ASYMMETRIC INTEGRABLE QUAD-GRAPH EQUATIONS

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Abstract. Integrable difference equations commonly have more low-order conservation laws than occur for nonintegrable difference equations of similar complexity. We use this empirical observation to sift a large class of difference equations, in order to find candidates for integrability. It turns out that all such candidates have an equivalent affine form. These are tested by calculating their algebraic entropy. In this way, we have found several types of integrable equations, one of which seems to be entirely unrelated to any known discrete integrable system. We also list all single-tile conservation laws for the integrable equations in the above class.

1. Introduction

A *quad-graph equation* is a scalar difference equation for $u(k,l)$, where $(k,l) \in \mathbb{Z}^2$, which is of the form

$$\mathcal{F}(k,l,u_{00},u_{10},u_{01},u_{11}) = 0. \tag{1}$$

Here u_{ij} denotes $u(k+i,l+j)$ and we assume that $\mathcal{F}$ depends on all four of these values. Various approaches have been used to discover quad-graph equations that are integrable. Having developed the bilinear formalism for continuous integrable systems, Hirota discretized the bilinear operators for several known integrable systems, obtaining difference equations that had soliton solutions built-in [1–3]. By contrast, Capel *et al.* focused on discretizations of plane wave factors for the singular integral equations that are ubiquitous features of continuous integrable systems [4–6]. Whereas these approaches used discretizations of problems that were known to be integrable, Adler, Bobenko and Suris (ABS) dealt directly with quad-graph equations without reference to continuous systems. They obtained a classification of all integrable quad-graph equations that are consistent on a cube (and thus admit a Lax pair), subject to certain nondegeneracy conditions [7,8]. The idea that consistency on a cube is a sufficient condition for integrability was proposed independently by Nijhoff [9] and Bobenko and Suris [10].

To make further progress, we adopt a different strategy. There is a systematic method for constructing conservation laws of difference equations; this has been used to identify low-order conservation laws of many integrable quad-graph equations [11, 12]. From this work, we observe that integrable difference equations tend to have more low-order conservation laws than nonintegrable equations of similar complexity. Although this observation is purely empirical, we use it to sift a large class of quad-graph equations, in order to find equations that admit 'extra' conservation laws. (This approach is dual to that of Levi and Yamilov, who recently obtained some necessary conditions for the existence of higher symmetries – which again indicate integrability – for certain types of quad-graph equations [13]). Having obtained a shortlist of possible candidates for integrability, we test their algebraic entropy.

Zero algebraic entropy is a signature of integrability [14–16]. This occurs for affine linear quad-graph equations when an arbitrary set of initial conditions produces polynomial growth in degree as one moves away from the initial points (see §3 for details). Linear growth in degree implies that the quad-graph equation is linearizable; all known integrable quad-graph equations that are not linearizable exhibit quadratic growth. The calculation of algebraic entropy is a diagnostic test, rather than a constructive method. For instance, Hietarinta discovered a quad-graph equation that is consistent on a cube, but

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does not appear in the ABS list [17]. A calculation of algebraic entropy showed that growth in degree for this quad-graph equation is linear; separately, Ramani *et al.* found a clever linearization [18].

In the next section, we determine conditions for the existence of extra conservation laws for a large class of quad-graph equations. Algebraic entropy is calculated in §3, and we find that most of the sifted quad-graph equations exhibit quadratic growth in degree. For completeness, we list the conservation laws in §4, before discussing our results and their consequences in §5.

2. Classification of integrable cases via conservation laws

In this section, we examine the conservation laws of equations of the form

$$u_{11} = \epsilon_1 u_{00} + A(u_{10}) - \epsilon_2 A(u_{01}). \tag{2}$$

Here each ϵ_i is either 1 or -1, and A is a nonlinear complex-valued function that is assumed to be 'differentiable enough' (so that as many derivatives as needed are well-defined, at least locally). Equations in this class lack the D_4 symmetry of the ABS equations. The class includes some known integrable equations (such as the Lattice KdV equation) and is simple enough for a complete classification of integrable cases to be possible. Henceforth, we use A_{ij} to denote $A(u_{ij})$. Conservation laws on a single tile satisfy the determining equation

$$F\big(k+1,\, l,\, u_{10},\, \omega\big) - F\big(k,\, l,\, u_{00},\, u_{01}\big) + G\big(k,\, l+1,\, u_{01},\, \omega\big) - G\big(k,\, l,\, u_{00},\, u_{10}\big) = 0, \tag{3}$$

where ω denotes the right-hand side of (2). We solve (3) by deriving a sequence of its differential consequences, each of which eliminates at least one unknown function from the previous equation in the sequence (see [11] for a fuller explanation). This leads to an overdetermined system of functional–differential equations that can be solved completely. Specifically, we apply the commuting differential operators

$$\mathcal{L}_1 = \partial_{10} - \epsilon_1 A'_{10}\partial_{00}, \qquad \mathcal{L}_2 = \partial_{01} + \epsilon_1\epsilon_2 A'_{01}\partial_{00},$$

(where ∂_{ij} denotes $\partial/\partial u_{ij}$) to obtain

$$\big(\epsilon_2 A'_{10}A'_{01}\partial_{00} + \epsilon_1 A'_{10}\partial_{01}\big)\partial_{00}F\big(k,\, l,\, u_{00},\, u_{01}\big) + \big(\epsilon_2 A'_{10}A'_{01}\partial_{00} - \epsilon_1\epsilon_2 A'_{01}\partial_{10}\big)\partial_{00}G\big(k,\, l,\, u_{00},\, u_{10}\big) = 0. \tag{4}$$

Dividing by $\epsilon_2 A'_{10}A'_{01}$, then differentiating with respect to u_{01} yields the partial differential equation

$$\partial_{01}\Big(\partial_{00} + \frac{\epsilon_1\epsilon_2}{A'_{01}}\partial_{01}\Big)\partial_{00}F\big(k,\, l,\, u_{00},\, u_{01}\big) = 0, \tag{5}$$

whose general solution is

$$F\big(k,\, l,\, u_{00},\, u_{01}\big) = f_1\big(k,\, l,\, \epsilon_1 u_{00} - \epsilon_2 A_{01}\big) + f_2\big(k,\, l,\, u_{00}\big) + f_3\big(k,\, l,\, u_{01}\big). \tag{6}$$

Without loss of generality, set $f_3 = 0$ (absorbing the resulting trivial conservation law into f_2 and G). Then (4) amounts to

$$\Big(\partial_{00} - \frac{\epsilon_1}{A'_{10}}\partial_{10}\Big)\partial_{00}G\big(k,\, l,\, u_{00},\, u_{10}\big) = -\partial_{00}^2 f_2\big(k,\, l,\, u_{00}\big),$$

whose general solution is

$$G\big(k,\, l,\, u_{00},\, u_{10}\big) = g_1\big(k,\, l,\, \epsilon_1 u_{00} + A_{10}\big) + g_2\big(k,\, l,\, u_{10}\big) - f_2\big(k,\, l,\, u_{00}\big). \tag{7}$$

At this stage, it is convenient to substitute (6) and (7) into the determining equation (3), using the difference equation (2) to eliminate u_{00}. This puts the determining equation in the form

$$\begin{aligned} &f_1\Big(k+1,\, l,\, \epsilon_1 u_{10} - \epsilon_2 A_{11}\Big) - f_1\Big(k,\, l,\, u_{11} - A_{10}\Big) + f_2\big(k+1,\, l,\, u_{10}\big) - f_2\big(k,\, l+1,\, u_{01}\big) \\ &\quad + g_1\Big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\Big) - g_1\Big(k,\, l,\, u_{11} + \epsilon_2 A_{01}\Big) + g_2\big(k,\, l+1,\, u_{11}\big) - g_2\big(k,\, l,\, u_{10}\big) = 0. \end{aligned} \tag{8}$$

Applying $\partial_{01}\partial_{11}$ to (8), we obtain

$$\epsilon_1 A'_{11} g''_1\Big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\Big) = \epsilon_2 A'_{01} g''_1\Big(k,\, l,\, u_{11} + \epsilon_2 A_{01}\Big), \tag{9}$$

where g''_1 is the second derivative of g with respect to its third argument. This condition holds trivially if g_1 is linear in the third argument, which leads to two 'universal' conservation laws for which F and G are each linear in A_{ij}. These are

$$\begin{aligned} F_1\big(k,\, l,\, u_{00},\, u_{01}\big) &= \big(\sqrt{\epsilon_1\epsilon_2}\big)^{k+l-1}\epsilon_2^{\,l}\big(\epsilon_2 u_{00} - \sqrt{\epsilon_1\epsilon_2}A_{01}\big),\\ G_1\big(k,\, l,\, u_{00},\, u_{10}\big) &= \big(\sqrt{\epsilon_1\epsilon_2}\big)^{k+l}\epsilon_2^{\,l}\big(\epsilon_2 u_{10} + A_{00}\big), \end{aligned} \tag{10}$$

and

$$\begin{aligned} F_2\big(k,\, l,\, u_{00},\, u_{01}\big) &= \big(-\sqrt{\epsilon_1\epsilon_2}\big)^{k+l-1}\epsilon_2^{\,l}\big(\epsilon_2 u_{00} + \sqrt{\epsilon_1\epsilon_2}A_{01}\big),\\ G_2\big(k,\, l,\, u_{00},\, u_{10}\big) &= \big(-\sqrt{\epsilon_1\epsilon_2}\big)^{k+l}\epsilon_2^{\,l}\big(\epsilon_2 u_{10} + A_{00}\big), \end{aligned} \tag{11}$$

In order to find all functions A for which there are additional conservation laws on a tile, we now restrict attention to the case $g''_1 \neq 0$.[1] Dividing (9) by A'_{01}, then applying the operator $\partial_{01} - \epsilon_2 A'_{01}\partial_{11}$ and rearranging the result, we obtain

$$\frac{g'''_1\big(k,\, l+1, \epsilon_1 u_{01} + A_{11}\big)}{g''_1\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)} = \frac{A'_{11}A''_{01} + \epsilon_2\big(A'_{01}\big)^2 A''_{11}}{A'_{01}A'_{11}\big(\epsilon_1 - \epsilon_2 A'_{01}A'_{11}\big)}.$$

It is convenient to write $A'_{ij} = B\big(A_{ij}\big) \equiv B_{ij}$, so that $A''_{ij} = B_{ij}B'_{ij}$ (which is nonzero, as A_{ij} is a nonlinear function of u_{ij}) and $A'''_{ij} = \big(B_{ij}\big)^2 B''_{ij} + B_{ij}\big(B'_{ij}\big)^2$. Then

$$\frac{g'''_1\big(k,\, l+1, \epsilon_1 u_{01} + A_{11}\big)}{g''_1\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)} = \frac{B'_{01} + \epsilon_2 B_{01}B'_{11}}{\epsilon_1 - \epsilon_2 B_{01}B_{11}}. \tag{12}$$

Applying the operator

$$\partial_{01} - \frac{\epsilon_1}{A'_{11}}\,\partial_{11} = B_{01}\frac{\partial}{\partial A_{01}} - \epsilon_1\frac{\partial}{\partial A_{11}}$$

to (12) gives (after simplification)

$$\big(1 - \epsilon_1\epsilon_2 B_{01}B_{11}\big)\big(B''_{01} - \epsilon_1\epsilon_2 B''_{11}\big) - B_{01}\big(B'_{11}\big)^2 + \epsilon_1\epsilon_2 B_{11}\big(B'_{01}\big)^2 = 0. \tag{13}$$

This is the classifying equation that yields all functions A for which there exist conservation laws other than (10) and (11). As (13) stands, the functions B_{01} and B_{11} are thoroughly entwined, but this can be resolved by one further differentiation, which yields the necessary condition

$$\big(B'''_{ij}/B'_{ij}\big)' = 0.$$

A simple calculation shows that $\big(B'_{ij}\big)^2$ is a nonzero quadratic function of B_{ij}; substituting this into (13) and solving the resulting conditions gives

$$\big(B'_{ij}\big)^2 = c_1^2\big(B_{ij}^2 + 1\big) + (1 + \epsilon_1\epsilon_2)c_2 B_{ij}, \qquad c_1, c_2 \in \mathbb{C}. \tag{14}$$

(Here and henceforth, arbitrary constants are denoted c or c_i.) This splits into four cases, as follows.

Case I: $c_1 = 0$.

In this case, we require $\epsilon_2 = \epsilon_1$ and $c_2 \neq 0$, in order that B'_{ij} is nonzero. Then

$$A'_{ij} = B_{ij} = \frac{c_2}{2}\big(A_{ij} + c_3\big)^2,$$

[1] If $g''_1 = 0$ but $f''_1 \neq 0$, similar calculations lead to precisely the same classifying equation (14), so nothing is lost by this assumption.

so

$$A_{ij} = \frac{c}{(c_4 - u_{ij})} - c_3, \qquad \text{where} \quad c = 2/c_2 \neq 0. \tag{15}$$

Then the solution of (12), after absorbing the linear terms into f_2 and g_2, is

$$g_1\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big) = c_5 \ln\big(\epsilon_1 c_4 - c_3 - (\epsilon_1 u_{01} + A_{11})\big), \qquad c_5 \neq 0. \tag{16}$$

This satisfies (9) if $\epsilon_1 = \epsilon_2 = 1$, but if $\epsilon_1 = \epsilon_2 = -1$ then (9) gives the further constraint $c_3 = c_4$. So this case leads to two possible equations, namely

$$u_{11} = u_{00} - c\Big(\frac{1}{u_{10}} - \frac{1}{u_{01}}\Big), \tag{17}$$

and

$$u_{11} = -u_{00} - c\Big(\frac{1}{u_{10}} + \frac{1}{u_{01}}\Big), \tag{18}$$

where the constant c_4 has been absorbed into u_{ij}. The point transformation $u_{00} \mapsto (-1)^k u_{00}$ maps (18) into (17), which is the lattice KdV equation (simplified slightly from the form stated in [19]). By solving (8) for the remaining unknown functions, we obtain five conservation laws for (17), which are listed later (after all integrable quad-graph equations of the form (2) have been identified). The corresponding conservation laws for (18) follow from the above transformation.

Case II: $c_1 \neq 0$, $\epsilon_2 = \epsilon_1$, $c_2^2 = c_1^4$.

In this case set $c_2 = \epsilon_3 c_1^2$, where $\epsilon_3 = \pm 1$. Then the general solution of (14) leads to the result

$$e^{c_1 A_{ij} + c_3} = \frac{\epsilon_3}{1 - z_{ij}^{\epsilon_3}}, \tag{19}$$

where the notation $z_{ij} = e^{c_1 u_{ij} + c_4}$ is used henceforth. Then (12) amounts to

$$\frac{g_1'''\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)}{g_1''\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)} = \frac{c_1\big[\epsilon_1 + z_{01}^{\epsilon_3} e^{c_1 A_{11} + c_3}\big]}{\epsilon_3 - z_{01}^{\epsilon_3} e^{c_1 A_{11} + c_3}}\,. \tag{20}$$

If $\epsilon_3 = \epsilon_1$, the general solution of (20) is

$$g_1''\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big) = \frac{a(k, l+1)\exp\big\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\big\}}{\big[1 - \epsilon_1 \exp\big\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\big\}\big]^2}\,,$$

where $a(k, l)$ is an arbitrary nonzero function. So when $\epsilon_3 = \epsilon_1 = 1$, the condition (9) gives only $a(k, l) = \alpha(k)$, whereas when $\epsilon_3 = \epsilon_1 = -1$ it also gives the constraint $e^{2(c_3 - c_4)} = 1$. Writing (2) in terms of z_{ij}, we obtain

$$\frac{z_{11}}{z_{00}} = \frac{z_{01} - 1}{z_{10} - 1}\,, \qquad \text{(when } \epsilon_3 = \epsilon_1 = 1\text{)}, \tag{21}$$

and

$$z_{00} z_{11} = \frac{z_{10} z_{01}}{(z_{10} - 1)(z_{01} - 1)}\,, \qquad \text{(when } \epsilon_3 = \epsilon_1 = -1\text{)}. \tag{22}$$

In [20], Ramani *et al.* show that (21) is equivalent (under a point transformation) to the 'discrete Lotka–Volterra equation of type I' that was discovered by Hirota and Tsujimoto [21]. Levi and Yamilov recently found higher symmetries, a Lax pair and two conservation laws for a variant of this equation [13].

When $\epsilon_3 = -\epsilon_1$, equation (20) yields

$$g_1''(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}) = a(k, l+1)\exp\big\{ - c_1(\epsilon_1 u_{01} + A_{11}) - c_3 - \epsilon_1 c_4\big\}$$

Equation (9) produces the constraint $a(k,l) = \alpha(k)$ when $\epsilon_3 = -\epsilon_1 = -1$, but when $\epsilon_3 = -\epsilon_1 = 1$ it gives $a(k,l) = 0$. So we obtain only one further equation, namely

$$\frac{z_{11}}{z_{00}} = \frac{z_{10}(z_{01}-1)}{z_{01}(z_{10}-1)}, \qquad \text{(when } \epsilon_3 = -\epsilon_1 = -1\text{)}. \tag{23}$$

Note that (21), (22) and (23) are affine linear in each z_{ij}.

Case III: $c_1 \neq 0$, $\epsilon_2 = \epsilon_1$, $c_2^2 \neq c_1^4$.

In this case

$$B_{ij} = \bar{c}_2 \sinh\big(c_1 A_{ij} + c_3\big) - \tilde{c}_2$$

where $\tilde{c}_2 = c_2/c_1^2$ and $\bar{c}_2^2 = 1 - \tilde{c}_2^2 \neq 0$. Then the general solution of $A'_{ij} = B_{ij}$ is

$$e^{c_1 A_{ij} + c_3} = \frac{1 + \tilde{c}_2 + (1-\tilde{c}_2) z_{ij}}{\bar{c}_2 (1 - z_{ij})},$$

and therefore (12) amounts to

$$\frac{g_1'''\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)}{g_1''\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)} = \frac{c_1\Big[1 + \epsilon_1\tilde{c}_2 + \epsilon_1\bar{c}_2 \exp\big\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\big\}\Big]}{\Big[1 + \epsilon_1\tilde{c}_2 - \epsilon_1\bar{c}_2 \exp\big\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\big\}\Big]}.$$

Hence

$$g_1''(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}) = \frac{a(k,l+1)\exp\big\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\big\}}{\Big[1 + \epsilon_1\tilde{c}_2 - \epsilon_1\bar{c}_2 \exp\big\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\big\}\Big]^2},$$

and so (9) produces the constraint $a(k,l) = \alpha(k)$ when $\epsilon_1 = 1$. The resulting difference equation is

$$\frac{z_{11}}{z_{00}} = \frac{(z_{10}+c)(z_{01}-1)}{(z_{01}+c)(z_{10}-1)}, \qquad c \notin \{-1, 0\}, \tag{24}$$

where $c = (1+\tilde{c}_2)/(1-\tilde{c}_2)$. This is equivalent under a point transformation to the lattice MKdV equation[2] (see [22–24]); in particular, a Lax pair for this equation is given in [24].

When $\epsilon_1 = -1$, the condition (9) yields $a(k,l) = \alpha(k)$, together with $e^{2(c_4 - c_3)} = c$. This leads to the difference equation

$$z_{00} z_{11} = \frac{(z_{10}+c)(z_{01}+c)}{(z_{10}-1)(z_{01}-1)}, \qquad c \notin \{-1, 0\}. \tag{25}$$

The point transformation

$$z_{ij} \mapsto (-c)^{k+i} z_{ij}^{(-1)^{k+i}} \tag{26}$$

maps (25) to (24). Note that when $c = 0$, (24) reduces to (23). Furthermore, (21) is the limit of (24) as $c \to \infty$ with z_{ij} fixed. So (21) and (23) are each singular limits of the lattice MKdV equation. Moreover, the point transformation

$$z_{ij} \mapsto 1/z_{ji} \tag{27}$$

maps (23) to the Lotka-Volterra type equation (21).[3]

Case IV: $c_1 \neq 0$, $\epsilon_2 = -\epsilon_1$.

This is similar to Case III; the solution of (14) is

$$B_{ij} = \sinh(c_1 A_{ij} + c_3).$$

[2] We thank Frank Nijhoff and Kenichi Maruno for alerting us to this.

[3] We are grateful to an anonymous referee for this observation.

Therefore
$$e^{c_1 A_{ij}+c_3} = \frac{1+z_{ij}}{1-z_{ij}}\,, \tag{28}$$
and so
$$\frac{g_1'''\big(k,\, l+1,\, \epsilon_1 u_{01}+A_{11}\big)}{g_1''\big(k,\, l+1,\, \epsilon_1 u_{01}+A_{11}\big)} = \frac{c_1\big[1-\epsilon_1 \exp\big\{c_1(\epsilon_1 u_{01}+A_{11})+c_3+\epsilon_1 c_4\big\}\big]}{\big[1+\epsilon_1 \exp\big\{c_1(\epsilon_1 u_{01}+A_{11})+c_3+\epsilon_1 c_4\big\}\big]}\,.$$
Hence
$$g_1''\big(k,\, l+1,\, \epsilon_1 u_{01}+A_{11}\big) = \frac{a(k,l+1)\exp\big\{c_1(\epsilon_1 u_{01}+A_{11})+c_3+\epsilon_1 c_4\big\}}{\Big[1+\epsilon_1 \exp\big\{c_1(\epsilon_1 u_{01}+A_{11})+c_3+\epsilon_1 c_4\big\}\Big]^2}\,.$$
When $\epsilon_1 = 1$, (9) gives the constraints $a(k,l) = \alpha(k)$ and $e^{2c_3} = -1$, and the resulting difference equation is
$$\frac{z_{11}}{z_{00}} = -\frac{(z_{10}+1)(z_{01}+1)}{(z_{10}-1)(z_{01}-1)}\,. \tag{29}$$
When $\epsilon_1 = -1$, we obtain similarly $a(k,l) = \alpha(k)$, $e^{2c_4} = -1$, which leads to
$$z_{00}z_{11} = -\frac{(z_{10}+1)(z_{01}-1)}{(z_{10}-1)(z_{01}+1)}\,. \tag{30}$$

Once again, the process has produced affine linear equations. It turns out that (29) can be mapped to (30) by the point transformation (26) with $c = 1$.

3. Algebraic entropy

To test the integrability of the previous lattice maps, we evaluate their algebraic entropy [25–27]. The system has an infinite dimensional space of initial conditions. We choose initial conditions on a diagonal regular staircase, which is shown in Figure 1.
$$\Delta = \big\{u_{nm}:\ n+m \in \{0,1\}\big\}. \tag{31}$$
This defines a forward evolution towards the upper right corner of the lattice, and a backward evolution towards the lower left corner.

The method is to let the system evolve, calculating u_{nm} away from the diagonal by using (recursively) the defining relation on an elementary tile of the lattice. Each u_{nm} is a rational polynomial in terms of the initial conditions; the degree of the denominator is evaluated. The space of initial conditions is infinite-dimensional but, for any quad-graph equation, we need to specify only $2k+1$ initial conditions to evaluate k iterates. This gives a sequence of degrees $\{d_n\}$, as shown in Figure 1. The growth of that sequence gives the entropy
$$\epsilon = \lim_{n\to\infty} \frac{1}{n}\ \ln(d_n). \tag{32}$$
Vanishing of the entropy is the hallmark of integrability [14–16].

Although we are able to calculate only a limited number of terms of the sequence, it is possible to infer the exact value of the entropy. The reason is the existence of a finite recurrence with integer coefficients that is satisfied by the sequence of degrees. The most efficient way to find this recurrence is to fit the sequence with a Padé approximant. The existence of the recurrence on the degrees ensures that the generating function for the sequence of degrees is a rational fraction.

Table 1 gives the sequences of degrees and the corresponding entropy for the various quad-graph equations in §2. For comparison, we also include a nonintegrable equation that is only slightly different to (25), namely
$$z_{00}z_{11} = \frac{(z_{10}+2)(z_{01}+2)}{2(z_{10}-1)(z_{01}-1)}. \tag{33}$$

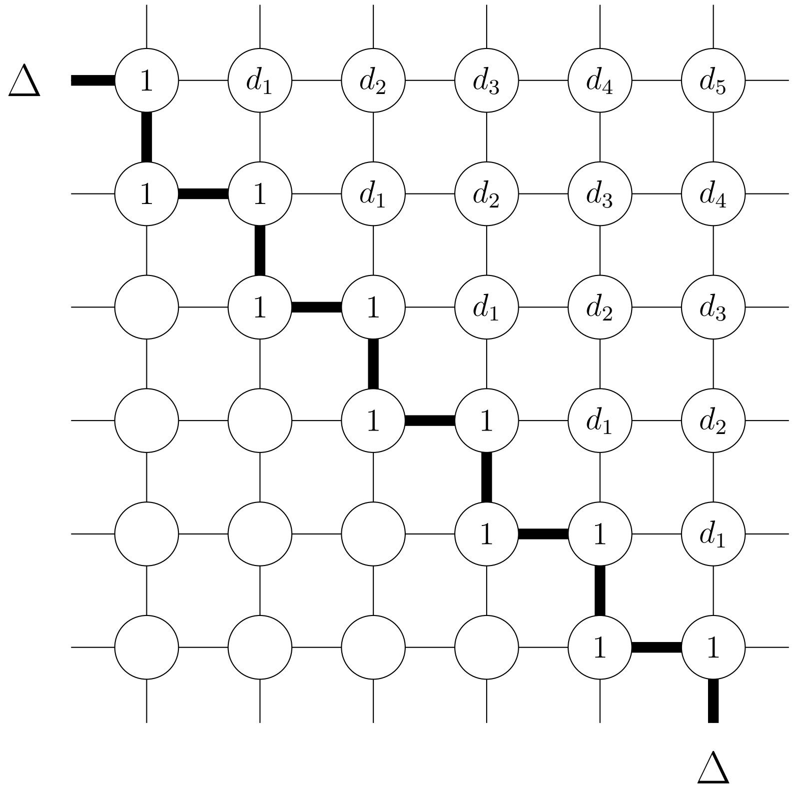

FIGURE 1. The distribution of degrees over the lattice.

Equation	Sequence	$\{d_n\}$	ϵ
(17)	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0
(21)	$1, 2, 4, 7, 11, 16, 22, 29, \ldots$	$1 + (n^2 + n)/2$	0
(22)	$1, 3, 6, 10, 14, 18, 22, 26, \ldots$	$4n - 2,\ (n \geq 2)$	0
(23)	$1, 3, 6, 11, 18, 27, 38, 51, \ldots$	$n^2 + 2, (n \geq 1)$	0
(24)	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0
(30)	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0
(33)	$1, 3, 7, 17, 41, 99, 239, 577, \ldots$	$((1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1})/2$	$\ln(1 + \sqrt{2})$

TABLE 1. The sequence of degrees for each equation.

Equation (22) has linear growth of the degree, which indicates that this equation is linearizable. This result is confirmed by the existence of an infinite family of conservation laws on a single tile (see §4). However, we have not been able to discover a linearizing transformation. The point transformation $z_{ij} \mapsto 1/z_{ij}$ simplifies (22) to

$$z_{00}z_{11} = (z_{10} - 1)(z_{01} - 1), \tag{34}$$

but this is no more tractable than the original equation.

All other cases that have more than two conservation laws exhibit quadratic growth of the degree, and therefore are claimed to be integrable, but not linearizable. This raises the question of whether any of the new integrable quad-graph equations can be mapped to any known equation. This will be discussed in §5.

4. The conservation laws

Although the lattice KdV equation (17), the lattice MKdV equation (24) and the discrete Lotka-Volterra equation (21) are not new, their conservation laws (on a single tile) have not previously been listed. Throughout this section, the universal conservation laws (10) and (11) span the first two conservation laws in each list (up to the addition of trivial conservation laws). So the 'extra' conservation laws are (F_i, G_i), where $i \geq 3$; here F_i denotes $F_i(k,\, l,\, z_{00},\, z_{01})$ and G_i denotes $G_i(k,\, l,\, z_{00},\, z_{10})$. Lattice KdV has the following conservation laws.

$$F_1 = u_{00} + c/u_{01}, \qquad G_1 = u_{10} - c/u_{10};$$

$$F_2 = (-1)^{k+l}\big(u_{00} + c/u_{01}\big), \qquad G_2 = (-1)^{k+l+1}\big(u_{10} + c/u_{10}\big);$$

$$F_3 = \ln\big(u_{00} + c/u_{01}\big), \qquad G_3 = \ln(u_{10});$$

$$F_4 = \ln(u_{00}), \qquad G_4 = \ln\big(u_{10} - c/u_{00}\big);$$

$$F_5 = -k\ln(u_{00}) + l\ln\big(u_{00} + c/u_{01}\big), \qquad G_5 = -k\ln\big(u_{10} - c/u_{00}\big) + (l-1)\ln(u_{10}).$$

For lattice MKdV, the single-tile conservation laws are:

$$F_1 = \ln\left(\frac{z_{00}(z_{01}-1)}{z_{01}+c}\right), \qquad G_1 = \ln\left(\frac{z_{10}(z_{10}+c)}{z_{10}-1}\right);$$

$$F_2 = (-1)^{k+l}\ln\left(\frac{z_{01}+c}{z_{00}(z_{01}-1)}\right), \qquad G_2 = (-1)^{k+l}\ln\left(\frac{z_{10}(z_{10}-1)}{z_{10}+c}\right);$$

$$F_3 = \ln\left(\frac{z_{00}z_{01} - z_{00} - z_{01} - c}{z_{01}+c}\right), \qquad G_3 = \ln(z_{10}+c);$$

$$F_4 = \ln\left(\frac{z_{00}+c}{z_{00}}\right), \qquad G_4 = \ln\left(\frac{z_{00}z_{10} + cz_{00} + cz_{10} - c}{z_{10}(z_{00}+c)}\right);$$

$$F_5 = k\ln\left(\frac{z_{00}(z_{01}+c)}{(z_{01}-1)(z_{00}+c)^2}\right) + l\ln\left(\frac{(z_{00}z_{01} - z_{00} - z_{01} - c)^2}{z_{00}(z_{01}+c)(z_{01}-1)}\right),$$

$$G_5 = k\ln\left(\frac{z_{10}(z_{10}-1)(z_{00}+c)^2}{(z_{10}+c)(z_{00}z_{10} + cz_{00} + cz_{10} - c)^2}\right) + l\ln\left(\frac{(z_{10}+c)(z_{10}-1)}{z_{10}}\right) + \ln\left(\frac{z_{10}}{(z_{10}+c)^2}\right).$$

The single-tile conservation laws for the discrete Lotka-Volterra equation (21) are

$$F_1 = \ln\big(z_{00}(z_{01}-1)\big), \qquad G_1 = \ln\left(\frac{z_{10}}{z_{10}-1}\right);$$

$$F_2 = (-1)^{k+l}\ln\big(z_{00}(z_{01}-1)\big), \qquad G_2 = (-1)^{k+l+1}\ln\big(z_{10}(z_{10}-1)\big);$$

$$F_3 = z_{00}(1-z_{01}), \qquad G_3 = z_{10};$$

$$F_4 = \ln(z_{00}), \qquad G_4 = \ln\left(\frac{z_{10}}{z_{00}+z_{10}-1}\right);$$

$$F_5 = k\ln\left(\frac{z_{00}}{z_{01}-1}\right) + l\ln\big(z_{00}(z_{01}-1)\big), \qquad G_5 = k\ln\left(\frac{z_{10}(z_{10}-1)}{(z_{00}+z_{10}-1)^2}\right) + l\ln\left(\frac{z_{10}-1}{z_{10}}\right) + \ln(z_{10}).$$

Levi and Yamilov [13] recently derived an alternative form of (21) and listed two of its conservation laws, which are equivalent to (F_1, G_1) and (F_4, G_4).

We now list the conservation laws corresponding to the remaining affine linear quad-graph equations that we have derived. Each of our equations that is not equivalent to Lattice KdV, Lattice MKdV or the Lotka-Volterra type equation is equivalent to either the linearizable equation (22) or the new equation (30).

Equation (22)

This equation has an infinite set of conservation laws, which depend upon two arbitrary functions α, β:

$$F_\alpha = \alpha(l+1)\ln\left(\frac{z_{00}z_{01}-z_{00}-z_{01}}{z_{00}(z_{01}-1)}\right) + \alpha(l)\ln\left(\frac{z_{00}z_{01}-z_{00}-z_{01}}{z_{01}}\right), \qquad G_\alpha = \alpha(l)\ln(1-z_{10});$$

$$F_\beta = \beta(k)\ln(1-z_{01}), \qquad G_\beta = \beta(k+1)\ln\left(\frac{z_{00}z_{10}-z_{00}-z_{10}}{z_{00}(z_{10}-1)}\right) + \beta(k)\ln\left(\frac{z_{00}z_{10}-z_{00}-z_{10}}{z_{10}}\right).$$

This is a further indicator that, unlike the other quad-graph equations in our class, (22) is linearizable.

Equation (30)

$$F_1 = (-1)^{(k+l)(k+l-1)/2}\ln\left(\tfrac{z_{00}(z_{01}+1)}{z_{01}-1}\right), \qquad G_1 = \cos\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\tfrac{z_{10}(1-z_{10})}{z_{10}+1}\right) + \sin\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\tfrac{z_{10}-1}{z_{10}(z_{10}+1)}\right);$$

$$F_2 = (-1)^{(k+l)(k+l+1)/2}\ln\left(\tfrac{z_{00}(z_{01}+1)}{z_{01}-1}\right), \qquad G_2 = -\sin\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\tfrac{z_{10}(1-z_{10})}{z_{10}+1}\right) + \cos\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\tfrac{z_{10}-1}{z_{10}(z_{10}+1)}\right);$$

$$F_3 = \ln\left(\frac{(z_{00}+1)^2(z_{01}-1)}{z_{00}(z_{01}+1)}\right), \qquad G_3 = \ln\left(\frac{(-1)^l(z_{00}z_{10}-z_{00}+z_{10}+1)^2(z_{10}+1)}{z_{10}(z_{00}+1)^2(z_{10}-1)}\right);$$

$$F_4 = \ln\left(\frac{(z_{00}z_{01}+z_{00}-z_{01}+1)^2}{z_{00}(z_{01}+1)(z_{01}-1)}\right), \qquad G_4 = \ln\left(\frac{(-1)^l(z_{10}+1)(z_{10}-1)}{z_{10}}\right);$$

$$F_5 = k\ln\left(\frac{z_{00}(z_{01}+1)}{(z_{00}+1)^2(z_{01}-1)}\right) + l\ln\left(\frac{(-1)^k(z_{00}z_{01}+z_{00}-z_{01}+1)^2}{z_{00}(z_{01}+1)(z_{01}-1)}\right),$$

$$G_5 = k\ln\left(\frac{(-1)^l z_{10}(z_{00}+1)^2(z_{10}-1)}{(z_{10}+1)(z_{00}z_{10}-z_{00}+z_{10}+1)^2}\right) + l\ln\left(\frac{(z_{10}+1)(z_{10}-1)}{z_{10}}\right) + \ln\left(\frac{z_{10}}{(z_{10}+1)^2}\right).$$

5. Comments

Remarkably, although the original Ansatz (2) contained an arbitrary function A, each of the equations that we have found by sifting can be written in affine form, using a simple change of dependent variable. This has made the entropy calculation possible, because it gives rational evolution.

It is natural to ask at this point how the equations that we have derived compare to the known affine linear quad-graph equations. We have already seen that in most cases, our Ansatz yields an equation that is equivalent under a point transformation to a known equation. Therefore it is important to characterize this equivalence, which can be done using the appoach introduced in [8]. Any affine linear quad-graph equation can be written in polynomial form:

$$Q(v_1, v_2, v_3, v_4) = 0,$$

where $v_i,\ i = 1 \ldots 4$, are the values (of u_{ij} or z_{ij} as appropriate) at the four corners. For any choice of a pair of indices $1 \leq i < j \leq 4$, define h_{ij} by

$$h_{ij}(v_k, v_l) = \partial_{v_i} Q \cdot \partial_{v_j} Q - Q \cdot \partial_{v_i} \partial_{v_j} Q, \qquad i \neq j \neq k \neq l \tag{35}$$

It is then possible to associate to each of the four corners a polynomial

$$r_k(v_k) = (\partial_{v_l} h_{ij})^2 - 2\, h_{ij}\, (\partial^2_{v_l} h_{ij}). \tag{36}$$

These polynomials play a central role in the classification of [8], because (after a Möbius tranformation, if necessary), they can take one of six canonical forms, according to their root distribution.

For example, the lattice MKdV equation (24) yields (adjusting the notation for clarity)

$$\begin{aligned}
h_{z_{00}z_{01}} &= (1+c)\,(z_{10}-1)\,(z_{10}+c)\,z_{11};\\
h_{z_{00}z_{10}} &= -(1+c)\,(z_{01}-1)\,(z_{01}+c)\,z_{11};\\
h_{z_{00}z_{11}} &= -(z_{01}-1)\,(z_{01}+c)\,(z_{10}-1)\,(z_{10}+c);\\
h_{z_{01}z_{10}} &= -(1+c)^2\,z_{00}\,z_{11};\\
h_{z_{01}z_{11}} &= -(1+c)\,(z_{10}-1)\,(z_{10}+c)\,z_{00};\\
h_{z_{10}z_{11}} &= (1+c)\,(z_{01}-1)\,(z_{01}+c)\,z_{00}.
\end{aligned}$$

All of the functions h_{ij} are products of linear factors; this is the case for every equation in our classification. In other words, all of these equations are 'degenerate' in the sense used in [8]. Moreover

$$\begin{aligned}
r_{00} &= (1+c)^4\,z_{00}^2;\\
r_{11} &= (1+c)^4\,z_{11}^2;\\
r_{10} &= (1+c)^2\,(1-z_{10})^2\,(z_{10}+c)^2;\\
r_{01} &= (1+c)^2\,(1-z_{01})^2\,(z_{01}+c)^2.
\end{aligned}$$

These are in the canonical forms, but are not in any of the cases that were classified in Theorem 2 of [8]. Hence none of the equations that we have studied are equivalent to any equation in the ABS classification.

In summary, it is feasible to look for new integrable difference equations by searching for equations that admit 'extra' conservation laws. The class that we have studied has been particularly fruitful, although only one of the equations (up to equivalence) seems to be unknown. A useful by-product is that one obtains a list of conservation laws, most of which are new (even for the known equations). The calculation of algebraic entropy is a clear indicator of integrability and linearizability.

It is worth noting that our new equation (30) is the only one with maximal asymmetry within the form of Ansatz (2), because $\epsilon_1 = -\epsilon_2$ in this case alone.

Two particularly important questions remain: does the new equation have a Lax pair description, and is it 3D-consistent? If we wanted to check directly the consistency around the cube, we should first choose an Ansatz for the form of the relations we want to use on the six faces of a cube. This leads one to ask which deformations of our models will be integrable. These might be Möbius transformations or other deformations which do not lie within the assumed Ansatz (2). The analysis of the singularity pattern may be a way to tackle this problem. One should be prepared to accept deformed equations that are not affine; however, this is beyond the scope of the current paper.

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