Symmetries of integrable difference equations on the quad-graph.

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Abstract

This paper describes symmetries of all integrable difference equations that belong to the famous Adler-Bobenko-Suris classification. For each equation, the characteristics of symmetries satisfy a functional equation, which we solve by reducing it to a system of partial differential equations. In this way, all five-point symmetries of integrable equations on the quad-graph are found. These include mastersymmetries, which allow one to construct infinite hierarchies of local symmetries. We also demonstrate a connection between the symmetries of quad-graph equations and those of the corresponding Toda type difference equations.

1 Introduction

Partial difference equations (PΔE's) on the quad-graph have recently attracted much interest, especially from the integrable systems community. The first quad-graphs were derived in the works of Hirota [13, 14]. Since then, Lax pairs have been derived for these and many other quad-graph equations [6, 7, 21]. Some conservation laws have recently been discovered [17, 30, 31].

Adler, Bobenko & Suris classified integrable equations on the quad-graph by using the observation that Lax pairs arise from consistency on the cube [1]. However, they did not solve all questions about integrability of quad-graph equations. Recently Hietarinta [11] found an example of a quad-graph equation that has a Lax pair but lacks the tetrahedron property, so it is not included in [1]. However, it has recently been shown that this equation can be linearized by a potential transformation [28]. It is not yet known whether there exists a quad-graph equation that is consistent on a cube and lacks the tetrahedron property, but is not linearizable.

Symmetries of PΔE's first appeared as similarity constraints for integrable lattices. In [23], it is shown that discrete analogues of the Painlevé equations arise from similarity constraints. Similarity constraints and reductions to discrete Painlevé equations for the cross-ratio, discrete Korteweg-de Vries (dKdV) and discrete potential modified Korteweg-de Vries equations were later considered in [4, 10, 20, 21, 22, 24, 25]. One notable feature of similarity constraints for quad-graphs is that circle patterns can be formed for certain initial conditions [2, 3, 5, 7]. Recently, Tongas *et al.* pointed out that the similarity constraints for quad-graph equations obtained previously are equivalent to characteristics of symmetries [34]. They used an indirect method to discover mastersymmetries and higher-order symmetries for the dKdV equation. Symmetries of several other quad-graph equations have been found in [27, 33].

An alternative approach to symmetries of difference equations (not only integrable ones) is to try to construct discretizations of differential equations that retain all Lie point symmetries of the original system [8, 18, 19]. This requires a non-constant grid; in effect, the original continuous independent variables become extra dependent variables in the discretized system.

For a given difference equation, whether it is integrable or not, the main problem in finding symmetries is solving the linearized symmetry condition, which is a functional equation. Hydon developed a direct method of solving such functional equations by creating an associated system of differential equations that can be solved [15, 17]. We have recently improved the efficiency of this method, and have used it to construct conservation laws of many integrable quad-graph equations [30, 31]. The advantage of this method is that it gives a complete list of solutions; the disadvantage is that it requires massive calculations, for which we have used MAPLE 9.5.

The purpose of this paper is to classify symmetries for all quad-graph equations that are listed in the classification by Adler, Bobenko & Suris [1].We shall refer to these equations as the equations from the ABS classification. We list all symmetries that depend on values of the dependent variable within a 3×3 square. The complete classification of symmetries is a highly complex task, as some of the ABS equations are quite complicated. The classification includes mastersymmetries for each of the ABS equations; these allow the construction of infinite hierarchies of local symmetries. Thus we have shown that each of the ABS equations admits infinitely many symmetries, a property that is characteristic of integrability in continuous systems.

The paper begins with an introduction to the theory that is the basis for our calculations. In §3, we explain the method of finding local symmetries. §4 lists all symmetries on the 3×3 square for equations from the ABS classification; mastersymmetries are given in §5. In §6, we explain the connection between symmetries of equations from the ABS classification and those of the corresponding Toda type equations. To illustrate one of the most common applications of symmetries, §7 describes the construction of a group-invariant solution. In the final section, we draw conclusions and describe some open problems.

2 Symmetries of quad-graph equations

The general form of ABS equations on the quad-graph is

$$\omega(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \alpha_k, \beta_l) = 0. \tag{1}$$

Here k and l are independent variables, $u_{0,0} = u(k, l)$ is a dependent variable that is defined on the domain $\mathbb{Z}^2$. We denote the values of this variable on other points by

$u_{i,j} = u(k+i, l+j) = S_k^i S_l^j u_{0,0}$, where S_k, S_l are the unit forward shift operators. The ABS equations contain functions $\alpha_k = \alpha(k)$ and $\beta_l = \beta(l)$ that play the roles of edge parameters. The transformation

$$\Gamma: \ (k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \alpha_k, \beta_l) \mapsto (k, l, \hat{u}_{0,0}, \hat{u}_{1,0}, \hat{u}_{0,1}, \hat{u}_{1,1}, \hat{\alpha}_k, \hat{\beta}_l)$$

is a symmetry for (1) if

$$\omega(k, l, \hat{u}_{0,0}, \hat{u}_{1,0}, \hat{u}_{0,1}, \hat{u}_{1,1}, \hat{\alpha}_k, \hat{\beta}_l) = 0, \tag{2}$$

whenever (1) holds. Lie symmetries are obtained by linearizing the symmetry condition about the identity, as follows. We seek one-parameter (local) Lie groups of symmetries of the form

$$\begin{aligned} \hat{u}_{0,0} &= u_{0,0} + \epsilon\eta + O(\epsilon^2), \\ \hat{\alpha_k} &= \alpha_k + \epsilon\xi_1(k, \alpha_k) + O(\epsilon^2), \\ \hat{\beta}_l &= \beta_l + \epsilon\xi_2(l, \beta_l) + O(\epsilon^2). \end{aligned} \tag{3}$$

The functions η, ξ_1 and ξ_2 are components of the characteristic $\mathbf{Q}$ of the one-parameter group. The function η depends on finitely many shifts of $u_{0,0}$; this is discussed in the next section. By shifting (3) in the k and l directions we obtain

$$\begin{aligned} \hat{u}_{i,j} &= u_{i,j} + \epsilon S_k^i S_l^j \eta + O(\epsilon^2), \\ \hat{\alpha}_{k+i} &= \alpha_{k+i} + \epsilon\xi_1(k+i, \alpha_{k+i}) + O(\epsilon^2), \\ \hat{\beta}_{l+j} &= \beta_{l+j} + \epsilon\xi_2(l+j, \beta_{l+j}) + O(\epsilon^2), \end{aligned}$$

for every $i, j \in \mathbb{Z}$. Expanding (2) to first order in ϵ yields the linearized symmetry condition

$$X\omega = 0 \qquad \text{whenever (1) holds,}$$

where

$$X = \eta\frac{\partial}{\partial u_{0,0}} + (S_k\eta)\frac{\partial}{\partial u_{1,0}} + (S_l\eta)\frac{\partial}{\partial u_{0,1}} + (S_kS_l\eta)\frac{\partial}{\partial u_{1,1}} + \xi_1\frac{\partial}{\partial \alpha_k} + \xi_2\frac{\partial}{\partial \beta_l}. \tag{4}$$

3 The method

If we were to seek only Lie point symmetries, then η would be of the form

$$\eta = \eta(k, l, u_{0,0}, \alpha_k, \beta_l).$$

However, we shall consider higher symmetries that depend upon the values of the dependent variable on a 3×3 square that is centred on (k, l). By using the quad-graph equation to eliminate the corner nodes (Figure 1), we simplify η to the following form:

$$\eta = \eta(k, l, u_{-1,0}, u_{0,-1}, u_{0,0}, u_{1,0}, u_{0,1}, \alpha_k, \beta_l). \tag{5}$$

As η depends on five values of the dependent variable, we call such symmetries *five-point symmetries.*

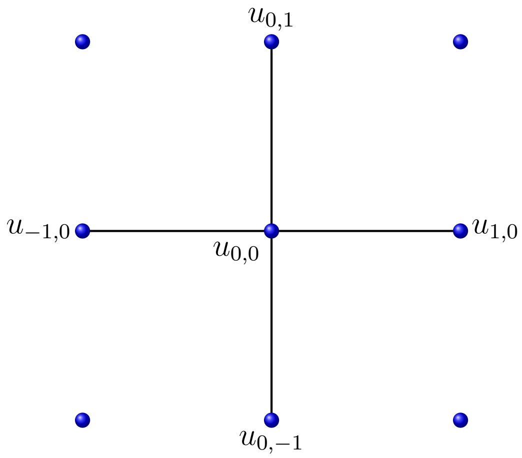

Figure 1: Form of a five-point symmetry

In fact η can be simplified still further. To show this, we apply a symmetry generator (4) to (1) and obtain the linearized symmetry condition:

$$\eta\omega_{u_{0,0}} + S_k\eta\omega_{u_{1,0}} + S_l\eta\omega_{u_{0,1}} + S_kS_l\eta\omega_{u_{1,1}} + \xi_1\omega_{\alpha_k} + \xi_2\omega_{\beta_l} = 0. \tag{6}$$

This expression has to be satisfied by all solutions of (1). Let

$$\bar{u}_{0,0}(u_{1,0}, u_{0,1}, u_{1,1}), \quad \bar{u}_{1,0}(u_{0,0}, u_{0,1}, u_{1,1}), \quad \bar{u}_{0,1}(u_{0,0}, u_{1,0}, u_{1,1}), \quad \bar{u}_{1,1}(u_{0,0}, u_{1,0}, u_{0,1}),$$

denote the result of solving (1) for $u_{0,0}, u_{1,0}, u_{0,1}$ and $u_{1,1}$ respectively. In the following, to save space, we suppress the dependence on k, l, α_k and β_l, and we use $\tilde{u}_{1,1}$ to denote $\bar{u}_{1,1}(u_{0,0}, u_{1,0}, u_{0,1})$. To write out the linearized symmetry condition explicitly, we substitute

$$\begin{aligned} u_{-1,0} =& \bar{u}_{0,0}(u_{0,0}, u_{-1,1}, u_{0,1}),\\ u_{0,-1} =& \bar{u}_{0,0}(u_{1,-1}, u_{0,0}, u_{1,0}),\\ u_{2,1} =& \bar{u}_{1,1}(u_{1,0}, u_{2,0}, \tilde{u}_{1,1}),\\ u_{1,2} =& \bar{u}_{1,1}(u_{0,1}, \tilde{u}_{1,1}, u_{0,2}),\\ u_{1,1} =& \tilde{u}_{1,1}, \end{aligned}$$

into (6), to obtain

$$\begin{aligned} &\eta(\bar{u}_{0,0}(u_{0,0}, u_{-1,1}, u_{0,1}), \bar{u}_{0,0}(u_{1,-1}, u_{0,0}, u_{1,0}), u_{0,0}, u_{1,0}, u_{0,1})\omega_{u_{0,0}}\\ &\qquad + \eta(u_{0,0}, u_{1,-1}, u_{1,0}, u_{2,0}, \tilde{u}_{1,1})\omega_{u_{1,0}} + \eta(u_{-1,1}, u_{0,0}, u_{0,1}, \tilde{u}_{1,1}, u_{0,2})\omega_{u_{0,1}}\\ &\quad + \eta(u_{0,1}, u_{1,0}, \tilde{u}_{1,1}, \bar{u}_{1,1}(u_{1,0}, u_{2,0}, \tilde{u}_{1,1}), \bar{u}_{1,1}(u_{0,1}, \tilde{u}_{1,1}, u_{0,2}))\omega_{u_{1,1}} + \xi_1\omega_{\alpha_k} + \xi_2\omega_{\beta_l} = 0. \end{aligned} \tag{7}$$

By differentiating (7) with respect to $u_{-1,1}$ and $u_{1,-1}$, we obtain the necessary condition

$$\omega_{u_{0,0}} \frac{\partial^2}{\partial u_{-1,1}\partial u_{1,-1}} \eta(\bar{u}_{0,0}(u_{0,0}, u_{-1,1}, u_{0,1}), \bar{u}_{0,0}(u_{1,-1}, u_{0,0}, u_{1,0}), u_{0,0}, u_{1,0}, u_{0,1}) = 0. \tag{8}$$

The coefficient of η is nonzero, so the solution of (8) shows that η can be split into the sum of two functions which have a simpler form than (5). New conditions for η can be obtained, for instance, by differentiating (7) with respect to $u_{-1,0}$ and $u_{0,-1}$. Taken together, all such conditions give a system of PDE's with the following solution

$$\eta_{cross}(k,l,u_{-1,0},u_{0,-1},u_{0,0},u_{1,0},u_{0,1},\alpha_k,\beta_l) = \eta_k(k,l,u_{-1,0},u_{0,0},u_{1,0},\alpha_k,\beta_l) + \eta_l(k,l,u_{0,-1},u_{0,0},u_{0,1},\alpha_k,\beta_l),$$

where η_k and η_l are functions which have to be found. Therefore we have demonstrated that η is of the form η_{cross}, which is the sum of the terms in the k and l direction separately. Similarly it is possible to show that for any higher-order symmetry generator, if η is simplified to depend only on values on a cross, it consists of two separate terms in the k and l direction respectively.

We now explain the method for calculating the characteristics $\mathbf{Q}$ for a given quad-graph equation. By substituting η_{cross} into (7) we obtain the following determining equation for η_k, η_l, ξ_1 and ξ_2 (again, we suppress k, l, α_k, β_l for brevity).

$$\begin{aligned}(\eta_k(\bar{u}_{0,0}(u_{0,0},u_{-1,1},u_{0,1}),u_{0,0},u_{1,0}) + \eta_l(\bar{u}_{0,0}(u_{1,-1},u_{0,0},u_{1,0}),u_{0,0},u_{0,1}))\omega_{u_{0,0}} \\ + (\eta_k(u_{0,0},u_{1,0},u_{2,0}) + \eta_l(u_{1,-1},u_{1,0},\bar{u}_{1,1}))\,\omega_{u_{1,0}} \\ + (\eta_k(u_{-1,1},u_{0,1},\bar{u}_{1,1}) + \eta_l(u_{0,0},u_{0,1},u_{0,2})\,\omega_{u_{0,1}} + (\eta_k(u_{0,1},\bar{u}_{1,1},\bar{u}_{1,1}(u_{1,0},u_{2,0},\bar{u}_{1,1})) \\ + \eta_l(u_{1,0},\bar{u}_{1,1},\bar{u}_{1,1}(u_{0,1},\bar{u}_{1,1},u_{0,2})))\omega_{u_{1,1}} + \xi_1\omega_{\alpha_k} + \xi_2\omega_{\beta_l} = 0. \quad (9)\end{aligned}$$

To solve this functional equation we use an idea which is described in [30, 31], namely we reduce it to a PDE. By differentiating (9) with respect to $u_{2,0}$ we obtain

$$\omega_{u_{1,0}}\frac{\partial}{\partial u_{2,0}}\eta_k(u_{0,0},u_{1,0},u_{2,0}) + \omega_{u_{1,1}}\frac{\partial}{\partial u_{2,0}}\eta_k(u_{0,1},\bar{u}_{1,1},\bar{u}_{1,1}(u_{1,0},u_{2,0},\bar{u}_{1,1})) = 0. \quad (10)$$

This is a functional-differential equation, but it contains fewer sets of arguments than (9) does. The first term can be eliminated by dividing by $\omega_{u_{1,0}}$ and then differentiating with respect to $u_{0,1}$, to obtain

$$\frac{\partial}{\partial u_{0,1}}\left(\frac{\omega_{u_{1,1}}}{\omega_{u_{1,0}}}\frac{\partial}{\partial u_{2,0}}\eta_k(u_{0,1},\bar{u}_{1,1},\bar{u}_{1,1}(u_{1,0},u_{2,0},\bar{u}_{1,1}))\right) = 0.$$

After making the substitution

$$u_{0,0} = \bar{u}_{0,0}, \quad u_{2,0} = S_k\bar{u}_{1,0},$$

we get a PDE for the function η_k and solve it. The constraints for the function η_l can be found in a similar way.

So far, we have differentiated the determining equations (9) twice; this has created a hierarchy of functional-differential equations that every five-point symmetry must satisfy. The unknown functions η_k, η_l, ξ_1 and ξ_2 can be found completely by going up the hierarchy, a step at a time, to determine more constraints. As the constraints are solved sequentially, more and more information is gained about the functions. At the highest stage, the determining equation is satisfied.

4 Five-point symmetries of integrable equations on the quad-graph

In this section we present all five-point symmetries for integrable equations on the quad-graph that are listed in [1]; these were found by the method described in the previous section.

The ABS equations depend on two arbitrary functions $\alpha = \alpha_k$ and $\beta = \beta_l$. First we shall consider symmetries of autonomous equations (α and β are constants); non-autonomous cases will be discussed later. We use the same titles for equations as were used in [1], except that **Q4** is given in the equivalent form due to Hietarinta [12]. The equations from the ABS classification are

$$\begin{aligned}
&\mathbf{Q1}: \ \alpha(u_{0,0}-u_{0,1})(u_{1,0}-u_{1,1})-\beta(u_{0,0}-u_{1,0})(u_{0,1}-u_{1,1})+\delta^2\alpha\beta(\alpha-\beta)=0,\\
&\mathbf{Q2}: \ \alpha(u_{0,0}-u_{0,1})(u_{1,0}-u_{1,1})-\beta(u_{0,0}-u_{1,0})(u_{0,1}-u_{1,1})\\
&\qquad +\alpha\beta(\alpha-\beta)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0,\\
&\mathbf{Q3}: \ (\beta^2-\alpha^2)(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})+\beta(\alpha^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})\\
&\qquad -\alpha(\beta^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})-\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta)=0,\\
&\mathbf{Q4}: \ \mathrm{sn}(\alpha)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\mathrm{sn}(\beta)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})-\mathrm{sn}(\alpha-\beta)(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})\\
&\qquad +\mathrm{sn}(\alpha-\beta)\mathrm{sn}(\alpha)\mathrm{sn}(\beta)(1+K^2u_{0,0}u_{1,0}u_{0,1}u_{1,1})=0,\\
&\mathbf{H1}: \ (u_{0,0}-u_{1,1})(u_{1,0}-u_{0,1})+\beta-\alpha=0,\\
&\mathbf{H2}: \ (u_{0,0}-u_{1,1})(u_{1,0}-u_{0,1})+(\beta-\alpha)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})+\beta^2-\alpha^2=0,\\
&\mathbf{H3}: \ \alpha(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\beta(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})+\delta^2(\alpha^2-\beta^2)=0,\\
&\mathbf{A1}: \ \alpha(u_{0,0}+u_{0,1})(u_{1,0}+u_{1,1})-\beta(u_{0,0}+u_{1,0})(u_{0,1}+u_{1,1})-\delta^2\alpha\beta(\alpha-\beta)=0,\\
&\mathbf{A2}: \ (\beta^2-\alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1}+1)+\beta(\alpha^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})\\
&\qquad -\alpha(\beta^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})=0.
\end{aligned}$$

Here $\mathrm{sn}(\alpha)=\mathrm{sn}(\alpha;K)$ is a Jacobi elliptic function with modulus K. All five-point symmetries for these equation are listed in Table 1. In these tables $\mathrm{cn}(\alpha)=\mathrm{cn}(\alpha;K)$ and $\mathrm{dn}(\alpha)=\mathrm{dn}(\alpha;K)$ are Jacobi elliptic functions with modulus K. In [7] it was shown that when $K=0$, equation **Q4** is equivalent to the case $\mathbf{Q3}_{\delta=1}$. When $K=0$, all symmetries for equation **Q4** are equivalent to the symmetries for equation $\mathbf{Q3}_{\delta=1}$. We omit the details of our calculations, because they are massive and it is impossible to present them in any suitable form. Five-point symmetries for **H1**, $\mathbf{H3}_{\delta=0}$ and $\mathbf{Q1}_{\delta=0}$ have already appeared in [4, 10, 20, 21, 22, 23, 24, 25, 27, 33, 34].

Note that each equation from the ABS classification has two nonpoint symmetries in the k direction and two nonpoint symmetries in the l direction. In each case, one of these symmetries in the k direction depends explicitly on k; in the next section, we denote this symmetry by X_{km}. The other symmetry in the k direction does not depend on k; we will denote it by X_k. Similarly, we will denote the nonpoint symmetries in the l direction by X_{lm} and X_l.

So far we have seen symmetries only for autonomous equations, for which α and β are constants. The same point symmetries occur even when α and β are not constant. However, there are no other five-point symmetries in the k (respectively l) direction if α (respectively β) is not constant.

Table 1: **Symmetry generators for equations from the ABS classification**

Equations	Generators
$\mathbf{Q1}_{\delta=0}$	$X_1=\alpha\partial_\alpha+\beta\partial_\beta,\quad X_2=\partial_{u_{0,0}},\quad X_3=u_{0,0}\partial_{u_{0,0}},\quad X_4=(u_{0,0})^2\partial_{u_{0,0}},\quad X_5=(u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_6=(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_7=k(u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}+\alpha\partial_\alpha,$ $X_8=l(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}+\beta\partial_\beta,$
$\mathbf{Q1}_{\delta=1}$	$X_1=\partial_{u_{0,0}},\quad X_2=u_{0,0}\partial_{u_{0,0}}+\alpha\partial_\alpha+\beta\partial_\beta,\quad X_3=\{\alpha^2+(u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_4=\{\beta^2+(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_5=k\{\alpha^2+(u_{1,0}-u_{0,0})(u_{0,0}-u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}+\alpha\partial_\alpha,$ $X_6=l\{\beta^2+(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}+\beta\partial_\beta,$
$\mathbf{Q2}$	$X_1=2u_{0,0}\partial_{u_{0,0}}+\alpha\partial_\alpha+\beta\partial_\beta,\quad X_2=\{(u_{0,0}-u_{1,0})(u_{0,0}-u_{-1,0})-\alpha^2(2u_{0,0}+u_{1,0}+u_{-1,0})+\alpha^4\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_3=\{(u_{0,0}-u_{0,1})(u_{0,0}-u_{0,-1})-\beta^2(2u_{0,0}+u_{0,1}+u_{0,-1})+\beta^4\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$ $X_4=k\{(u_{0,0}-u_{1,0})(u_{0,0}-u_{-1,0})-\alpha^2(2u_{0,0}+u_{1,0}+u_{-1,0})+\alpha^4\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}-\alpha\partial_\alpha,$ $X_5=l\{(u_{0,0}-u_{0,1})(u_{0,0}-u_{0,-1})-\beta^2(2u_{0,0}+u_{0,1}+u_{0,-1})+\beta^4\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}-\beta\partial_\beta,$
$\mathbf{Q3}_{\delta=0}$	$X_1=u_{0,0}\partial_{u_{0,0}},\quad X_2=\{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0})-2\alpha((u_{0,0})^2+u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_3=\{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1})-2\beta((u_{0,0})^2+u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$ $X_4=k\{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0})-2\alpha((u_{0,0})^2+u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}+\alpha(\alpha^2-1)\partial_\alpha,$ $X_5=l\{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1})-2\beta((u_{0,0})^2+u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}+\beta(\beta^2-1)\partial_\beta,$

Continued on next page

Equations	Generators
Q3$_{\delta=1}$	$X_1=\{2\alpha(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0})-4\alpha^2(u_{1,0}u_{-1,0}+(u_{0,0})^2)-(1-\alpha^2)^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}$, $X_2=\{2\beta(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1})-4\beta^2(u_{0,1}u_{0,-1}+(u_{0,0})^2)-(1-\beta^2)^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}$, $X_3=k\{2\alpha(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0})-4\alpha^2(u_{1,0}u_{-1,0}+(u_{0,0})^2)-(1-\alpha^2)^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}+2\alpha^2(\alpha^2-1)\partial_\alpha$, $X_4=l\{2\beta(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1})-4\beta^2(u_{0,1}u_{0,-1}+(u_{0,0})^2)-(1-\beta^2)^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}+2\beta^2(\beta^2-1)\partial_\beta$,
Q4	$X_1=\{\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)u_{0,0}(u_{1,0}+u_{-1,0})+\mathrm{sn}^2(\alpha)(1+K^2u_{-1,0}(u_{0,0})^2u_{1,0})-u_{-1,0}u_{1,0}-(u_{0,0})^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}$, $X_2=\{\mathrm{cn}(\beta)\mathrm{dn}(\beta)u_{0,0}(u_{0,1}+u_{0,-1})+\mathrm{sn}^2(\beta)(1+K^2u_{0,-1}(u_{0,0})^2u_{0,1})-u_{0,-1}u_{0,1}-(u_{0,0})^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}$, $X_3=k\{\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)u_{0,0}(u_{1,0}+u_{-1,0})+\mathrm{sn}^2(\alpha)(1+K^2u_{-1,0}(u_{0,0})^2u_{1,0})-u_{-1,0}u_{1,0}-(u_{0,0})^2\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}+\mathrm{sn}(\alpha)\partial_\alpha$, $X_4=l\{\mathrm{cn}(\beta)\mathrm{dn}(\beta)u_{0,0}(u_{0,1}+u_{0,-1})+\mathrm{sn}^2(\beta)(1+K^2u_{0,-1}(u_{0,0})^2u_{0,1})-u_{0,-1}u_{0,1}-(u_{0,0})^2\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}+\mathrm{sn}(\beta)\partial_\beta$, when $K=\pm1$ $\quad X_5=\{1-(u_{0,0})^2\}\partial_{u_{0,0}}$,
H1	$X_1=\partial_\alpha+\partial_\beta,\quad X_2=\partial_{u_{0,0}},\quad X_3=(-1)^{k+l}\partial_{u_{0,0}},\quad X_4=u_{0,0}\partial_{u_{0,0}}+2\alpha\partial_\alpha+2\beta\partial_\beta,\quad X_5=(-1)^{k+l}u_{0,0}\partial_{u_{0,0}}$, $X_6=(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},\quad X_7=(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_8=k(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}-\partial_\alpha,\quad X_9=l(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}-\partial_\beta$,
H2	$X_1=\partial_{u_{0,0}}-2\partial_\alpha-2\partial_\beta,\quad X_2=(-1)^{k+l}\partial_{u_{0,0}},\quad X_3=u_{0,0}\partial_{u_{0,0}}+\alpha\partial_\alpha+\beta\partial_\beta,\quad X_4=(u_{1,0}+u_{-1,0}+2u_{0,0}+2\alpha)(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}$, $X_5=(u_{0,1}+u_{0,-1}+2u_{0,0}+2\beta)(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},\quad X_6=k\,(u_{1,0}+u_{-1,0}+2u_{0,0}+2\alpha)(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}-\partial_\alpha$, $X_7=l\,(u_{0,1}+u_{0,-1}+2u_{0,0}+2\beta)(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}-\partial_\beta$,

Continued on next page

Equations	Generators
H3$_{\delta=0}$	$X_1 = \alpha\partial_\alpha + \beta\partial_\beta, \quad X_2 = u_{0,0}\partial_{u_{0,0}}, \quad X_3 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}}, \quad X_4 = u_{0,0}(u_{1,0}+u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_5 = u_{0,0}(u_{0,1}+u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}, \quad X_6 = ku_{0,0}(u_{1,0}+u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,$ $X_7 = lu_{0,0}(u_{0,1}+u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
H3$_{\delta=1}$	$X_1 = u_{0,0}\partial_{u_{0,0}} + 2\alpha\partial_\alpha + 2\beta\partial_\beta, \quad X_2 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}}, \quad X_3 = (2\alpha + u_{0,0}u_{1,0} + u_{0,0}u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_4 = (2\beta + u_{0,0}u_{0,1} + u_{0,0}u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}, \quad X_5 = k(2\alpha + u_{0,0}u_{1,0} + u_{0,0}u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha,$ $X_6 = l(2\beta + u_{0,0}u_{0,1} + u_{0,0}u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
A1$_{\delta=0}$	$X_1 = \alpha\partial_\alpha + \beta\partial_\beta, \quad X_2 = (-1)^{k+l}\partial_{u_{0,0}}, \quad X_3 = u_{0,0}\partial_{u_{0,0}}, \quad X_4 = (-1)^{k+l}(u_{0,0})^2\partial_{u_{0,0}},$ $X_5 = (u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}}, \quad X_6 = (u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$ $X_7 = k(u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha\partial_\alpha, \quad X_8 = l(u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta\partial_\beta,$
A1$_{\delta=1}$	$X_1 = (-1)^{k+l}\partial_{u_{0,0}}, \quad X_2 = u_{0,0}\partial_{u_{0,0}} + \alpha\partial_\alpha + \beta\partial_\beta, \quad X_3 = \{\alpha^2 - (u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_4 = \{\beta^2 - (u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}}, \quad X_5 = k\{\alpha^2 - (u_{0,0}+u_{1,0})(u_{0,0}+u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} - \alpha\partial_\alpha,$ $X_6 = l\{\beta^2 - (u_{0,0}+u_{0,1})(u_{0,0}+u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} - \beta\partial_\beta,$
A2	$X_1 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}}, \quad X_2 = \{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 2\alpha(1+(u_{0,0})^2u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}},$ $X_3 = \{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 2\beta(1+(u_{0,0})^2u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}},$ $X_4 = k\{(1+\alpha^2)u_{0,0}(u_{1,0}+u_{-1,0}) - 2\alpha(1+(u_{0,0})^2u_{1,0}u_{-1,0})\}(u_{1,0}-u_{-1,0})^{-1}\partial_{u_{0,0}} + \alpha(1-\alpha^2)\partial_\alpha,$ $X_5 = l\{(1+\beta^2)u_{0,0}(u_{0,1}+u_{0,-1}) - 2\beta(1+(u_{0,0})^2u_{0,1}u_{0,-1})\}(u_{0,1}-u_{0,-1})^{-1}\partial_{u_{0,0}} + \beta(1-\beta^2)\partial_\beta.$

5 Mastersymmetries

For continuous integrable systems, Fuchssteiner [9] has explained the link between mastersymmetries and symmetries that are linear in the independent variables. Furthermore, [34] showed that the dKdV equation also has mastersymmetries that are linear in the independent variables. For each of the ABS equations with constant α and β, the generators X_{km} and X_{lm} have this property, which suggests that they may be mastersymmetries. An algebraic approach to mastersymmetries gives the following criterion [29, 32].

Definition 1. A symmetry X_m is a mastersymmetry for the symmetry X if it satisfies

$$[X_m, X] \neq 0, \qquad [[X_m, X], X] = 0. \tag{11}$$

Here $[\cdot, \cdot]$ denotes the commutator.

By checking these properties for all symmetries from Table 1 we find that X_{km} is a mastersymmetry for X_k and X_{lm} is a mastersymmetry for X_l for each equation in the ABS classification. Therefore we can obtain a hierarchy of symmetries in the k direction:

$$X_{k_1} = [X_{km}, X_k], \quad X_{k_2} = [X_{km}, X_{k_1}], \quad ..., \quad X_{k_{n+1}} = [X_{km}, X_{k_n}].$$

Similarly, there is a hierarchy of symmetries in the l direction:

$$X_{l_1} = [X_{lm}, X_l], \quad X_{l_2} = [X_{lm}, X_{l_1}], \quad ..., \quad X_{l_{n+1}} = [X_{lm}, X_{l_n}].$$

As an example, consider the autonomous equation $\mathbf{Q1}_{\delta=0}$. The commutator of symmetries X_7 and X_5 gives us a new symmetry:

$$X_9 = \frac{(u_{1,0} - u_{0,0})^2 (u_{0,0} - u_{-1,0})^2}{(u_{1,0} - u_{-1,0})^2} \left(\frac{1}{u_{2,0} - u_{0,0}} + \frac{1}{u_{0,0} - u_{-2,0}} \right) \partial_{u_{0,0}}. \tag{12}$$

This symmetry cannot be reduced to any lower-order symmetry, for its characteristic depends on $u_{2,0}, u_{-2,0}$. The symmetry (12) lies on a line of five points; if we apply the mastersymmetry a second time we will obtain an expression which lies on a seven-point line, and so on. The same situation occurs for each of the ABS equations, namely the order of a symmetry increases by two each time one applies a mastersymmetry, creating hierarchies with the following dependencies:

$$X_{k_n} = \eta_{k_n}(u_{-n,0}, u_{-n+1,0}, ..., u_{n-1,0}, u_{n,0}) \partial_{u_{0,0}},$$
$$X_{l_n} = \eta_{l_n}(u_{0,-n}, u_{0,-n+1}, ..., u_{0,n-1}, u_{0,n}) \partial_{u_{0,0}}.$$

6 Symmetries of Toda type equations

The connection between integrable quad-graph equations and Toda type systems is now well-known [1, 6, 7]; we can use it to transform symmetries of quad-graph equations into symmetries of the corresponding Toda type systems. A Toda system can be obtained from any equation in the ABS classification

$$\omega(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, \alpha, \beta) = 0 \tag{13}$$

by the substitution

$$u_{0,1} = S_k^{-1}\tilde{u}_{1,1}, \quad u_{1,0} = S_l^{-1}\tilde{u}_{1,1}, \quad u_{-1,0} = \bar{u}_{0,1}(u_{-1,-1}, u_{0,-1}, u_{0,0}). \tag{14}$$

Here, we are using the notation introduced in §3. Note: it is necessary to make the substitution $u_{-1,0} = \bar{u}_{0,1}(u_{-1,-1}, u_{0,-1}, u_{0,0})$ after the substitution $u_{0,1} = S_k^{-1}\tilde{u}_{1,1}$ because equation (13) does not depend on $u_{-1,0}$.

We have verified that each of listed symmetries for the ABS classification can be transformed to a symmetry for the corresponding Toda type system by the substitution (14).

As an example consider the Toda type system that corresponds to the autonomous equations **H1** and $\mathbf{Q1}_{\delta=0}$:

$$\frac{1}{u_{1,1}-u_{0,0}} - \frac{1}{u_{-1,1}-u_{0,0}} - \frac{1}{u_{1,-1}-u_{0,0}} + \frac{1}{u_{-1,-1}-u_{0,0}} = 0. \tag{15}$$

The characteristics of symmetries for (15) can be obtained by transformation of the characteristics of the symmetries for **H1** and $\mathbf{Q1}_{\delta=0}$ by (14). Note that for **H1** and $\mathbf{Q1}_{\delta=0}$ the substitutions (14) are different.

The point symmetries stay the same after substitution (14) for both **H1** and $\mathbf{Q1}_{\delta=0}$; they are

$$\begin{array}{l} X_1 = \partial_{u_{0,0}}, \quad X_2 = (-1)^{k+l}\partial_{u_{0,0}}, \quad X_3 = u_{0,0}\partial_{u_{0,0}}, \\ X_4 = (-1)^{k+l}u_{0,0}\partial_{u_{0,0}}, \quad X_5 = {u_{0,0}}^2\partial_{u_{0,0}}. \end{array} \tag{16}$$

(We have omitted the components ξ_1 and ξ_2, because (15) does not depend on α or β.) The commutators of (16) yield one more symmetry generator:

$$X_6 = (-1)^{k+l}{u_{0,0}}^2\partial_{u_{0,0}}.$$

The rescaled remaining five-point symmetries of **H1** transform by (14) to

$$\begin{aligned} X_7 &= \frac{(u_{0,0}-u_{-1,-1})(u_{1,-1}-u_{0,0})}{u_{1,-1}-u_{-1,-1}}\partial_{u_{0,0}}, \\ X_8 &= \frac{(u_{0,0}-u_{-1,-1})(u_{-1,1}-u_{0,0})}{u_{-1,1}-u_{-1,-1}}\partial_{u_{0,0}}, \\ X_9 &= \frac{k(u_{0,0}-u_{-1,-1})(u_{1,-1}-u_{0,0})}{u_{1,-1}-u_{-1,-1}}\partial_{u_{0,0}}, \\ X_{10} &= \frac{l(u_{0,0}-u_{-1,-1})(u_{-1,1}-u_{0,0})}{u_{-1,1}-u_{-1,-1}}\partial_{u_{0,0}}. \end{aligned}$$

The same result is obtained from the symmetries for $\mathbf{Q1}_{\delta=0}$. The Toda system (15) also has mastersymmetries. As expected, X_9 is the mastersymmetry for X_7 and X_{10} is the mastersymmetry for X_8. Two hierarchies of the local symmetries therefore can be constructed.

In the same way, each Toda system for the other quad-graph equations has mastersymmetries that can be obtained from the mastersymmetries of the corresponding quad-graph equations.

Note that the five-point symmetries for (15) lie on the same five-point cross on which the Toda system is defined (Figure 2), not on the one which is in Figure 1.

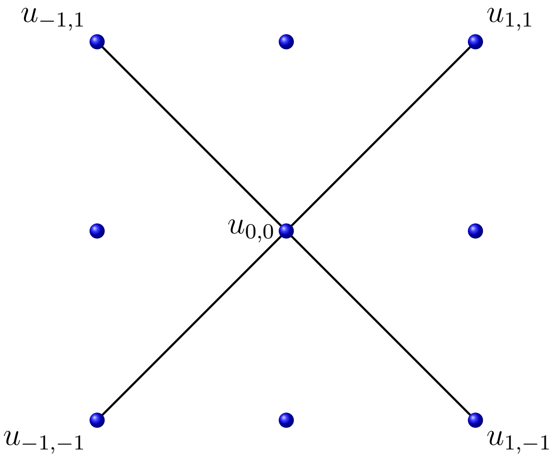

Figure 2: Five-point symmetries for (15)

7 Similarity solutions

One way of finding similarity solutions of quad-graph equations was considered in [2, 3, 5, 7]. The authors define three initial points, from which they construct symmetric initial conditions for a given quad-graph equation. This approach typically yields an integrable map.

Another approach is to reduce the number of variables by requiring that the solution is invariant under the symmetries generated by a characteristic. This method is widely used for continuous systems [16, 26], and has been applied to the dKdV equation in [34].

We shall illustrate the method by seeking nonzero solutions of $\mathbf{H3}_{\delta=0}$ that are invariant under the symmetries generated by

$$Q = Q_4 - aQ_2 = \frac{u_{0,0}(u_{1,0}+u_{-1,0})}{u_{1,0}-u_{-1,0}} - au_{0,0}, \qquad a>1.$$

The general solution of the invariance condition $Q=0$ (with $u_{0,0}\neq 0$) is

$$u_{0,0} = (f_1(l) + (-1)^k f_2(l)) \left(\sqrt{\frac{a+1}{a-1}}\right)^k, \tag{17}$$

where f_1 and f_2 are arbitrary functions. By substituting (17) into $\mathbf{H3}_{\delta=0}$ we find that

$$f_1(l) = c_1(-1)^l f_2(l),$$

where c_1 is an arbitrary constant. If $c_1^2\neq 1$ then f_2 satisfies the following ordinary difference equation:

$$\alpha\sqrt{a^2-1}\left(\frac{f_2(l+1)}{f_2(l)}\right)^2 + 2a\beta\,\frac{f_2(l+1)}{f_2(l)} + \alpha\sqrt{a^2-1} = 0, \tag{18}$$

which yields a large family of exact solutions, including

$$u_{0,0} = (\bar{c}_1 + c_2(-1)^{k+l})\left(\sqrt{\frac{a+1}{a-1}}\right)^k \left(\frac{a\beta + \sqrt{\alpha^2 + a^2\beta^2 - a^2\alpha^2}}{\alpha\sqrt{a^2-1}}\right)^l,$$

where $\bar{c}_1 = c_1 c_2$.

If $c_1 = \pm 1$ then there are no further constraints, so there are two families of invariant solutions

$$u_{0,0} = f_2(l)\left((-1)^l \pm (-1)^k\right)\left(\sqrt{\frac{a+1}{a-1}}\right)^k. \tag{19}$$

These belong to the following degenerate class of solutions of $\mathbf{H3}_{\delta=0}$:

$$u_{0,0} = F(k,l)\left((-1)^l \pm (-1)^k\right), \tag{20}$$

where F is an arbitrary function.

8 Conclusion and discussion

The main result of this work is the derivation of the complete set of five-point symmetries for equations from the ABS classification. We found all symmetries by a generalization of the method which is described in [15]. This confirms that this method can be used in a systematic way without making restrictive assumptions about the form of symmetries.

The symmetries that we have found have various applications. For instance, symmetries can be used to obtain group-invariant reductions that lead to exact solutions of the quad-graph equations. We have only considered a single example of such a reduction (for $\mathbf{H3}_{\delta=0}$). However we have shown that all ABS equations have infinitely many symmetries, any of which could be used to construct invariant solutions. Five-point and other higher symmetries can also be used for the generation of new conservation laws. It is notable that all equations from [1] have four five-point symmetries that have similar forms.

Mastersymmetries for integrable equations on the quad-graph have been derived. These mastersymmetries allow us to construct infinite hierarchies of local symmetries. It is important to allow mastersymmetries to act on α and β; otherwise the mastersymmetries for $Q3$ and $Q4$ would not have been found. The existence of mastersymmetries shows the similarity of structures for continuous and difference equations.

We have discussed the relation between the symmetries of quad-graph equations and symmetries for Toda type systems. We have also verified that for each symmetry of the integrable quad-graph equation there is a corresponding symmetry of the related Toda type system. It is not yet known whether this relationship is true for all symmetries of integrable quad-graph equations.

Our work raises an important question about symmetries for nonautonomous quad-graph equations. Why are there no five-point symmetries in the k direction when $\alpha = \alpha_k$ and five-point symmetries in the l direction when $\beta = \beta_l$? Is it possible that there are no nonpoint local symmetries of any order?

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