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Discrete Symmetries of Differential Equations

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ABSTRACT. The discrete point symmetries of a given differential equation cannot usually be found directly from the symmetry condition. However, an easy indirect method has been devised recently. This enables the user to obtain all discrete point symmetries by first classifying their adjoint actions on the Lie algebra of Lie point symmetry generators. The current paper outlines the new method, which is used to solve a problem posed by Ibragimov, namely to find all point symmetries of a particular multi-sheeted differential equation.

1. Introduction

For many systems that are modelled by differential equations, it is useful to know all symmetries of a particular class. For simplicity, we shall focus on the problem of obtaining all point symmetries of a given ordinary differential equation (ODE),

$$y^{(n)} = \omega\left(x, y, y', \dots, y^{(n-1)}\right), \qquad n \geq 2. \tag{1.1}$$

Each symmetry is a diffeomorphism,

$$\Gamma : (x, y) \mapsto \big(\hat{x}(x, y),\, \hat{y}(x, y)\big),$$

that satisfies the *symmetry condition*:

$$\hat{y}^{(n)} = \omega\left(\hat{x}, \hat{y}, \dots, \hat{y}^{(n-1)}\right) \qquad \text{when (1.1) holds.} \tag{1.2}$$

Usually, this condition yields a highly-coupled system of nonlinear partial differential equations (PDEs) for the functions $\hat{x}(x, y)$ and $\hat{y}(x, y)$. It is seldom possible to solve the symmetry condition directly (see [**16**] for one exception). However, all one-parameter Lie groups of point symmetries can be obtained fairly easily by linearizing the symmetry condition about the identity transformation to derive the Lie algebra of point symmetry generators. Each generator can be exponentiated to yield a one-parameter (local) Lie group of symmetries [**2, 11, 14, 17**]. The set of all such 'continuous' symmetries is a normal (local) subgroup of the group of all point symmetries. When the continuous symmetries are factored out, the remaining point symmetries form a discrete group. If (1.2) is intractable, these discrete symmetries cannot be calculated directly. Nevertheless, discrete symmetries have

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many applications (as discussed in §4), and it is desirable to be able to find them systematically. Until recently, the ansatz-based approach was regarded as the easiest way to obtain discrete symmetries (see [**5**]). However this approach does not guarantee that all discrete point symmetries will be found. Reid *et al.* [**16**] used computer algebra to simplify the symmetry condition for one second-order ODE, enabling a complete list of symmetries to be generated. However, to the best of my knowledge, this is the only instance in which the symmetry condition for a nonlinear ODE has been solved directly.

This paper describes an easy indirect method for obtaining the complete set of discrete symmetries of a given ODE that has Lie symmetries. The method is introduced in §2, and is used (in §3) to solve a problem proposed by Ibragimov [**12**]. We indicate how to generalize the indirect method to contact symmetries. Generalizations to other classes of symmetries of ODEs and PDEs are described in [**7, 8, 9, 10**].

2. How to determine the discrete symmetries

In this section, we describe the indirect method for obtaining all inequivalent discrete symmetries. We restrict attention to real transformations and to real-valued Lie group parameters. For brevity, the main results are stated without proof; readers should consult [**11**] for details. Suppose that the Lie algebra, $\mathcal{L}$, of Lie point symmetry generators for a given ODE (1.1) has a basis

$$X_i = \xi_i(x,y)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1,\dots,R. \tag{2.1}$$

In this basis, the commutator relations are

$$[X_i, X_j] = c_{ij}^k X_k. \tag{2.2}$$

The abstract structure of $\mathcal{L}$ is reflected in the set of structure constants, c_{ij}^k. Suppose that

$$\Gamma : (x,y) \mapsto \big(\hat{x}(x,y),\, \hat{y}(x,y)\big)$$

is a symmetry of (1.1). The adjoint action of this symmetry induces an automorphism of $\mathcal{L}$, as a consequence of the following lemma.

LEMMA 2.1. *For each one-parameter Lie group of symmetries,*

$$\Gamma_\delta = e^{\delta X}, \qquad X \in \mathcal{L},$$

there is an associated Lie group of symmetries,

$$\hat{\Gamma}_\delta = \Gamma e^{\delta X} \Gamma^{-1},$$

whose infinitesimal generator is

$$\hat{X} = \Gamma X \Gamma^{-1}.$$

In particular, for each generator X_i in the basis (2.1),

$$\hat{X}_i = \xi_i(\hat{x},\hat{y})\partial_{\hat{x}} + \eta_i(\hat{x},\hat{y})\partial_{\hat{y}}. \tag{2.3}$$

The generators $\hat{X}_i$, $i = 1,\dots,R$, constitute a basis for $\mathcal{L}$, with the same structure constants as the basis (2.1).

Therefore there exists a nonsingular constant matrix $B = (b_i^l)$ that describes the original basis in terms of the transformed basis:

$$X_i = b_i^l \hat{X}_l, \qquad i = 1, \ldots, R. \tag{2.4}$$

(The usual summation convention is adopted.) By substituting (2.4) into the commutator relations

$$[\hat{X}_i, \hat{X}_j] = c_{ij}^k \hat{X}_k \qquad \text{when (2.2) holds,} \tag{2.5}$$

we obtain the following result.

LEMMA 2.2. *The elements of B satisfy the nonlinear contraints*

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n\,, \qquad 1 \le i < j \le R, \quad 1 \le n \le R. \tag{2.6}$$

The automorphism of $\mathcal{L}$ induced by Γ is

$$\Gamma : X_i \mapsto b_i^l X_l, \qquad i = 1, \ldots, R. \tag{2.7}$$

In particular, if $\Gamma = e^{\epsilon X_j}$ for a given value of ϵ then $B = A(j, \epsilon)$, where

$$A(j, \epsilon) = \exp\left\{\epsilon C(j)\right\}, \qquad \left(C(j)\right)_i^k = c_{ij}^k\,. \tag{2.8}$$

Multiplication in the symmetry group corresponds to multiplication of the matrices B that represent the associated automorphisms. Two symmetries Γ and $\tilde{\Gamma}$ are *equivalent* if there exists $X \in \mathcal{L}$ such that

$$\tilde{\Gamma} = e^{\epsilon X}\Gamma$$

for some real ϵ. (Note: we require that $e^{\epsilon X}$ is in the connected component of the identity.) Moreover, if Γ and $\tilde{\Gamma} = e^{\epsilon X}\Gamma$ induce automorphisms with matrices B and $\tilde{B}$ respectively then

$$\tilde{B} = A(1, \epsilon_1)A(2, \epsilon_2) \cdots A(R, \epsilon_R)B,$$

for some parameters ϵ_j.

To find the discrete symmetries, we must factor out the continuous symmetries, leaving a set of inequivalent symmetries. This is most easily done in several stages. First, it is possible to create a set of inequivalent automorphisms by factoring out the continuous symmetries whose generators are not in the centre of the Lie algebra. If X_j is not in the centre (i.e. if $C(j)$ is non-zero), replace B by either $BA(j, \epsilon)$ or $A(j, \epsilon)B$ and choose ϵ to be a value that simplifies at least one entry in the new matrix. Each $A(j, \epsilon)$ is used once, so that all continuous symmetries that are not in the centre are factored out. Generally speaking, it is best to use all $A(j, \epsilon)$ that have nonzero off-diagonal elements first; these matrices can create zeros in B. Then use the diagonal $A(j, \epsilon)$ matrices to rescale elements of B. By this process, it is possible to solve the nonlinear constraints (2.6) and obtain a complete set of inequivalent automorphisms. No continuous symmetry whose generator is in the centre can be factored out at this stage, because its adjoint action on $\mathcal{L}$ is the identity mapping.

For each inequivalent automorphism, the identity (2.4) can be applied to the unknown functions $\hat{x}(x, y)$ and $\hat{y}(x, y)$, yielding a system of $2R$ *determining equations*:

$$X_i \hat{x} = b_i^l \xi_l(\hat{x}, \hat{y}), \qquad X_i \hat{y} = b_i^l \eta_l(\hat{x}, \hat{y}). \tag{2.9}$$

This system of first-order quasilinear PDEs may be solved by the method of characteristics or (if $R \ge 3$) by algebraic means. The solutions are the point transformations that correspond to each inequivalent automorphism. Now the symmetries whose generators are in the centre of the Lie algebra can be factored out, leaving a set of point transformations that are not equivalent to one another under any continuous symmetry. (Usually this set consists of a few fairly simple transformations.) Finally, the symmetry condition (1.2) is used to determine which of these transformations are symmetries. (It is easy to check whether or not a given point transformation satisfies the symmetry condition.)

EXAMPLE 2.3. Consider the ODE

$$y''' = y''(1 - y''), \tag{2.10}$$

whose Lie point symmetries are generated by

$$X_1 = \partial_x\,, \qquad X_2 = x\partial_y\,, \qquad X_3 = \partial_y\,. \tag{2.11}$$

(Although this ODE is easy to solve exactly, it provides a simple context in which all of the main features of the indirect method can be seen. Some less easy ODEs are studied later.) For the Lie algebra spanned by (2.11), the only nonzero structure constants are

$$c_{12}^3 = 1, \qquad c_{21}^3 = -1, \tag{2.12}$$

so X_3 is in the centre of the Lie algebra. The matrices corresponding to the automorphisms generated by X_1 and X_2 are

$$A(1,\epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\epsilon \\ 0 & 0 & 1 \end{bmatrix}, \qquad A(2,\epsilon) = \begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The constraints (2.6) with $n = 1$ amount to

$$c_{ij}^k b_k^1 = 0, \qquad 1 \le i < j \le 3.$$

These yield only one constraint, which is obtained by setting $(i,j) = (1,2)$, namely

$$b_3^1 = 0.$$

Similarly, the $n = 2$ constraints amount to

$$b_3^2 = 0,$$

so $b_3^3 \neq 0$ (because B is nonsingular). To simplify B further, premultiply it by $A(1, b_2^3/b_3^3)$ to replace b_2^3 by zero. Then premultiply B by $A(2, -b_1^3/b_3^3)$ to replace b_1^3 by zero, so that now

$$B = \begin{bmatrix} b_1^1 & b_1^2 & 0 \\ b_2^1 & b_2^2 & 0 \\ 0 & 0 & b_3^3 \end{bmatrix}. \tag{2.13}$$

We have not yet used the nonlinear constraints with $n = 3$; the above simplifications have reduced these constraints to the single equation

$$b_1^1 b_2^2 - b_1^2 b_2^1 = b_3^3. \tag{2.14}$$

The matrices (2.13) satisfying (2.14) represent the inequivalent automorphisms of the abstract 3-dimensional Lie algebra whose only nonzero structure constants are (2.12). To find out which of these automorphisms can be realized as real point transformations of the plane, we must solve the determining equations,

$$\begin{bmatrix} \hat{x}_x & \hat{y}_x \\ x\hat{x}_y & x\hat{y}_y \\ \hat{x}_y & \hat{y}_y \end{bmatrix} = B \begin{bmatrix} 1 & 0 \\ 0 & \hat{x} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2\hat{x} \\ b_2^1 & b_2^2\hat{x} \\ 0 & b_3^3 \end{bmatrix}.$$

Taking (2.14) into account, the general solution of the determining equations is

$$\hat{x} = b_1^1 x, \qquad \hat{y} = \tfrac{1}{2} b_1^1 b_1^2 x^2 + b_1^1 b_2^2 y + c, \qquad b_1^1 b_2^2 \neq 0, \tag{2.15}$$

where c is a constant of integration. Note that the determining equations require that $b_2^1 = 0$, so not every automorphism can be realized as a point transformation of the plane. Now we factor out the one-parameter Lie group generated by X_3, setting $c = 0$ for simplicity. Finally, we must check the symmetry condition (1.2) for the ODE (2.10) to see which of the inequivalent point transformations (2.15) are symmetries. From (2.15),

$$\hat{y}'' = \frac{b_1^2}{b_1^1} + \frac{b_2^2}{b_1^1} y'',$$

$$\hat{y}''' = \frac{b_2^2}{(b_1^1)^2} y'''.$$

Substituting these results into the symmetry condition (1.2) yields

$$\frac{b_2^2}{(b_1^1)^2}\big(y''(1-y'')\big) = \left(\frac{b_1^2}{b_1^1} + \frac{b_2^2}{b_1^1} y''\right)\left\{1 - \frac{b_1^2}{b_1^1} - \frac{b_2^2}{b_1^1} y''\right\}.$$

By equating powers of y'', we find that either

$$(\hat{x}, \hat{y}) = (x, y),$$

or

$$(\hat{x}, \hat{y}) = \big(-x,\ \tfrac{1}{2}x^2 - y\big).$$

So the group of inequivalent discrete symmetries of the ODE (2.10) is generated by

$$\Gamma_1 : (x, y) \mapsto \big(-x,\ \tfrac{1}{2}x^2 - y\big).$$

This group is isomorphic to $\mathbb{Z}_2$, because $(\Gamma_1)^2$ is the identity transformation.

The same approach can also be used to obtain contact symmetries. In this case, $\hat{x}$, $\hat{y}$, and

$$\hat{y'} \equiv \frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} \tag{2.16}$$

are allowed to be functions of (x, y, y'). It turns out that the Lie algebra of Lie contact symmetry generators is spanned by (2.11), so all Lie contact symmetries of the ODE (2.10) are point symmetries. The determining equations for the discrete contact symmetries are now based on mappings of (x, y, y')-space, so the first prolongation of each generator X_i is used in place of X_i. Thus the determining equations are

$$\begin{bmatrix} \hat{x}_x & \hat{y}_x & \hat{y'}_x \\ x\hat{x}_y + \hat{x}_{y'} & x\hat{y}_y + \hat{y}_{y'} & x\hat{y'}_y + \hat{y'}_{y'} \\ \hat{x}_y & \hat{y}_y & \hat{y'}_y \end{bmatrix} = B \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{x} & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where B is given by (2.13), (2.14). The general solution of this system and the constraint (2.16) is

$$\begin{aligned} \hat{x} &= b_1^1 x + b_2^1 y', \\ \hat{y} &= (b_1^1 b_2^2 - b_1^2 b_2^1) y + \tfrac{1}{2} b_1^1 b_1^2 x^2 + b_1^2 b_2^1 x y' + \tfrac{1}{2} b_2^1 b_2^2 (y')^2 + c, \\ \hat{y'} &= b_1^2 x + b_2^2 y'. \end{aligned}$$

Upon substituting this solution into the symmetry condition, and factoring out the symmetries generated by X_3, the following result is obtained. The group of inequivalent discrete contact symmetries of (2.10) is isomorphic to the dihedral group

$$D(3) = \langle \Gamma_1, \Gamma_2 \,:\, (\Gamma_1)^2 = \mathrm{id}, \quad (\Gamma_2)^3 = \mathrm{id}, \quad \Gamma_2\Gamma_1 = \Gamma_1(\Gamma_2)^2 \rangle,$$

where id is the identity map. This symmetry group is generated by

$$\begin{aligned} \Gamma_1 : (x, y, y') &\mapsto \big(-x,\, \tfrac{1}{2}x^2 - y,\, y' - x\big), \\ \Gamma_2 : (x, y, y') &\mapsto \big(-y',\, y - xy' + \tfrac{1}{2}(y')^2,\, x - y'\big). \end{aligned}$$

The indirect method breaks the problem of obtaining the discrete symmetries into several steps, each of which is straightforward in principle. In practice, the most complicated part of the calculation is the determination of all inequivalent automorphisms of a given Lie algebra. These automorphisms depend only on the abstract Lie algebra structure, and it is wise to use a basis in which the structure constants are as simple as possible. Even so, if the dimension of $\mathcal{L}$ is large, computer algebra may be needed.

In order to alleviate this problem, Laine-Pearson & Hydon have classified the inequivalent automorphisms of Lie algebras that occur for Lie point symmetries of scalar ODEs [**13**]. Their classification, which includes the inequivalent realizations as point transformations, is based upon Olver's list of vector fields that act on the real plane [**15**]. The automorphisms and their realizations are presented as a look-up table, to enable users to obtain the inequivalent discrete symmetries of a given scalar ODE, with minimal effort.

EXAMPLE 2.4. The fourth-order ODE

$$y^{(iv)} = \frac{2y'''}{y}\big(1 - y'\big) \tag{2.17}$$

has the Lie algebra of point symmetry generators spanned by

$$X_1 = \partial_x\,, \qquad X_2 = x\partial_x + y\partial_y\,, \qquad X_3 = x^2\partial_x + 2xy\partial_y\,. \tag{2.18}$$

This Lie algebra is a realization of $\mathfrak{sl}(2)$, for which the nonlinear constraints (2.6) are fairly complicated. However, by using the look-up table in [**13**], we find that the group of inequivalent real-valued automorphisms of $\mathfrak{sl}(2)$ is generated by

$$\begin{aligned} \Gamma_1 : (X_1, X_2, X_3) &\mapsto (-X_1,\, X_2,\, -X_3), \\ \Gamma_2 : (X_1, X_2, X_3) &\mapsto (X_3,\, -X_2,\, X_1). \end{aligned}$$

This group is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. The general solution of the determining equations for the realization of these automorphisms is

$$\begin{aligned} \Gamma_1 : (x, y) &\mapsto (-x,\, c_1 y), \\ \Gamma_2 : (x, y) &\mapsto \left(-\frac{1}{x},\, \frac{c_2 y}{x^2}\right), \end{aligned}$$

where each c_i is an arbitary real constant. The only such transformations that satisfy the symmetry condition belong to the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group generated by

$$\begin{aligned} \tilde{\Gamma}_1 : (x, y) &\mapsto (-x,\, -y), \\ \tilde{\Gamma}_2 : (x, y) &\mapsto \left(-\frac{1}{x},\, \frac{y}{x^2}\right). \end{aligned} \tag{2.19}$$

In this example, the dimension of the Lie algebra is less than the order of the ODE. Furthermore, the Lie algebra is not solvable, so the method of successive reduction of order cannot be used. In such circumstances, it is unusual to be able to obtain the general solution of the ODE, but it may be possible to find group-invariant solutions. (For this particular ODE, the general solution can be obtained in parametric form, using a technique developed by Clarkson & Olver [**4**] — see [**11**] for details.) Every solution that is invariant under a one-parameter group of Lie point symmetries is of the form

$$y = c_1 + c_2 x + c_3 x^2, \tag{2.20}$$

where each c_i is a constant. The standard method for finding solutions of an ODE that are invariant under a Lie group is well-known and easy to apply. By contrast, it is not usually easy to find solutions that are merely invariant under a discrete symmetry group. Every solution that is invariant under the discrete group generated by (2.19) is of the form

$$y = x f(\ln|x|),$$

where f is an *odd* function. The ODE (2.17) is satisfied if

$$f'''' - f'' = -\frac{2}{f}\left(f''' - f'\right)\left(f' - 1\right).$$

(Here $'$ denotes the derivative with respect to $\ln|x|$.) This ODE is not easily solved, but it has some obvious singular solutions. The odd solutions of

$$f''' - f' = 0$$

correspond to solutions of (2.17) that are of the form (2.20). However, the odd solution of

$$f' - 1 = 0$$

produces a new group-invariant solution of (2.17), namely

$$y = x \ln|x|. \tag{2.21}$$

This example shows that writing an ODE in terms of invariants of a discrete group can yield invariant solutions that are unobtainable by the standard method.

3. A multi-sheeted differential equation

Ibragimov [**12**, page 235] posed the following problem, which illustrates one of the difficulties of studying ODEs that are not solved for the highest derivative. Find the maximal point symmetry group for the ODE

$$(y'')^2 = \left(\frac{y'}{x} - e^y\right)^2. \tag{3.1}$$

This ODE is two-sheeted; it can be split into the pair of ODEs

$$y'' = \frac{y'}{x} - e^y; \tag{3.2}$$

$$y'' = -\frac{y'}{x} + e^y. \tag{3.3}$$

The sets of solutions of (3.2) and (3.3) are disjoint, because no curve satisfies both ODEs. The Lie point symmetries of (3.2) are generated by

$$X_1 = x\partial_x - 2\partial_y, \tag{3.4}$$

whereas those of (3.3) are generated by

$$X_1 = x\partial_x - 2\partial_y, \qquad X_2 = x\ln|x|\partial_x - 2(1+\ln|x|)\partial_y. \tag{3.5}$$

The Lie algebras spanned by (3.4) and (3.5) respectively have different dimensions, so there is no point symmetry that maps the solutions of (3.2) to those of (3.3) or *vice versa*. Therefore every point symmetry of (3.1) is a point symmetry of both (3.2) and (3.3) separately. This means that we are free to seek the discrete symmetries of either of these equations, and then find out which are also symmetries of the remaining ODE. As (3.3) has the larger Lie algebra, it is convenient to study this ODE first.

The nonzero structure constants for the basis (3.5) are

$$c^1_{12} = 1, \qquad c^1_{21} = -1. \tag{3.6}$$

Therefore the nonlinear constraints (2.6) are

$$\begin{aligned} b^1_1 b^2_2 - b^2_1 b^1_2 &= b^1_1, \\ 0 &= b^2_1, \end{aligned}$$

and so (bearing in mind that B is nonsingular)

$$B = \begin{bmatrix} b^1_1 & 0 \\ b^1_2 & 1 \end{bmatrix}, \qquad\qquad b^1_1 \neq 0.$$

We now factor out the continuous symmetries, using the matrices

$$A(1,\epsilon) = \begin{bmatrix} 1 & 0 \\ -\epsilon & 1 \end{bmatrix}, \qquad A(2,\epsilon) = \begin{bmatrix} e^\epsilon & 0 \\ 0 & 1 \end{bmatrix}.$$

Postmultiplying B by $A(1, b^1_2)$, we can replace b^1_2 by zero. Then, postmultiplying B by $A(2, -\ln|b^1_1|)$, we obtain

$$B = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{where} \quad \alpha \in \{-1, 1\}.$$

Thus the inequivalent automorphisms form a group that is isomorphic to $\mathbb{Z}_2$, which is generated by

$$\Gamma_1 : (X_1, X_2) \mapsto (-X_1, X_2).$$

To solve the determining equations, it is helpful to have at least one generator in the canonical form. This is achieved by introducing new variables

$$t = \ln|x|, \qquad z = y + 2\ln|x|,$$

which reduces the ODE (3.3) to

$$\ddot{z} = e^z. \tag{3.7}$$

In the new variables, the symmetry generators (3.5) are

$$X_1 = \partial_t, \qquad X_2 = t\partial_t - 2\partial_z\,.$$

The determining equations (2.9) amount to

$$\hat{t}_t = \alpha, \qquad \hat{z}_t = 0, \qquad t\hat{t}_t - 2\hat{t}_z = \hat{t}, \qquad t\hat{z}_t - 2\hat{z}_z = -2.$$

By the method of the characteristics, the general solution of these equations is

$$\hat{t} = \alpha t + c_1 e^{-z/2}, \qquad \hat{z} = z + c_2,$$

where c_1 and c_2 are constants. The symmetry condition for (3.7) is satisfied if and only if $c_1 = c_2 = 0$. Reverting to the original variables, the inequivalent point symmetries of (3.3) are given by

$$(\hat{x}, \hat{y}) = (\beta x^\alpha,\, y + 2(1-\alpha)\ln|x|), \qquad \alpha = \pm 1, \quad \beta = \pm 1.$$

These symmetries form a discrete group that is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, which is generated by

$$\Gamma_0 : (x, y) \mapsto (-x, y), \qquad \Gamma_1 : (x, y) \mapsto (1/x,\, y + 4\ln|x|).$$

Having found all point symmetries of (3.3), we can solve Ibragimov's problem by finding out which of these are also point symmetries of (3.2). From the symmetry condition for this ODE we find that, up to the continuous symmetries generated by X_1, the only remaining symmetries are the reflections generated by Γ_0. Thus we have derived the following result.

THEOREM 3.1. *The maximal real point symmetry group of the ODE (3.1) consists of all point transformations of the form*

$$\Gamma : (x, y) \mapsto (\alpha e^\epsilon x,\, y - 2\epsilon), \qquad \alpha = \pm 1, \quad \epsilon \in \mathbb{R}.$$

Note that we could also seek all inequivalent discrete point symmetries of (3.2) by using the automorphisms of the one-dimensional Lie algebra spanned by (3.4). After a messy but straightforward calculation, it is possible to show that (3.2) has no point symmetries other than those in the maximal group for (3.1).

4. Discussion

The method described in this paper is both systematic and exhaustive − a complete list of discrete symmetries can be obtained provided that the Lie algebra of Lie point symmetry generators is known (and is nontrivial). In particular, it is not necessary to be able to obtain the general solution of the ODE. Thus we were able to solve the problem posed by Ibragimov, without knowing the general solution of (3.2).

Discrete point symmetries are used in several branches of applied mathematics. Equivariant bifurcation theory describes the behaviour of nonlinear systems with symmetries [**6**]. To analyse the bifurcations correctly, it is essential that all point symmetries are known. Nonlinear systems occur in many different contexts (see [**1**] and [**3**] for some examples).

Various numerical methods for boundary value problems can be made more efficient by using discrete symmetries. For example, if the solution is known to be unique, it can be approximated by a spectral method, using trial functions that are invariant under the discrete symmetries. Sometimes it is possible to obtain exact solutions that are invariant under a discrete symmetry group, as in Example 2.4.

There are many other applications of discrete symmetries, including those that are not point symmetries. For brevity, the reader is referred to [**11**] for details.

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