{k+2} + b_{k+2}^1 X_1\big).$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\ b_1^1 y\big), \qquad b_1^1 \neq 0.$$

Also, if $\alpha = k$,

$$\Gamma_2 : (x, y) \mapsto \big(x,\ y + \tfrac{1}{k} b_{k+1}^k x^k\big),$$

Note that Γ_2' is not realizable for $b_{k+2}^1 \neq 0$.

Lie algebra: $\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^k)$

Basis (Olver's no. 1.8, $k \geq 3$):

$$X_i = x^{i-1}\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = \partial_x, \quad X_{k+2} = x\partial_x + (ky + x^k)\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_i, X_{k+1}, X_{k+2}) \mapsto \big((b_4^4)^{k+1-i} X_i,\ b_4^4 X_{k+1},\ X_{k+2}\big), \qquad b_4^4 \neq 0.$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(b_4^4 x,\ (b_4^4)^k y\big), \qquad b_4^4 \neq 0.$$

Lie algebra: $(\mathfrak{a}(1) \oplus \mathbb{R}) \ltimes \mathbb{R}^k$

Basis (Olver's no. 1.9, $k \geq 3$):

$$X_i = x^{i-1}\partial_y, \quad i = 1, \ldots, k, \quad X_{k+1} = \partial_x, \quad X_{k+2} = x\partial_x, \quad X_{k+3} = y\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}) \mapsto \big((-1)^{i-1} X_i,\ X_{k+1},\ X_{k+2},\ X_{k+3}\big);$$

$$\Gamma_2 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}) \mapsto \big(-X_i,\ X_{k+1},\ X_{k+2},\ X_{k+3}\big).$$

The group of inequivalent automorphisms is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\ y\big);$$

$$\Gamma_2 : (x, y) \mapsto \big(x,\ -y\big).$$

This group of realizations is $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, with $\bar{\Gamma}_1 = \Gamma_1,\ \bar{\Gamma}_2 = \Gamma_2$.

Lie algebra: $\mathfrak{sl}(2) \ltimes \mathbb{R}^k$

Basis (Olver's no. 1.10, $k \geq 3$):

$$X_i = x^{i-1}\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = \partial_x,$$

$$X_{k+2} = x\partial_x + \tfrac{1}{2}(k-1)y\partial_y, \qquad X_{k+3} = x^2\partial_x + (k-1)xy\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}) \mapsto \big(b_1^1 X_i,\, X_{k+1},\, X_{k+2},\, X_{k+3}\big), \quad b_1^1 \neq 0;$$

$$\Gamma_2 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}) \mapsto \big((-1)^{i-1} X_i,\, -X_{k+1},\, X_{k+2},\, -X_{k+3}\big);$$

$$\Gamma_3 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}) \mapsto \big((-1)^{i-1} X_{k+1-i},\, X_{k+3},\, -X_{k+2},\, X_{k+1}\big).$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(x,\, b_1^1 y\big) \qquad b_1^1 \neq 0;$$

$$\Gamma_2 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_3 : (x, y) \mapsto \Big(-\frac{1}{x},\, (-x)^{1-k} y\Big).$$

Lie algebra: $(\mathfrak{sl}(2) \oplus \mathbb{R}) \ltimes \mathbb{R}^k$

Basis (Olver's no. 1.11, $k \geq 3$):

$$X_i = x^{i-1}\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = \partial_x,$$

$$X_{k+2} = x\partial_x + \tfrac{1}{2}(k-1)y\partial_y, \quad X_{k+3} = x^2\partial_x + (k-1)xy\partial_y, \quad X_{k+4} = y\partial_y.$$

Inequivalent automorphisms:

$$\Gamma_1 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}, X_{k+4}) \mapsto \big((-1)^{i-1} X_i,\, -X_{k+1},\, X_{k+2},\, -X_{k+3},\, X_{k+4}\big);$$

$$\Gamma_2 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}, X_{k+4}) \mapsto \big((-1)^{i-1} X_{k+1-i},\, X_{k+3},\, -X_{k+2},\, X_{k+1},\, X_{k+4}\big).$$

Also, if k is odd,

$$\Gamma_3 : (X_i, X_{k+1}, X_{k+2}, X_{k+3}, X_{k+4}) \mapsto \big(-X_i,\, X_{k+1},\, X_{k+2},\, X_{k+3},\, X_{k+4}\big).$$

If k is even, the group of inequivalent automorphisms is $D(4)$, with

$$\bar{\Gamma}_1 = \Gamma_2, \quad \bar{\Gamma}_2 = \Gamma_1.$$

Otherwise, the group is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

Inequivalent realizations:

$$\Gamma_1 : (x, y) \mapsto \big(-x,\, y\big);$$

$$\Gamma_2 : (x, y) \mapsto \Big(-\frac{1}{x},\, (-x)^{1-k} y\Big).$$

Also, if k is odd,

$$\Gamma_3 : (x, y) \mapsto \big(x,\, -y\big).$$

If k is even, this group of realizations is $D(4)$, with

$$\bar{\Gamma}_1 = \Gamma_2, \quad \bar{\Gamma}_2 = \Gamma_1.$$

Otherwise, the group is $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, with

$$\bar{\Gamma}_1 = \Gamma_1, \quad \bar{\Gamma}_2 = \Gamma_2, \quad \bar{\Gamma}_3 = \Gamma_3.$$

So far, we have have discussed only five out of the nine classes with an ideal of arbitrary dimension. In the other four classes, the $\mathbb{R}^k$ ideal has a basis

$$X_i = \eta_i(x)\partial_y, \qquad i = 1, \ldots, k, \tag{4.2}$$

where the functions $\eta_i(x)$ are constrained only by the requirement that the Lie algebra is closed. For $k = 1$ and $k = 2$, this constraint is sufficient to allow us to include all four classes in §3. However, for each $k \geq 3$, a complete classification of inequivalent automorphisms is possible for only two of the four classes. We do not include any of these classifications, which have numerous special cases and are very lengthy. If necessary, the automorphisms and realizations can be calculated as shown in §2, once a basis for the Lie algebra is known. We end by describing a simple choice of basis for each of the remaining four classes of Lie algebras.

Lie algebra: $\mathbb{R} \ltimes \mathbb{R}^k$

Basis (Olver's no. 1.5, $k \geq 3$):

$$X_i = \eta_i(x)\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = \partial_x. \tag{4.3}$$

Lie algebra: $\mathbb{R}^2 \ltimes \mathbb{R}^k$

Basis (Olver's no. 1.6, $k \geq 3$):

$$X_i = \eta_i(x)\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = \partial_x, \qquad X_{k+2} = y\partial_y. \tag{4.4}$$

For (4.3) and (4.4), the Lie algebra is closed if and only if

$$\eta_i'(x) = t_i^j \eta_j(x), \qquad i = 1, \ldots, k,$$

for some matrix $T = (t_i^j)$. If all eigenvalues of T are real, the commutator relations may be simplified by using a basis in which T is in Jordan normal form. If some eigenvalues are complex, a real 2×2 block Jordan form can be achieved on the space spanned by the corresponding generalized eigenvectors, as follows. If $\lambda = \mu + i\nu$ is an eigenvalue of multiplicity L, then so is its complex conjugate, $\lambda^* = \mu - i\nu$. Then

$$\tilde{\eta}_{2l-1}(x) = x^{L-l} e^{\mu x} \cos(\nu x), \qquad \tilde{\eta}_{2l}(x) = x^{L-l} e^{\mu x} \sin(\nu x), \qquad l = 1, \ldots, L,$$

produce the required real block. A further slight simplification may be achieved by rescaling x. These simplifications have been used in §3 to determine the results for $k \leq 2$.

Lie algebra: $\mathbb{R}^k$

Basis (Olver's no. 3.1, $k \geq 3$):

$$X_i = \eta_i(x)\partial_y, \quad i = 1, \ldots, k. \tag{4.5}$$

Lie algebra: $\mathbb{R} \ltimes \mathbb{R}^k$

Basis (Olver's no. 3.2, $k \geq 3$):

$$X_i = \eta_i(x)\partial_y, \quad i = 1, \ldots, k, \qquad X_{k+1} = y\partial_y. \tag{4.6}$$

The Lie algebras (4.5) and (4.6) are closed for all functions $\eta_i(x)$. The only possible simplification is obtained by introducing new variables

$$\tilde{x} = \frac{\eta_2(x)}{\eta_1(x)}, \quad \tilde{y} = \frac{y}{\eta_1(x)}.$$

Therefore, without loss of generality, we can restrict attention to Lie algebras with

$$\eta_1(x) = 1, \qquad \eta_2(x) = x.$$

Even so, for $k \geq 3$, the functions $\eta_i(x)$, $i \geq 3$, can be arbitrary. Therefore it is only possible to find the inequivalent automorphisms and realizations once these functions are known.

5. Conclusion

The classification that is presented in §3 enables the reader to obtain all inequivalent discrete symmetries of a given scalar ODE whose Lie algebra of point symmetries is known. If the Lie algebra is not included in §3, because it belongs to a family that has ideals of arbitrary dimension, the results in §4 should be used. A wide range of applications of this method can be found in Hydon's papers that were cited in the introduction.

Earlier, we stated that the classification of inequivalent automorphisms of a particular Lie algebra can be useful for PDEs and systems of differential equations that have that Lie algebra. To illustrate this, consider *Burgers' equation*,

$$u_t + uu_x = u_{xx}, \tag{5.1}$$

which has a five-dimensional Lie algebra of point symmetry generators that is isomorphic to $\mathfrak{sa}(2)$, which is the Lie algebra **5a** in §3. One basis in which the structure constants are the same as in **5a** is

$$X_1 = -\tfrac{1}{2}\partial_t, \quad X_2 = \tfrac{1}{2}x\partial_x + t\partial_t - \tfrac{1}{2}u\partial_u,$$

$$X_3 = -2tx\partial_x - 2t^2\partial_t + 2(tu - x)\partial_u, \quad X_4 = \partial_x, \quad X_5 = 2t\partial_x + 2\partial_u. \tag{5.2}$$

From **5a**, the inequivalent real automorphisms of this Lie algebra are generated by

$$\Gamma_1 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_1,\, X_2,\, X_3,\, e^{\lambda}X_4,\, e^{\lambda}X_5\big), \qquad \lambda \in \mathbb{R};$$

$$\Gamma_2 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(-X_1,\, X_2,\, -X_3,\, X_4,\, -X_5\big);$$

$$\Gamma_3 : (X_1, X_2, X_3, X_4, X_5) \mapsto \big(X_3,\, -X_2,\, X_1,\, X_5,\, -X_4\big).$$

Note that the Lie group generated by (5.2) is transitive in (x, t, u)-space, whereas the Lie group generated by each basis in **5a** is only two-dimensional. Therefore there is no point transformation mapping either realization in **5a** to the realization of the inequivalent discrete transformations for Burgers' equation. Consequently, it is necessary to calculate this realization directly from the analogue of the determining equations (2.2). Each generator in the basis (5.2) is of the form

$$X_i = \xi_i(x, t, u)\partial_x + \tau_i(x, t, u)\partial_t + \eta_i(x, t, u)\partial_u\,,$$

so the determining equations are

$$X_i\hat{x} = b_i^l \xi_l(\hat{x}, \hat{t}, \hat{u}), \quad X_i\hat{t} = b_i^l \tau_l(\hat{x}, \hat{t}, \hat{u}), \quad X_i\hat{u} = b_i^l \eta_l(\hat{x}, \hat{t}, \hat{u}),$$

where the coefficients b_i^l can be read off from each Γ_j in turn. By solving the determining equations for $(\hat{x}, \hat{t}, \hat{u})$, we find that each of the generators Γ_j can be realized as a point transformation, as follows:

$$\begin{aligned}
\Gamma_1 : (x,t,u) &\mapsto \left(e^{\lambda}x, t, e^{\lambda}u\right), \qquad \lambda \in \mathbb{R};\\
\Gamma_2 : (x,t,u) &\mapsto \left(x, -t, -u\right);\\
\Gamma_3 : (x,t,u) &\mapsto \left(-\frac{x}{2t}, -\frac{1}{4t}, 2(x-tu)\right).
\end{aligned}$$

This is the complete list of real point transformations that produce automorphisms of the Lie algebra spanned by (5.2), up to equivalence under the adjoint action of the one-parameter subgroups. At this stage, we must check to see which of the above transformations are symmetries of Burgers' equation. It turns out that Γ_1 does not generate symmetries of Burgers' equation (expect in the trivial case $\lambda = 0$). Furthermore, neither Γ_2 nor $\Gamma_1\Gamma_2$ generate symmetries. However, Γ_3 generates a four-element group of discrete symmetries which is isomorphic to $\mathbb{Z}_4$. These are the inequivalent real discrete symmetries of Burgers' equation.

For many differential equations the inequivalent discrete symmetries are all real-valued. Burgers' equation is an exception; its inequivalent complex-valued discrete symmetries form a group of order 8 that is isomorphic to the quaternion group Q_2 (see [14] for details).

Appendix

The following standard presentations of finite groups are used in the main text. Here 1 denotes the identity element.

Cyclic group and its direct products

$$\mathbb{Z}_2 : \quad \bar{\Gamma}_1^2 = 1.$$

This group has two elements.

$$\mathbb{Z}_2 \otimes \mathbb{Z}_2 : \quad \bar{\Gamma}_1^2 = \bar{\Gamma}_2^2 = 1, \quad \bar{\Gamma}_1\bar{\Gamma}_2 = \bar{\Gamma}_2\bar{\Gamma}_1.$$

This group has four elements.

$$\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 : \ \bar{\Gamma}_1^2 = \bar{\Gamma}_2^2 = \bar{\Gamma}_3^2 = 1, \quad \text{all generators commute.}$$

This group has eight elements.

Dihedral groups

$$D(4) : \quad \bar{\Gamma}_1^4 = \bar{\Gamma}_2^2 = 1, \quad \bar{\Gamma}_2\bar{\Gamma}_1 = \bar{\Gamma}_1^3\bar{\Gamma}_2.$$

This group has eight elements.

$$\mathrm{dih}(\mathbb{Z}_4 \otimes \mathbb{Z}_4) : \quad \bar{\Gamma}_1^4 = \bar{\Gamma}_2^4 = \bar{\Gamma}_3^2 = 1, \quad \bar{\Gamma}_2\bar{\Gamma}_1 = \bar{\Gamma}_1\bar{\Gamma}_2, \quad \bar{\Gamma}_3\bar{\Gamma}_1 = \bar{\Gamma}_1^3\bar{\Gamma}_3, \quad \bar{\Gamma}_3\bar{\Gamma}_2 = \bar{\Gamma}_2^3\bar{\Gamma}_3.$$

This group has thirty-two elements.

Symmetric group and its direct products

$$S(4) : \quad \bar{\Gamma}_1^4 = \bar{\Gamma}_2^2 = 1, \quad (\bar{\Gamma}_1\bar{\Gamma}_2)^3 = 1.$$

This group has twenty-four elements.

$$S(4) \otimes \mathbb{Z}_2 : \quad \bar{\Gamma}_1^4 = \bar{\Gamma}_2^2 = \bar{\Gamma}_3^2 = 1, \quad (\bar{\Gamma}_1\bar{\Gamma}_2)^3 = 1, \quad \bar{\Gamma}_1\bar{\Gamma}_3 = \bar{\Gamma}_3\bar{\Gamma}_1, \quad \bar{\Gamma}_2\bar{\Gamma}_3 = \bar{\Gamma}_3\bar{\Gamma}_2.$$

This group has forty-eight elements.

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