# CONSERVATION LAWS OF PARTIAL DIFFERENCE EQUATIONS WITH TWO INDEPENDENT VARIABLES

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ABSTRACT. This paper introduces a technique for obtaining the conservation laws of a given scalar partial difference equation with two independent variables. Unlike methods that are based on Nöther's Theorem, the new technique does not use symmetries. Neither does it require the difference equation to have any special structure, such as a Lagrangian, Hamiltonian, or multisymplectic formulation. Instead, it uses a discrete analogue of the variational complex.

## 1. INTRODUCTION

Conservation laws are ubiquitous in applied mathematics. In some cases, they express conservation of physical quantities. Even when they do not, they are usually of mathematical interest. Much attention has been given to integrable systems that have infinite hierarchies of conservation laws, which are related to generalized symmetries by Nöther's Theorem. Conservation laws of integrable and nonintegrable systems can be used in many ways, such as to prove existence and uniqueness theorems, to derive shock conditions, and to check that numerical methods are not producing spurious results (at least qualitatively). If a differential equation is to be approximated using a finite difference method, it seems desirable that the discretized equation should retain as much of the original structure as possible, including discrete analogues of the conservation laws. Thus it would be useful to have a systematic method for constructing conservation laws of a given difference equation, that does not require the equation to be integrable. The purpose of this paper is to introduce such a method in the simplest possible context.

Nöther's Theorem provides the best-known method of constructing conservation laws of any partial differential equation (PDE) that is the Euler-Lagrange equation for a variational problem [11]. This method uses variational symmetries, which form a subset of the set of generalized (or Lie-Bäcklund) symmetries of the PDE. Generalized symmetries of a particular order can be found systematically from the symmetry condition, which amounts to an overdetermined system of PDEs (see [1, 12] for a modern introduction). Nöther's Theorem has been extended to Hamiltonian PDEs [12] and, in a restricted form, to multisymplectic PDEs [2]. However, Nöther's Theorem does not apply to all PDEs, but only to those that have at least one of the special structures listed above.

Some partial difference equations (P$\Delta$Es) have a variational formulation. Nöther's Theorem has been adapted to P$\Delta$Es [3, 8], but there is a substantial drawback: the symmetry condition is a functional equation, rather than a system of PDEs. It is often possible to obtain series solutions of the symmetry condition [4, 9], but such solutions may be nonlocal or hard to write in closed form. Recently, a method was developed that uses repeated differentiation to derive overdetermined systems of PDEs from the symmetry condition [6]. This method usually requires the assistance of computer algebra, but it can yield all symmetries of a given order (in closed form).

There is another way to construct conservation laws of a given PDE, that uses neither symmetries nor any structural properties of the PDE. It is based on the variational complex [12], which has a homotopy operator. This operator has two main uses. First, it is used to prove that the complex is exact on topologically trivial domains. In particular, a function is a conservation law (that is, a total divergence) if and only if it is in the kernel of the Euler operator. Second, the homotopy operator provides a systematic means of constructing conservation laws. The main drawback is that the explicit formula for the homotopy operator is very cumbersome.

The discrete analogue of the variational complex was discovered recently [7, 10]. A homotopy operator has been found, so for topologically trivial domains (see §2 for details) the complex is exact. At least in principle, it is possible to construct conservation laws systematically using the homotopy operator, but the complexity of the calculations is even more fearsome than for PDEs! There are two main reasons for this. First, the space of independent variables is continuous for PDEs but is discrete for PΔEs, so the homotopy operator involves sums rather than integrals. Second, functional equations occur in the discrete case, making the governing equations more complicated than in the continuous case. It seems likely that a general homtopy-based method would have to be implemented as a computer algebra package; this represents a substantial computational challenge. Nevertheless, the homotopy operator does not use symmetries or any special structures. In particular, the PΔE need not be integrable.

For PΔEs with two independent variables, considerable simplification is possible. This paper introduces the first systematic technique for obtaining conservation laws for such equations that does not require a Lagrangian or Hamiltonian structure. For clarity and brevity, we shall restrict attention to scalar PΔEs; it is straightforward (but slightly messy) to extend the technique to systems of PΔEs. The structure of the rest of the paper is as follows: §2 describes the technique and summarizes the underlying theory. The implementation of the technique is discussed in §3. To illustrate this, we find conservation laws of a wave equation and the discrete potential modified Korteweg-deVries equation. To conclude, some extensions of the technique are outlined in §4. For completeness, the homotopy formula is included in the Appendix.

## 2. The method

The domain of a given PDE can be regarded as a fibre bundle $M = X \times U$, where $X$ is the base space of independent variables and $U$ is the vertical space, i. e. the fibre of dependent variables $\mathbf{u}$ over each $\mathbf{x} \in X$. The direct method for constructing conservation laws of PDEs requires the domain $M$ to be topologically trivial, which occurs if each fibre $U$ and the base space $X$ are star-shaped [12].

For a given PΔE, we again write the domain as $M = X \times U$, but now $X$ is the set of integer-valued multi-indices $\mathbf{n}$ that label each lattice point. (We assume that the lattice points are labelled sequentially, without jumps; this does not require the lattice to be uniform.) The label space $X$ is said to be cube-shaped if, given any two points $\mathbf{n}_1, \mathbf{n}_2 \in X$, the PΔE is well-defined for each $\mathbf{n}$ that lies in the (hyper-) cube whose opposite corners are $\mathbf{n}_1$ and $\mathbf{n}_2$.

**Definition 1.** *For a given PΔE, the domain $M = X \times U$ is topologically trivial if $X$ is cube-shaped and each fibre $U$ is star-shaped.*

We restrict attention to domains that are topologically trivial. The reason for doing this is to exclude 'holes' in the lattice, which are points at which the PΔE is singular.

In this section and in §3, we consider scalar P$\Delta$Es that are second-order in one variable. The integer-valued labels $(m, n)$ are the independent variables, and the value of the dependent variable $u$ at the lattice point $(m, n)$ is denoted by $u_m^n$. The shift operators,

$$S_m : (m,n) \mapsto (m+1,\, n), \qquad S_n : (m,n) \mapsto (m,\, n+1),$$

induce the following mappings on the dependent variables:

$$S_m : u_m^n \mapsto u_{m+1}^n, \qquad S_n : u_m^n \mapsto u_m^{n+1}.$$

We assume that the labels have been chosen in such a way that the P$\Delta$E is of the form

$$u_{m+2}^{n+p} = \omega(m, n, \mathbf{u}_m, \mathbf{u}_{m+1}), \tag{2.1}$$

for a given function $\omega$ and a given integer $p$. Here each $\mathbf{u}_i$ denotes all variables of the form $u_i^{n+j}$. We shall always choose $p$ so that $j \geq 0$ and (2.1) depends nontrivially upon at least one of $u_m^n$, $u_{m+1}^n$ and $u_{m+2}^n$.

The form (2.1) is analogous to Kovalevskaya form for PDEs. It is achieved by a suitable choice of variables. To illustrate this, consider the discrete potential modified Korteweg-deVries (dpmKdV) equation:

$$u_{k+1}^{l+1} = u_k^l \left( \frac{\nu(k,l) u_{k+1}^l - u_k^{l+1}}{\nu(k,l) u_k^{l+1} - u_{k+1}^l} \right), \tag{2.2}$$

which is not in the required form. However, there is only one second-order term, which appears on the left-hand side. By choosing the new independent variables $m = k + l$ and $n = l$, the dpmKdV equation is equivalent to

$$u_{m+2}^{n+1} = u_m^n \left( \frac{\mu(m,n) u_{m+1}^n - u_{m+1}^{n+1}}{\mu(m,n) u_{m+1}^{n+1} - u_{m+1}^n} \right), \quad \text{where} \quad \mu(m,n) = \nu(m-n, n). \tag{2.3}$$

This P$\Delta$E has the form (2.1).

A conservation law for the P$\Delta$E (2.1) is an expression of the form

$$(S_m - \mathrm{id})F + (S_n - \mathrm{id})G = 0 \tag{2.4}$$

that is satisfied by all solutions of the equation. Here id is the identity mapping, and $F, G$ are functions of the dependent and independent variables. A conservation law is trivial if it holds identically (not just on solutions of the P$\Delta$E), or if $F$ and $G$ both vanish on all solutions of (2.1). We aim to find nontrivial conservation laws, so we assume without loss of generality that $F$ and $G$ depend only on $m, n$ and a finite subset of the variables $\mathbf{u}_m$, $\mathbf{u}_{m+1}$. Note that the only place where (2.1) can be substituted into (2.4) is in the term $S_m F$. Therefore $F$ must depend upon at least one of the variables $\mathbf{u}_{m+1}$. To keep things as simple as possible, we shall only look for conservation laws for which $F$ depends on exactly one such variable. By applying $S_n$ or its inverse repeatedly to (2.4), we may assume that that variable is $u_{m+1}^{n+p}$.

Under the above restriction, the conservation law (2.4) amounts to

$$(S_m - \mathrm{id})F(m, n, \mathbf{u}_m, u_{m+1}^{n+p}) \;+\; (S_n - \mathrm{id})G(m, n, \mathbf{u}_m, \mathbf{u}_{m+1}) = 0$$

on solutions of (2.1). Therefore

$$(S_n - \mathrm{id})G(m, n, \mathbf{u}_m, \mathbf{u}_{m+1}) = F(m, n, \mathbf{u}_m, u_{m+1}^{n+p}) - F(m+1, n, \mathbf{u}_{m+1}, \omega), \tag{2.5}$$

where $\omega$ is the right-hand side of the P$\Delta$E (2.1). This constraint on $F$ and $G$ is the key to obtaining the conservation laws. It does not involve the shift operator $S_m$, so $m$ merely plays the role of a parameter. Therefore (2.5) can be regarded as a functional difference equation involving *one* independent variable, $n$, and *two* dependent variables, $u_m^n$ and $u_{m+1}^n$. (Note that all of the variables $(\mathbf{u}_m, \mathbf{u}_{m+1})$ can be obtained from $(u_m^n, u_{m+1}^n)$ by prolongation, that is, by shifting $n$.)

The operator $S_n - \mathrm{id}$ is a total difference operator (because $S_n$ treats the dependent variables as functions of $n$). Thus the left-hand side of (2.5) is a total difference, and so it lies within the kernel of the Euler operator (see [7, 8, 10] for details). For difference equations whose independent variable is $n$ and whose dependent variables are $(u_m^n, u_{m+1}^n)$, the Euler operator has two components:

$$\mathrm{E}_m = \sum_j (S_n)^{-j} \frac{\partial}{\partial u_m^{n+j}}\,, \tag{2.6}$$

$$\mathrm{E}_{m+1} = \sum_j (S_n)^{-j} \frac{\partial}{\partial u_{m+1}^{n+j}}\,. \tag{2.7}$$

By applying the Euler operator to (2.5), we obtain the following pair of linear functional equations for $F$.

$$\mathrm{E}_m\big\{F(m, n, \mathbf{u}_m, u_{m+1}^{n+p}) - F(m+1, n, \mathbf{u}_{m+1}, \omega)\big\} = 0\,, \tag{2.8}$$

$$\mathrm{E}_{m+1}\big\{F(m, n, \mathbf{u}_m, u_{m+1}^{n+p}) - F(m+1, n, \mathbf{u}_{m+1}, \omega)\big\} = 0\,. \tag{2.9}$$

This pair of *determining equations* can be solved using the technique of invariant differentiation, as described in §3. Next, the function $G$ can be reconstructed, as the following result shows.

**Theorem 1.** *Suppose that the domain $M$ for a given P$\Delta$E is topologically trivial. Then for every solution $F$ of (2.8), (2.9), there exists a function $G$ such that (2.5) holds.*

This theorem holds because the variational complex is exact on topologically trivial domains. (The proof of that result is long and complicated; details are given in [7].) In this case, the kernel of the Euler operator is the image of the total difference operator $S_n - \mathrm{id}$. The homotopy operator gives a systematic formula (which is written down in the Appendix) for constructing $G$, but it is almost always far easier to obtain $G$ by inspection, as we shall do in §3.

Note that if $F$ solves (2.8) and (2.9), then so does $F + B(m, n)$ for any function $B(m, n)$. This freedom merely adds a trivial conservation law, so we shall always use the simplest possible solutions $F$. Indeed, whenever the right-hand side of (2.5) depends only upon $m$ and $n$, the conservation law is trivial. Apart from such cases, there is a conservation law for each linearly independent solution $F$.

## 3. Implementation and examples

Before trying to solve the determining equations, it is necessary to decide how general to make the function $F$. The greater the number of variables that $F$ is allowed to depend upon, the greater is the difficulty of the calculation. On the other hand, if $F$ is restricted too much then some conservation laws will not be found. This dilemma is universal – it applies as much to PDEs as to P$\Delta$Es, and it occurs in the search for symmetries as well as conservation laws. Usually, a compromise must be made, based on the limits set by patience and computational power. The examples in this section were first calculated by hand, then checked with computer algebra. With the exception of the wave equation in the first example, details of most calculations are not included, for they are neither brief nor particularly illuminating!

The method of invariant differentiation has previously been applied to the problem of obtaining symmetries of difference equations [6]. The symmetry condition, like each of the above determining equations, is a functional equation. The idea is to use it to derive a set of PDEs by repeatedly applying first-order differential operators that eliminate parts of the functional equation at each step. The same idea can be used to solve the determining equations for $F$. The following straightforward example shows how the method works.

We shall solve the determining equations for the discrete wave equation

$$u_{m+2}^{n} = u_{m}^{n+1}. \tag{3.1}$$

In order to keep the calculations simple, let us seek solutions of the form

$$F = F(u_m^n, u_m^{n+1}, u_{m+1}^n). \tag{3.2}$$

Then the constraint (2.5) is

$$(S_n - \mathrm{id})G(m, n, \mathbf{u}_m, \mathbf{u}_{m+1}) = F(u_m^n, u_m^{n+1}, u_{m+1}^n) - F(u_{m+1}^n, u_{m+1}^{n+1}, u_m^{n+1}), \tag{3.3}$$

and the first determining equation (2.8) amounts to

$$F_{,1}(u_m^n, u_m^{n+1}, u_{m+1}^n) + F_{,2}(u_m^{n-1}, u_m^n, u_{m+1}^{n-1}) - F_{,3}(u_{m+1}^{n-1}, u_{m+1}^n, u_m^n) = 0, \tag{3.4}$$

where $F_{,k}$ denotes the partial derivative of $F$ with respect to its $k^{\text{th}}$ argument. Each of the functions in (3.4) takes a different set of arguments. The first function is the only one that depends upon $u_m^{n+1}$, so the remaining functions are invariant under the first-order differential operator $\partial/\partial u_m^{n+1}$. Applying this operator to (3.4) yields

$$F_{,12}(u_m^n, u_m^{n+1}, u_{m+1}^n) = 0,$$

and therefore there exist functions $A$ and $B$ such that

$$F(x, y, z) = A(x, z) + B(y, z).$$

Substituting this result into (3.4), we obtain

$$A_{,1}(u_m^n, u_{m+1}^n) + B_{,1}(u_m^n, u_{m+1}^{n-1}) - A_{,2}(u_{m+1}^{n-1}, u_m^n) - B_{,2}(u_{m+1}^n, u_m^n) = 0. \tag{3.5}$$

This completes the first step of the reduction. Now we iterate, keeping going until both determining equations are satisfied. The functional equation (3.5) is simplified by differentiation with respect to $u_{m+1}^n$:

$$A_{,12}(u_m^n, u_{m+1}^n) - B_{,12}(u_{m+1}^n, u_m^n) = 0.$$

Therefore

$$B(y, z) = A(z, y) + \alpha(y) + \beta(z),$$

for some functions $\alpha$ and $\beta$. Then (3.5) reduces to

$$\alpha'(u_m^n) - \beta'(u_m^n) = 0,$$

whose general solution is

$$\beta(z) = \alpha(z) + c,$$

where $c$ is an arbitrary constant. So, from the first determining equation, we have found that

$$F(u_m^n, u_m^{n+1}, u_{m+1}^n) = A(u_m^n, u_{m+1}^n) + A(u_{m+1}^n, u_m^{n+1}) + \alpha(u_m^{n+1}) + \alpha(u_{m+1}^n) + c.$$

Without loss of generality, we can set $\alpha$ and $c$ to be zero (redefining the arbitrary function $A$ if necessary). Perhaps surprisingly, the second determining equation (2.9), which amounts to

$$F_{,3}(u_m^n, u_m^{n+1}, u_{m+1}^n) - F_{,2}(u_{m+1}^{n-1}, u_{m+1}^n, u_m^n) - F_{,1}(u_{m+1}^n, u_{m+1}^{n+1}, u_m^{n+1}) = 0, \tag{3.6}$$

is satisfied by

$$F(u_m^n, u_m^{n+1}, u_{m+1}^n) = A(u_m^n, u_{m+1}^n) + A(u_{m+1}^n, u_m^{n+1}), \tag{3.7}$$

for any differentiable function $A$. Therefore (3.7) is the general solution of the determining equations that is of the form (3.2); here $A$ is an arbitrary function. From (3.3), we obtain

$$(S_n - \mathrm{id})G(m, n, \mathbf{u}_m, \mathbf{u}_{m+1}) = A(u_m^n, u_{m+1}^n) - A(u_m^{n+1}, u_{m+1}^{n+1}),$$

and hence (by inspection)

$$G = -A(u_m^n, u_{m+1}^n).$$

Summarizing these results, we have obtained an infinite set of independent conservation laws,

$$(S_m - \mathrm{id})\{A(u_m^n, u_{m+1}^n) + A(u_{m+1}^n, u_m^{n+1})\} + (S_n - \mathrm{id})\{-A(u_m^n, u_{m+1}^n)\} = 0. \tag{3.8}$$

Although this may seem surprising, it is analogous to conservation laws for the wave equation

$$u_t = -u_x\,.$$

For every nonconstant function $A(u)$, there is a conservation law

$$\big(A(u)\big)_t + \big(A(u)\big)_x = 0.$$

The above example is particularly easy, because the PDEs are found after very little differentiation. In general, information from both determining equations is needed to obtain reductions (see [6] for a detailed discussion of invariant differentiation). In the next example, the solutions of the determining equations are stated without the details of their derivation.

To find conservation laws of the discrete potential modified Korteweg-deVries (dpmKdV) equation,

$$u_{k+1}^{l+1} = u_k^l \left( \frac{\nu(k,l)u_{k+1}^l - u_k^{l+1}}{\nu(k,l)u_k^{l+1} - u_{k+1}^l} \right), \tag{3.9}$$

write it in the form (2.3):

$$u_{m+2}^{n+1} = u_m^n \left( \frac{\mu(m,n)u_{m+1}^n - u_{m+1}^{n+1}}{\mu(m,n)u_{m+1}^{n+1} - u_{m+1}^n} \right), \quad \text{where} \quad \mu(m,n) = \nu(m-n,n). \tag{3.10}$$

We shall seek solutions of the determining equations that are of the form

$$F = F(m, n, u_m^n, u_m^{n+1}, u_{m+1}^{n+1}), \qquad F_{,5} \not\equiv 0. \tag{3.11}$$

(This level of generality is close to the limit of what can be achieved by hand in one hour; with the aid of computer algebra, one could seek solutions of greater generality.)

Papageorgiou *et al.* [13] have shown that the singularities of the dpmKdV are confined if and only if $\nu$ is separable, i. e.

$$\nu(k,l) = \alpha(k)\beta(l) \tag{3.12}$$

for some functions $\alpha$ and $\beta$. (Singularity confinement is an indication that a given discrete system is integrable [5].) The determining equations have no solutions of the form (3.11) unless (3.12) holds, in which case there are four linearly independent solutions:

$$F^1 = u_{m+1}^{n+1}\left((S_n\alpha)u_m^{n+1} - \frac{u_m^n}{\beta}\right),$$

$$F^2 = \frac{1}{u_{m+1}^{n+1}}\left(\frac{S_n\alpha}{u_m^{n+1}} - \frac{1}{\beta u_m^n}\right),$$

$$F^3 = \frac{1}{u_{m+1}^{n+1}}\left(\frac{u_m^{n+1}}{S_n\alpha} - \beta u_m^n\right) + u_{m+1}^{n+1}\left(\frac{1}{(S_n\alpha)u_m^{n+1}} - \frac{\beta}{u_m^n}\right),$$

$$F^4 = (-1)^m\left\{\frac{1}{u_{m+1}^{n+1}}\left(\frac{u_m^{n+1}}{S_n\alpha} - \beta u_m^n\right) - u_{m+1}^{n+1}\left(\frac{1}{(S_n\alpha)u_m^{n+1}} - \frac{\beta}{u_m^n}\right)\right\},$$

where

$$S_n\alpha = \alpha(m-n-1), \qquad \beta = \beta(n).$$

For $F = F^1$, the corresponding $G = G^1$ can be reconstructed from (2.5), which (with $\alpha = \alpha(m-n)$) amounts to

$$\begin{aligned}(S_n - \mathrm{id})G^1 &= u_{m+1}^{n+1}\left((S_n\alpha)u_m^{n+1} - \frac{u_m^n}{\beta}\right)\\ &\quad - u_m^n\left(\frac{\alpha\beta u_{m+1}^n - u_{m+1}^{n+1}}{\alpha\beta u_{m+1}^{n+1} - u_{m+1}^n}\right)\left(\alpha u_{m+1}^{n+1} - \frac{u_{m+1}^n}{\beta}\right)\\ &= u_{m+1}^{n+1}\left((S_n\alpha)u_m^{n+1} - \frac{u_m^n}{\beta}\right) - u_m^n\left(\alpha u_{m+1}^n - \frac{u_{m+1}^{n+1}}{\beta}\right)\\ &= (S_n - \mathrm{id})(\alpha u_m^n u_{m+1}^n).\end{aligned}$$

Therefore

$$G^1 = \alpha u_m^n u_{m+1}^n.$$

Similarly, the remaining components of the other three conservation laws are

$$\begin{aligned}G^2 &= \frac{\alpha}{u_m^n u_{m+1}^n},\\ G^3 &= \frac{1}{\alpha}\left(\frac{u_m^n}{u_{m+1}^n} + \frac{u_{m+1}^n}{u_m^n}\right),\\ G^4 &= \frac{(-1)^m}{\alpha}\left(\frac{u_m^n}{u_{m+1}^n} - \frac{u_{m+1}^n}{u_m^n}\right).\end{aligned}$$

Finally, the four conservation laws can be rewritten in their original variables, as follows:

$$(S_k - \mathrm{id})\left\{-\frac{u_k^l u_k^{l+1}}{\beta}\right\} + (S_l - \mathrm{id})\left\{\alpha u_k^l u_{k+1}^l\right\} = 0,$$

$$(S_k - \mathrm{id})\left\{-\frac{1}{\beta u_k^l u_k^{l+1}}\right\} + (S_l - \mathrm{id})\left\{\frac{\alpha}{u_k^l u_{k+1}^l}\right\} = 0,$$

$$(S_k - \mathrm{id})\left\{-\beta\left(\frac{u_k^{l+1}}{u_k^l} + \frac{u_k^l}{u_k^{l+1}}\right)\right\} + (S_l - \mathrm{id})\left\{\frac{1}{\alpha}\left(\frac{u_k^l}{u_{k+1}^l} + \frac{u_{k+1}^l}{u_k^l}\right)\right\} = 0,$$

$$(S_k - \mathrm{id})\left\{(-1)^{k+l}\beta\left(\frac{u_k^{l+1}}{u_k^l} - \frac{u_k^l}{u_k^{l+1}}\right)\right\} +$$

$$(S_l - \mathrm{id})\left\{\frac{(-1)^{k+l}}{\alpha}\left(\frac{u_k^l}{u_{k+1}^l} - \frac{u_{k+1}^l}{u_k^l}\right)\right\} = 0.$$

## 4. Conclusions and extensions of the technique

The technique presented in this paper is a practical way of determining the conservation laws of a given form. The method of invariant differentiation enables the user to obtain closed-form solutions of the determining equations. Once these solutions have been found, the reconstruction of the conservation law is usually easy. The most complicated part of the technique is the derivation of PDEs by invariant differentiation, but this is not difficult if a reliable computer algebra system is used.

For brevity, several restrictions were imposed that are not needed for the technique to succeed. For example, there is no reason why $F$ should not depend upon more than one of the variables $\mathbf{u}_{m+1}$; in that case, $n$-shifts of $\omega$ will appear in several parts of the determining equations. It is also easy to generalize the technique to systems of P$\Delta$Es. If there are $q$ dependent variables then the Euler operator has $2q$ components; these are of the form (2.6) or (2.7), where $u$ is replaced by each dependent variable in turn. After the $2q$ determining equations have been solved, the reconstruction of the conservation law is straightforward. We have restricted attention to second-order equations, but the technique works just as well for higher-order P$\Delta$Es. Then $F$ should be chosen so that $S_m F$ is changed when the left-hand side of the P$\Delta$E is replaced by the right-hand side.

The only real obstacle to allowing more than two independent variables is the complexity of the calculations. It is still possible to write down a set of determining equations for one of the unknown functions in the conservation law, but (unless this function is heavily restricted) computer algebra is an essential tool for solving them. Furthermore, the reconstruction of the conservation law may not be obvious, as there will be more than one function to find. If all else fails, the homotopy formula will produce a reconstruction, but the calculations are usually messy. In [12], Olver includes the following comment in his discussion of conservation laws for PDEs. "In practice, it is often easier to determine the divergence form directly by inspection, using [the homotopy formula] only as a last resort." The same is true for P$\Delta$Es.

## Appendix: The homotopy formula

Here we present the homotopy formula for the class of P$\Delta$Es described in the main body of the paper. For details of the general homotopy operator for P$\Delta$Es with more independent or dependent variables, readers should consult [7]. If

$$\Omega = \Omega(m, n, \mathbf{u}_m, \mathbf{u}_{m+1})$$

is in the kernel of the Euler operator (2.6), (2.7) then

$$(S_n - \mathrm{id})G = \Omega \tag{A.1}$$

is solved as follows. Introduce the *higher Euler operators*,

$$\begin{aligned} \mathrm{E}^i_m &= \sum_{j\geq i} \binom{j}{i} S_n^{-j} \frac{\partial}{\partial u_m^{n+j}}\,, \\ \mathrm{E}^i_{m+1} &= \sum_{j\geq i} \binom{j}{i} S_n^{-j} \frac{\partial}{\partial u_{m+1}^{n+j}}\,. \end{aligned}$$

Then

$$G = \int_{\lambda=0}^{1} \lambda^{-1} \mathrm{K}(\Omega)\Big|_{u\,\mapsto\lambda u} \mathrm{d}\lambda + \sum_{k=n_0}^{n-1} \Omega\Big|_{u\,\mapsto 0} \tag{A.2}$$

solves (A.1), where

$$\mathrm{K}(\Omega) = \sum_{i\geq 1} (S_n - \mathrm{id})^{i-1} \left[ u_m^n \mathrm{E}^i_m(\Omega) + u_{m+1}^n \mathrm{E}^i_{m+1}(\Omega) \right]. \tag{A.3}$$

In the homotopy formula (A.2), $n_0$ is any convenient reference value of $n$, and the notation $u \mapsto \lambda u$ means that each $u_{m+i}^{n+j}$ is replaced by $\lambda u_{m+i}^{n+j}$. Some care is needed if $\Omega$ is singular when $u \mapsto 0$, but this is not a major difficulty.

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