# Conservation laws for integrable difference equations.

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**Abstract.** This paper deals with conservation laws for all integrable difference equations that belong to the famous Adler-Bobenko-Suris classification. All inequivalent three-point conservation laws are found, as are three five-point conservation laws for each equation. We also describe a method of generating conservation laws from known ones; this method can be used to generate higher-order conservation laws from those that are listed here.

## 1. Introduction

Generally speaking, it is much harder to calculate conservation laws for difference equations than for differential equations, because one has to solve a complicated functional equation rather than a system of overdetermined partial differential equations. The first systematic technique for obtaining all conservation laws of a given type was introduced by Hydon [5], who found all three-point conservation laws of the discrete potential modified Korteweg-de Vries (dpmKdV) equation. This technique was improved in [7] and used to find all three- and five-point conservation laws of the discrete Korteweg-de Vries (dKdV) equation. Rasin & Hydon also classified all three-point conservation laws for a family of integrable difference equations due to Nijhoff, Quispel & Capel, showing that every equation from this class has at least two nontrivial conservation laws [8]. The existence of three-point conservation laws was also used to derive conditions under which nonautonomous dKdV and dpmKdV equations are integrable [11].

Having found an practical direct approach to obtaining and classifying conservation laws, we set out to discover the conservation laws of all integrable quad-graph equations that are listed in the famous classification by Adler, Bobenko & Suris [1]. These are the primary examples of integrable difference equations; they are the focus of much current research. In particular, there is interest in nonautonomous quad-graph equations. Integrability conditions have been investigated [10, 11], as have reductions to ordinary difference equations [2], or to a series of q-discrete Painlevé equations [3]. This encouraged us to include nonautonomous quad-graph equations in our analysis.

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In the current paper, we present a complete classification of all three-point conservation laws for each of the ABS equations. In addition, three five-point conservation laws for each quad-graph are found. The existence of infinitely many conservation laws is a key feature of continuous integrable equations. We suggest that this question can be studied for integrable quad-graphs in the same way as for continuous equations: the action of a symmetry on a given conservation law can yield another conservation law [6]. This approach uses high-order symmetries of the ABS equations, which have recently been found [9].

The structure of this paper is as follows: §2 describes the direct method for calculating conservation laws of quad-graph equations. §3 lists all three-point conservation laws for each equation in the ABS classification; three five-point conservation laws for each ABS equation are given in §4. The method that allows to generate a new law from the known one is described in §5. We conclude in §6 with a brief discussion of some open problems.

## 2. The method

The general form of a scalar P$\Delta$E on the quad-graph is:

$$P(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0. \tag{1}$$

Here $k$ and $l$ are independent variables and $u_{0,0} = u(k, l)$ is a dependent variable that is defined on the domain $\mathbb{Z}^2$. We denote the values of this variable on other points by $u_{i,j} = u(k + i, l + j) = S_k^i S_l^j u_{0,0}$, where $S_k$, $S_l$ are the unit forward shift operators in $k$ and $l$ respectively.

A conservation law for any quad-graph equation (1) is an expression of the form

$$(S_k - \mathrm{id})F + (S_l - \mathrm{id})G = 0 \tag{2}$$

that is satisfied by all solutions of the equation. Here the functions $F$ and $G$ are the components of the conservation law and id is the identity mapping.

A conservation law is trivial if it holds identically (not just on solutions of the P$\Delta$E), or if $F$ and $G$ both vanish on all solutions of (1). For three-point conservation laws, the components $F$ and $G$ are of the form

$$F = F(k, l, u_{0,0}, u_{0,1}), \qquad G = G(k, l, u_{0,0}, u_{1,0}). \tag{3}$$

We use the method of calculating conservation laws described in [7, 8], which works as follows. The three-point conservation laws can be determined directly by substituting (1) into

$$F(k+1, l, u_{1,0}, u_{1,1}) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, u_{1,1}) - G(k, l, u_{0,0}, u_{1,0}) = 0, \tag{4}$$

and solving the resulting functional equation. Suppose that (1) can be solved for $u_{1,1}$ as follows:

$$u_{1,1} = \omega(k, l, u_{0,0}, u_{1,0}, u_{0,1}).$$

Then (4) amounts to

$$F(k+1, l, u_{1,0}, \omega) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, \omega) - G(k, l, u_{0,0}, u_{1,0}) = 0. \tag{5}$$

In order to solve this functional equation we have to reduce it to a system of partial differential equations. To do this, first eliminate functional terms $F(k+1,l,u_{1,0},\omega)$ and $G(k,l+1,u_{0,1},\omega)$ by applying each of the following (commuting) differential operators to (5):

$$L_1 = \frac{\partial}{\partial u_{0,1}} - \frac{\omega_{u_{0,1}}}{\omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}}\,, \qquad L_2 = \frac{\partial}{\partial u_{1,0}} - \frac{\omega_{u_{1,0}}}{\omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}}\,,$$

where $\omega_{u_{i,j}}$ denotes $\frac{\partial \omega}{\partial u_{i,j}}$. The operators $L_1$ and $L_2$ differentiate with respect to $u_{0,1}$ and $u_{1,0}$ respectively, keeping $\omega$ fixed, so

$$L_1\left(F(k+1,l,u_{1,0},\omega)\right) = 0\,, \qquad L_2\left(G(k,l+1,u_{0,1},\omega)\right) = 0.$$

This procedure does not depend upon the form of $\omega$; it can be applied equally to any quad-graph equation. Applying $L_1$ and $L_2$ to (5) yields

$$L_1 L_2\Big(F(k,l,u_{0,0},u_{0,1}) + G(k,l,u_{0,0},u_{1,0})\Big) = 0. \tag{6}$$

If (6) is divided by the factor that multiplies a particular derivative of $G(k,l,u_{0,0},u_{1,0})$ and is then differentiated with respect to $u_{0,1}$, we obtain a functional differential equation which is independent of that derivative. This process is repeated for each derivative of $G(k,l,u_{0,0},u_{1,0})$ and finally for $G(k,l,u_{0,0},u_{1,0})$ itself; this produces a PDE for $F(k,l,u_{0,0},u_{0,1})$. If the coefficients involve $u_{1,0}$, the PDE can be split into a system of PDEs.

Further information about $F$ may be found by substituting

$$u_{1,0} = \Omega(k,l,u_{0,0},u_{0,1},u_{1,1})$$

into (4). Here $u_{1,0} = \Omega$ is another representation of (1). Then (4) amounts to

$$F(k+1,l,\Omega,u_{1,1}) - F(k,l,u_{0,0},u_{0,1}) + G(k,l+1,u_{0,1},u_{1,1}) - G(k,l,u_{0,0},\Omega) = 0. \tag{7}$$

We eliminate the terms $F(k+1,l,\Omega,u_{1,1})$ and $G(k,l,u_{0,0},\Omega)$ by applying each of the following (commuting) differential operators to (7):

$$L_3 = \frac{\partial}{\partial u_{0,1}} - \frac{\Omega_{u_{0,1}}}{\Omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}}\,, \qquad L_4 = \frac{\partial}{\partial u_{1,1}} - \frac{\Omega_{u_{1,1}}}{\Omega_{u_{0,0}}}\frac{\partial}{\partial u_{0,0}}\,.$$

This yields

$$L_3 L_4\Big(-F(k,l,u_{0,0},u_{0,1}) + G(k,l+1,u_{0,1},u_{1,1})\Big) = 0.$$

This equation can also be reduced to a system of partial differential equations for $F(k,l,u_{0,0},u_{0,1})$ which (typically) is different from obtained previously.

Having differentiated the determining equation for a conservation law several times, we have created a hierarchy of functional differential equations that every three-point conservation law must satisfy. The functions $F$ and $G$ can be determined completely by going up the hierarchy, a step at a time, to determine the constraints that these equations place on the unknown functions. As the constraints are solved sequentially, more and more information is gained about the functions. At the highest stage, the determining equation (5) is satisfied, and the only remaining unknowns are the constants that multiply each conservation law. This is a straightforward but very lengthy process; for brevity, we shall omit the details of these calculations in our analysis of the ABS equations.

## 3. Three-point conservation laws for integrable equations on the quad-graph

In this section we present all three-point conservation laws for integrable equations on the quad-graph that are listed in [1]; these were found by the method described in the previous section.

The ABS equations depend on two arbitrary functions $\alpha = \alpha(k)$ and $\beta = \beta(l)$. The equations from the ABS classification are as follows; for convenience, we have used the form of **Q4** that was discovered by Hietarinta [4].

$$\begin{aligned}
&\mathbf{Q1}: && \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \delta^2\alpha\beta(\alpha - \beta) = 0,\\
&\mathbf{Q2}: && \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1})\\
& && + \alpha\beta(\alpha - \beta)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) = 0,\\
&\mathbf{Q3}: && (\beta^2 - \alpha^2)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + \beta(\alpha^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1})\\
& && - \alpha(\beta^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)/(4\alpha\beta) = 0,\\
&\mathbf{Q4}: && \mathrm{sn}(\alpha)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \mathrm{sn}(\beta)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \mathrm{sn}(\alpha - \beta)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1})\\
& && + \mathrm{sn}(\alpha - \beta)\mathrm{sn}(\alpha)\mathrm{sn}(\beta)(1 + K^2u_{0,0}u_{1,0}u_{0,1}u_{1,1}) = 0,\\
&\mathbf{H1}: && (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + \beta - \alpha = 0,\\
&\mathbf{H2}: && (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + (\beta - \alpha)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) + \beta^2 - \alpha^2 = 0,\\
&\mathbf{H3}: && \alpha(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) + \delta^2(\alpha^2 - \beta^2) = 0,\\
&\mathbf{A1}: && \alpha(u_{0,0} + u_{0,1})(u_{1,0} + u_{1,1}) - \beta(u_{0,0} + u_{1,0})(u_{0,1} + u_{1,1}) - \delta^2\alpha\beta(\alpha - \beta) = 0,\\
&\mathbf{A2}: && (\beta^2 - \alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1} + 1) + \beta(\alpha^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1})\\
& && - \alpha(\beta^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) = 0.
\end{aligned}$$

Here $\mathrm{sn}(\alpha) = \mathrm{sn}(\alpha; K)$ is a Jacobi elliptic function with modulus $K$. Without loss of generality, the parameter $\delta$ is restricted to the values 0 and 1.

All three-point conservation laws for these equations are listed in Table 1. We omit the details of our calculations, which were carried out using the computer algebra system Maple; they are very complex and it is impossible to present them in any suitable form. Three-point conservation laws for **H1**, $\mathbf{H3}_{\delta=0}$ and $\mathbf{Q1}_{\delta=0}$ have already appeared in [5, 7, 8]. One conservation law for **Q4** involves the following Jacobi elliptic functions with modulus $K$:

$$\mathrm{cn}(\alpha) = \mathrm{cn}(\alpha; K), \qquad \mathrm{ns}(\alpha) = \mathrm{ns}(\alpha; K), \qquad \mathrm{dn}(\alpha) = \mathrm{dn}(\alpha; K).$$

Note that in all three-point conservation laws for ABS equations, the component $F$ does not depend upon $\alpha$ and $G$ does not depend upon $\beta$. Therefore the conservation laws from Table 1 are valid for all functions $\alpha = \alpha(k)$ and $\beta = \beta(l)$.

Table 1: **Three-point conservation laws for equations from the ABS classification**

| **Eq.** | **Generators** |
|---|---|
| $\mathbf{Q1}_{\delta=0}$ | $F_1 = -\beta(u_{0,0}-u_{0,1})^{-1},\quad F_2 = -\beta u_{0,0}(u_{0,0}-u_{0,1})^{-1},\quad F_3 = -\beta u_{0,0}u_{0,1}(u_{0,0}-u_{0,1})^{-1},\quad F_4 = -(-1)^{k+l}\left(2\ln\left(u_{0,0}-u_{0,1}\right)-\ln(\beta)\right),$<br>$G_1 = \alpha(u_{0,0}-u_{1,0})^{-1},\quad G_2 = \alpha u_{1,0}(u_{0,0}-u_{1,0})^{-1},\quad G_3 = \alpha u_{0,0}u_{1,0}(u_{0,0}-u_{1,0})^{-1},\quad G_4 = (-1)^{k+l}\left(2\ln\left(u_{0,0}-u_{1,0}\right)-\ln(\alpha)\right),$ |
| $\mathbf{Q1}_{\delta=1}$ | $F_1 = \ln\left(\beta-u_{0,1}+u_{0,0}\right)-\ln\left(\beta-u_{0,0}+u_{0,1}\right),\quad F_2 = -(-1)^{k+l}\left(\ln\left(\beta-u_{0,1}+u_{0,0}\right)+\ln\left(\beta-u_{0,0}+u_{0,1}\right)-\ln(\beta)\right),$<br>$G_1 = \ln\left(\alpha-u_{0,0}+u_{1,0}\right)-\ln\left(\alpha-u_{1,0}+u_{0,0}\right),\quad G_2 = (-1)^{k+l}\left(\ln\left(\alpha-u_{0,0}+u_{1,0}\right)+\ln(\alpha+u_{0,0}-u_{1,0})-\ln(\alpha)\right),$ |
| $\mathbf{Q2}$ | $F_1 = -(-1)^{k+l}\left(\ln\left(u_{0,0}^2+(\beta^2-u_{0,1})^2-2u_{0,0}(\beta^2+u_{0,1})\right)-\ln(\beta)\right),$<br>$G_1 = (-1)^{k+l}\left(\ln\left(u_{0,0}^2+(\alpha^2-u_{1,0})^2-2u_{0,0}(\alpha^2+u_{1,0})\right)-\ln(\alpha)\right),$ |
| $\mathbf{Q3}_{\delta=0}$ | $F_1 = \ln\left(u_{0,1}-\beta u_{0,0}\right)-\ln\left(\beta u_{0,1}-u_{0,0}\right),\quad F_2 = -(-1)^{k+l}\left(\ln\left(u_{0,1}-\beta u_{0,0}\right)+\ln\left(\beta u_{0,1}-u_{0,0}\right)-\ln(\beta^2-1)\right),$<br>$G_1 = \ln\left(\alpha u_{1,0}-u_{0,0}\right)-\ln\left(u_{1,0}-\alpha u_{0,0}\right),\quad G_2 = (-1)^{k+l}\left(\ln\left(\alpha u_{1,0}-u_{0,0}\right)+\ln\left(u_{1,0}-\alpha u_{0,0}\right)-\ln(\alpha^2-1)\right),$ |
| $\mathbf{Q3}_{\delta=1}$ | $F_1 = -(-1)^{k+l}\left(\ln\left((1-\beta^2)^2+4\beta(u_{0,1}-\beta u_{0,0})(\beta u_{0,1}-u_{0,0})\right)-\ln\left(\beta(\beta^2-1)\right)\right),$<br>$G_1 = (-1)^{k+l}\left(\ln\left((1-\alpha^2)^2+4\alpha(u_{1,0}-\alpha u_{0,0})(\alpha u_{1,0}-u_{0,0})\right)-\ln\left(\alpha\left(\alpha^2-1\right)\right)\right),$ |
| $\mathbf{Q4}$<br><br>$K=\pm1$ | $F_1 = -(-1)^{k+l}\ln\left(\mathrm{ns}(\beta)\left(u_{0,1}^2+u_{0,0}^2\right)-K^2\mathrm{sn}(\beta)\,u_{0,1}^2u_{0,0}^2-2\mathrm{cn}(\beta)\mathrm{dn}(\beta)\mathrm{ns}(\beta)\,u_{0,1}u_{0,0}-\mathrm{sn}(\beta)\right),$<br>$G_1 = (-1)^{k+l}\ln\left(\mathrm{ns}(\alpha)\left(u_{1,0}^2+u_{0,0}^2\right)-K^2\mathrm{sn}(\alpha)\,u_{1,0}^2u_{0,0}^2-2\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)\mathrm{ns}(\alpha)\,u_{1,0}u_{0,0}-\mathrm{sn}(\alpha)\right),$<br>$F_2 = -\ln(\tanh(\beta)u_{0,0}u_{0,1}+u_{0,0}-u_{0,1}-\tanh(\beta))+\ln(\tanh(\beta)u_{0,0}u_{0,1}-u_{0,0}+u_{0,1}-\tanh(\beta)),$<br>$G_2 = \ln(\tanh(\alpha)u_{0,0}u_{1,0}+u_{0,0}-u_{1,0}-\tanh(\alpha))-\ln(\tanh(\alpha)u_{0,0}u_{1,0}-u_{0,0}+u_{1,0}-\tanh(\alpha)),$ |

*Continued on next page*

| Eq. | Generators |
| --- | --- |
| **H1** | $F_1 = -(-1)^{k+l}\,(2u_{0,0}u_{0,1}-\beta)\,,\quad F_2 = -\,(u_{0,0}-u_{0,1})\,(u_{0,0}u_{0,1}-\beta)\,,\quad F_3 = -(-1)^{k+l}\,(u_{0,0}+u_{0,1})\,(u_{0,0}u_{0,1}-\beta)\,,$ <br> $G_1 = (-1)^{k+l}\,(2u_{0,0}u_{1,0}-\alpha)\,,\quad G_2 = (u_{0,0}-u_{1,0})\,(u_{0,0}u_{1,0}-\alpha)\,,\quad G_3 = (-1)^{k+l}\,(u_{0,0}+u_{1,0})\,(u_{0,0}u_{1,0}-\alpha)\,,$ <br> $F_4 = -(-1)^{k+l}\left(2{u_{0,0}}^2{u_{0,1}}^2-4\beta u_{0,0}u_{0,1}+\beta^2\right),$ <br> $G_4 = (-1)^{k+l}\left(2{u_{0,0}}^2{u_{1,0}}^2-4\alpha u_{0,0}u_{1,0}+\alpha^2\right),$ |
| **H2** | $F_1 = -(-1)^{k+l}\left(2u_{0,0}u_{0,1}-\beta^2-2\beta u_{0,0}-2\beta u_{0,1}\right),\quad F_2 = -(-1)^{k+l}\ln(\beta+u_{0,0}+u_{0,1})\,,$ <br> $G_1 = (-1)^{k+l}\left(2u_{0,0}u_{1,0}-\alpha^2-2\alpha u_{0,0}-2\alpha u_{1,0}\right),\quad G_2 = (-1)^{k+l}\ln(\alpha+u_{0,0}+u_{1,0})\,,$ |
| **H3**$_{\delta=0}$ | $F_1 = -(-1)^{k+l}\beta u_{0,0}u_{0,1},\quad F_2 = -(-1)^{k+l}\beta(u_{0,0}u_{0,1})^{-1},\quad F_3 = ({u_{0,0}}^2-{u_{0,1}}^2)(\beta u_{0,0}u_{0,1})^{-1},\quad F_4 = -(-1)^{k+l}\left({u_{0,0}}^2+{u_{0,1}}^2\right)(\beta u_{0,0}u_{0,1})^{-1},$ <br> $G_1 = (-1)^{k+l}\alpha u_{0,0}u_{1,0},\quad G_2 = (-1)^{k+l}\alpha(u_{0,0}u_{1,0})^{-1},\quad G_3 = ({u_{1,0}}^2-{u_{0,0}}^2)(\alpha u_{0,0}u_{1,0})^{-1},\quad G_4 = (-1)^{k+l}\left({u_{0,0}}^2+{u_{1,0}}^2\right)(\alpha u_{0,0}u_{1,0})^{-1},$ |
| **H3**$_{\delta=1}$ | $F_1 = -(-1)^{k+l}\ln(\beta+u_{0,0}u_{0,1})\,,\quad F_2 = -(-1)^{k+l}\beta\,(\beta+2u_{0,0}u_{0,1})\,,$ <br> $G_1 = (-1)^{k+l}\ln(\alpha+u_{0,0}u_{1,0})\,,\quad G_2 = (-1)^{k+l}\alpha\,(\alpha+2u_{0,0}u_{1,0})\,,$ |
| **A1**$_{\delta=0}$ | $F_1 = -(-1)^{k+l}\beta(u_{0,0}+u_{0,1})^{-1},\quad F_2 = \beta u_{0,0}(u_{0,0}+u_{0,1})^{-1},\quad F_3 = -(-1)^{k+l}\beta u_{0,0}u_{0,1}(u_{0,0}+u_{0,1})^{-1},$ <br> $G_1 = (-1)^{k+l}\alpha(u_{0,0}+u_{1,0})^{-1},\quad G_2 = \alpha u_{1,0}(u_{0,0}+u_{1,0})^{-1},\quad G_3 = (-1)^{k+l}\alpha u_{0,0}u_{1,0}(u_{0,0}+u_{1,0})^{-1},$ <br> $F_4 = -(-1)^{k+l}\,(2\ln(u_{0,0}+u_{0,1})-\ln(\beta))\,,$ <br> $G_4 = (-1)^{k+l}\,(2\ln(u_{0,0}+u_{1,0})-\ln(\alpha))\,,$ |
| **A1**$_{\delta=1}$ | $F_1 = -(-1)^{k+l}\,(2\ln(u_{0,0}+u_{0,1}+\beta)-\ln(\beta))\,,\quad F_2 = -(-1)^{k+l}\,(2\ln(u_{0,0}+u_{0,1}-\beta)-\ln(\beta))\,,$ <br> $G_1 = (-1)^{k+l}\,(2\ln(u_{0,0}+u_{1,0}+\alpha)-\ln(\alpha))\,,\quad G_2 = (-1)^{k+l}\,(2\ln(u_{0,0}+u_{1,0}-\alpha)-\ln(\alpha))\,,$ |
| **A2** | $F_1 = -(-1)^{k+l}\left(2\ln(u_{0,0}u_{0,1}-\beta)-\ln(\beta^2-1)\right),\quad F_2 = -(-1)^{k+l}\left(2\ln(\beta u_{0,0}u_{0,1}-1)-\ln(\beta^2-1)\right),$ <br> $G_1 = (-1)^{k+l}\left(2\ln(u_{0,0}u_{1,0}-\alpha)-\ln(\alpha^2-1)\right),\quad G_2 = (-1)^{k+l}\left(2\ln(\alpha u_{0,0}u_{1,0}-1)-\ln(\alpha^2-1)\right).$ |

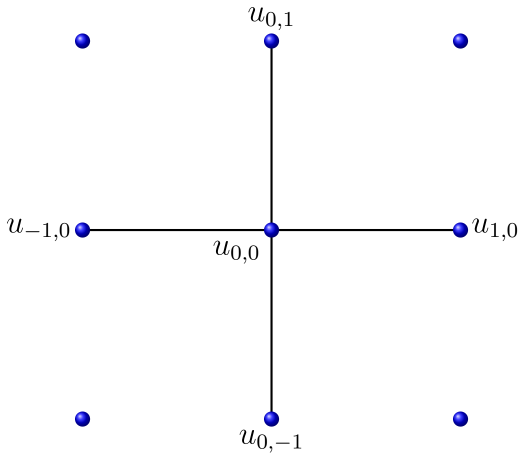


Figure 1: Form of a five-point conservation law

## 4. Five-point conservation laws

The simplest higher conservation laws are defined on five points (Figure 1). The functions $F$ and $G$ are of the form

$$F = F(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad G = G(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{1,0}). \tag{8}$$

Therefore the determining equation for the five-point conservation laws is

$$\begin{aligned} &F(k+1, l, u_{1,-1}, u_{0,0}, u_{1,0}, u_{1,1}) - F(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{0,1}) + \\ &\quad G(k, l+1, u_{0,0}, u_{-1,1}, u_{0,1}, u_{1,1}) - G(k, l, u_{0,-1}, u_{-1,0}, u_{0,0}, u_{1,0}) = 0. \end{aligned} \tag{9}$$

Shifted versions of each quad-graph equation are used to eliminate $u_{-1,1}, u_{1,-1}$ and $u_{1,1}$.

For **H1**, we have previously shown that there are three five-point conservation laws, apart from those that are equivalent to three-point conservation laws [7]. In the current investigation, the direct method also yields three five-point conservation laws for $\mathbf{H3}_{\delta=0}$.

For the other equations, the complexity of the calculations has prevented us from solving the determining equation (9) directly when $F$ and $G$ are of the form (8). However, for each of the equations **H1** and $\mathbf{H3}_{\delta=0}$, the three five-point conservation laws can be written in the form

$$\begin{aligned} F_1 &= F(k, l, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad & G_1 &= G(k, l, u_{-1,0}, u_{0,0}, u_{1,0}), \\ F_2 &= F(l, k, u_{0,-1}, u_{0,0}, u_{1,0}), & G_2 &= G(l, k, u_{0,-1}, u_{0,0}, u_{0,1}), \\ F_3 &= kF_1 + lF_2, & G_3 &= kG_1 + lG_2. \end{aligned}$$

This suggests that, for each of the remaining ABS equations, we should seek a five-point conservation law of the form

$$F = F(k, l, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad G = G(k, l, u_{-1,0}, u_{0,0}, u_{1,0}). \tag{10}$$

By substituting (10) into (2) we obtain a determining equation that is simpler than (9):

$$F(k+1,l,u_{0,0},u_{1,0},u_{1,1}) - F(k,l,u_{-1,0},u_{0,0},u_{0,1}) + \\ G(k,l+1,u_{-1,1},u_{0,1},u_{1,1}) - G(k,l,u_{-1,0},u_{0,0},u_{1,0}) = 0. \quad (11)$$

Shifted versions of the quad-graph equation are used to eliminate $u_{-1,1}$ and $u_{1,1}$.

By using the direct method to solve (11) we found one five-point conservation law for $\mathbf{Q1}_{\delta=0,1}$, $\mathbf{Q3}_{\delta=0}$, $\mathbf{H2}$, $\mathbf{H3}_{\delta=1}$, $\mathbf{A1}_{\delta=0,1}$, $\mathbf{A2}$. Let

$$F_1 = F_s(k,l,u_{-1,0},u_{0,0},u_{0,1}), \quad G_1 = G_s(k,l,u_{-1,0},u_{0,0},u_{1,0}) \quad (12)$$

be the solution of (11) for an ABS equation. All ABS equations are invariant under the transformation

$$k \to \tilde{l}, \quad l \to \tilde{k}.$$

Therefore each of the above equations has a second five-point conservation law,

$$F_2 = F_s(l,k,u_{0,-1},u_{0,0},u_{0,1}), \quad G_2 = G_s(l,k,u_{0,-1},u_{0,0},u_{1,0}). \quad (13)$$

For equations $\mathbf{Q2}$, $\mathbf{Q3}_{\delta=1}$ and $\mathbf{Q4}$, we could not solve the simplified determining equation (11) directly. However, we observed that each of the other ABS equations has two five-point conservation laws of the form:

$$\begin{aligned} F_1 &= -S_k^{-1}(-1)^{k+l}\tilde{G} + f(u_{-1,0},u_{0,0},u_{0,1}), \\ G_1 &= S_k^{-1}(-1)^{k+l}\tilde{G} - a\ln(u_{1,0}-u_{-1,0}), \\ F_2 &= S_l^{-1}(-1)^{k+l}\tilde{F} + a\ln(u_{0,1}-u_{0,-1}), \\ G_2 &= -S_l^{-1}(-1)^{k+l}\tilde{F} - f(u_{0,-1},u_{0,0},u_{1,0}). \end{aligned} \quad (14)$$

Here $f$ is a function and $a$ is a constant; furthermore, $\tilde{F}$ and $\tilde{G}$ are components of a three-point conservation law of the same equation, of the form

$$\tilde{F} = (-1)^{k+l}\ln(\ldots), \quad \tilde{G} = (-1)^{k+l}\ln(\ldots). \quad (15)$$

Table 1 shows that most equations from the ABS classification have a three-point conservation law of the form (15); the only exceptions are $\mathbf{H1}$, $\mathbf{H3}_{\delta=0}$, whose five-point conservation laws we have already found. Therefore we have sought two five-point conservation laws that can be written in the form (14) for each of the remaining equations $\mathbf{Q2}$, $\mathbf{Q3}_{\delta=1}$ and $\mathbf{Q4}$. By substituting $F_1$, $G_1$ from (14) into (2) we obtain the determining equation for $f$ and $a$. For each of $\mathbf{Q2}$, $\mathbf{Q3}_{\delta=1}$ and $\mathbf{Q4}$, this determining equation can be solved by the direct method.

So far we have described how to find two five-point conservation laws for all ABS equations. Our results for $\mathbf{H1}$ and $\mathbf{H3}_{\delta=0}$ suggest that other equations from the ABS classification may have a third conservation law that is related to the other two as follows:

$$F_3 = kF_1 + lF_2, \quad G_3 = kG_1 + lG_2. \quad (16)$$

We find that in (16) the two conservation laws must be written in the form

$$\begin{aligned}
F_1 &= -S_l S_k^{-1}(-1)^{k+l}\tilde{G} - S_k^{-1}(-1)^{k+l}\tilde{G} + a_1 f(u_{-1,0}, u_{0,0}, u_{0,1}) + a_2,\\
G_1 &= (-1)^{k+l}\tilde{G} + S_k^{-1}(-1)^{k+l}\tilde{G} - a_3 \ln(u_{1,0} - u_{-1,0}),\\
F_2 &= (-1)^{k+l}\tilde{F} + S_l^{-1}(-1)^{k+l}\tilde{F} + a_3 \ln(u_{0,1} - u_{0,-1}),\\
G_2 &= -S_k S_l^{-1}(-1)^{k+l}\tilde{F} - S_l^{-1}(-1)^{k+l}\tilde{F} - a_1 f(u_{0,-1}, u_{0,0}, u_{1,0}) + a_2. \qquad (17)
\end{aligned}$$

Here $\tilde{F}$, $\tilde{G}$ and $f$ are the same as in (14), and the constants $a_i$ can be found by substituting (16) into (2). The terms $S_l S_k^{-1}(-1)^{k+l}\tilde{G}$ and $S_k S_l^{-1}(-1)^{k+l}\tilde{F}$ depend on $u_{-1,1}, u_{1,-1}$, which do not lie on the cross (8); these variables can be eliminated by shifted versions of the quad-graph equation.

The results of the above are summarized in Table 2, in which we list the generators $\bar{F}_i$ and $\bar{G}_i$ of the five-point conservation laws for each of the ABS equations. The corresponding conservation laws are

$$\begin{aligned}
(F_1, G_1) &= (\bar{F}_1, \bar{G}_1),\\
(F_2, G_2) &= (\bar{F}_2, \bar{G}_2),\\
(F_3, G_3) &= (k\bar{F}_1 + l\bar{F}_2, k\bar{G}_1 + l\bar{G}_2).
\end{aligned}$$

In Table 2 we use $F_n$ and $G_n$ to denote the components of $n^{\text{th}}$ three-point conservation law for the same equation as given in Table 1. For **Q4** alone, we have presented the result without eliminating $u_{-1,1}, u_{1,-1}$, as this is far shorter than the result after elimination.

All of the three-point conservation laws apply to nonautonomous equations, for which $\alpha$ and $\beta$ are not constants. However, each equation from the ABS classification has only one five-point conservation law whose component $G$ does not depend upon $\alpha$ and one five-point conservation law for which $F$ does not depend upon $\beta$. Consequently, if exactly one of $\alpha$ and $\beta$ is constant then only one of the five-point conservation laws survives. If neither $\alpha$ nor $\beta$ is constant, none of the above conservation laws hold.

Table 2: **Generators for five-point conservation laws for equations from the ABS classification**

| Eq. | Generators |
|---|---|
| $\mathbf{Q1}_{\delta=0}$ | $\bar{F}_1=(-1)^{k+l}F_4-S_k^{-1}(-1)^{k+l}G_4+2\ln((u_{0,1}-u_{-1,0})(\alpha(u_{0,1}-u_{0,0})+\beta(u_{0,0}-u_{-1,0})))$, |
| | $\bar{G}_1=(-1)^{k+l}G_4+S_k^{-1}(-1)^{k+l}G_4-4\ln(u_{1,0}-u_{-1,0})$, |
| | $\bar{F}_2=(-1)^{k+l}F_4+S_l^{-1}(-1)^{k+l}F_4+4\ln(u_{0,1}-u_{0,-1})$, |
| | $\bar{G}_2=(-1)^{k+l}G_4-S_l^{-1}(-1)^{k+l}F_4-2\ln((u_{1,0}-u_{0,-1})(\alpha(u_{0,0}-u_{0,-1})+\beta(u_{1,0}-u_{0,0})))$, |
| $\mathbf{Q1}_{\delta=1}$ | $\bar{F}_1=-\ln((u_{0,1}-u_{0,0}+\beta)(u_{0,0}-u_{-1,0}-\alpha)(u_{0,1}-u_{-1,0}-\alpha+\beta)^{-1}(\alpha(u_{0,1}-u_{0,0})+\beta(u_{0,0}-u_{-1,0}))^{-1})$, |
| | $\bar{G}_1=\ln((u_{1,0}-u_{0,0}+\alpha)(u_{0,0}-u_{-1,0}-\alpha)(u_{1,0}-u_{-1,0})^{-2})$, |
| | $\bar{F}_2=-\ln((u_{0,0}-u_{0,-1}-\beta)(u_{0,1}-u_{0,0}+\beta)(u_{0,1}-u_{0,-1})^{-2})$, |
| | $\bar{G}_2=\ln((u_{0,0}-u_{0,-1}-\beta)(u_{1,0}-u_{0,0}+\alpha)(u_{1,0}-u_{0,-1}+\alpha-\beta)^{-1}(\beta(u_{1,0}-u_{0,0})+\alpha(u_{0,0}-u_{0,-1}))^{-1})$, |
| $\mathbf{Q2}$ | $\bar{F}_1=(-1)^{k+l}F_1-S_k^{-1}(-1)^{k+l}G_1+\ln((u_{-1,0}^2+((\alpha-\beta)^2-u_{0,1})^2-2u_{-1,0}((\alpha-\beta)^2+u_{0,1}))(\alpha u_{0,1}+(\alpha-\beta)(\alpha\beta-u_{0,0})-\beta u_{-1,0})^2)$, |
| | $\bar{G}_1=(-1)^{k+l}G_1+S_k^{-1}(-1)^{k+l}G_1-4\ln(u_{1,0}-u_{-1,0})$, |
| | $\bar{F}_2=(-1)^{k+l}F_1+S_l^{-1}(-1)^{k+l}F_1+4\ln(u_{0,1}-u_{0,-1})$, |
| | $\bar{G}_2=(-1)^{k+l}G_1-S_l^{-1}(-1)^{k+l}F_1-\ln((u_{0,-1}^2+((\alpha-\beta)^2-u_{1,0})^2-2u_{0,-1}((\alpha-\beta)^2+u_{1,0}))(\alpha u_{0,-1}+(\alpha-\beta)(\alpha\beta-u_{0,0})-\beta u_{1,0})^2)$, |
| $\mathbf{Q3}_{\delta=0}$ | $\bar{F}_1=-\ln\left((\alpha u_{-1,0}-u_{0,0})(\beta u_{0,1}-u_{0,0})(\beta u_{0,1}-\alpha u_{-1,0})^{-1}(\beta(1-\alpha^2)u_{0,1}+(\alpha^2-\beta^2)u_{0,0}-\alpha(1-\beta^2)u_{-1,0})^{-1}\right)$, |
| | $\bar{G}_1=\ln\left((\alpha u_{1,0}-u_{0,0})(u_{0,0}-\alpha u_{-1,0})(u_{1,0}-u_{-1,0})^{-2}\right)$, |
| | $\bar{F}_2=-\ln\left((\beta u_{0,1}-u_{0,0})(\beta u_{0,-1}-u_{0,0})(u_{0,1}-u_{0,-1})^{-2}\right)$, |
| | $\bar{G}_2=\ln\left((\alpha u_{1,0}-u_{0,0})(u_{0,0}-\beta u_{0,-1})(\alpha u_{1,0}-\beta u_{0,-1})^{-1}(\beta(1-\alpha^2)u_{0,-1}+(\alpha^2-\beta^2)u_{0,0}-\alpha(1-\beta^2)u_{1,0})^{-1}\right)$, |

*Continued on next page*

| Eq. | Generators |
| --- | --- |
| $\mathbf{Q3}_{\delta=1}$ | $\bar{F}_1 = (-1)^{k+l}F_1 - S_k^{-1}(-1)^{k+l}G_1 + \ln(((\alpha^2-\beta^2)^2 + 4\alpha\beta(\alpha u_{0,1} - \beta u_{-1,0})(\beta u_{0,1} - \alpha u_{-1,0}))(\beta(1-\alpha^2)u_{0,1} + (\alpha^2-\beta^2)u_{0,0} - \alpha(1-\beta^2)u_{-1,0})^2)$, |
| | $\bar{G}_1 = (-1)^{k+l}G_1 + S_k^{-1}(-1)^{k+l}G_1 - 4\ln(u_{1,0} - u_{-1,0})$, |
| | $\bar{F}_2 = (-1)^{k+l}F_1 + S_l^{-1}(-1)^{k+l}F_1 + 4\ln(u_{0,1} - u_{0,-1})$, |
| | $\bar{G}_2 = (-1)^{k+l}G_1 - S_l^{-1}(-1)^{k+l}F_1 - \ln(((\alpha^2-\beta^2)^2 + 4\alpha\beta(\alpha u_{1,0} - \beta u_{0,-1})(\beta u_{1,0} - \alpha u_{0,-1}))(\beta(1-\alpha^2)u_{0,-1} + (\alpha^2-\beta^2)u_{0,0} - \alpha(1-\beta^2)u_{1,0})^2)$, |
| $\mathbf{Q4}$ | $\bar{F}_1 = -S_l S_k^{-1}(-1)^{k+l}G_1 - S_k^{-1}(-1)^{k+l}G_1 +$ <br> $2\ln((\mathrm{sn}(\alpha-\beta)^2(1+K^2\mathrm{sn}(\beta)^2\mathrm{sn}(\alpha)^2) - \mathrm{sn}(\beta)^2 - \mathrm{sn}(\alpha)^2)u_{0,1}u_{-1,0} + \mathrm{sn}(\alpha)\mathrm{sn}(\beta)(u_{0,1}^2 + u_{-1,0}^2 - \mathrm{sn}(\alpha-\beta)^2(1+K^2u_{0,1}^2u_{-1,0}^2)))$, |
| | $\bar{G}_1 = (-1)^{k+l}G_1 + S_k^{-1}(-1)^{k+l}G_1 - 4\ln(u_{1,0} - u_{-1,0})$, |
| | $\bar{F}_2 = (-1)^{k+l}F_1 + S_l^{-1}(-1)^{k+l}F_1 + 4\ln(u_{0,1} - u_{0,-1})$, |
| | $\bar{G}_2 = -S_k S_l^{-1}(-1)^{k+l}F_1 - S_l^{-1}(-1)^{k+l}F_1 -$ <br> $2\ln((\mathrm{sn}(\alpha-\beta)^2(1+K^2\mathrm{sn}(\beta)^2\mathrm{sn}(\alpha)^2) - \mathrm{sn}(\beta)^2 - \mathrm{sn}(\alpha)^2)u_{1,0}u_{0,-1} + \mathrm{sn}(\alpha)\mathrm{sn}(\beta)(u_{1,0}^2 + u_{0,-1}^2 - \mathrm{sn}(\alpha-\beta)^2(1+K^2u_{1,0}^2u_{0,-1}^2)))$, |
| $\mathbf{H1}$ | $\bar{F}_1 = -\ln(u_{0,1} - u_{-1,0})\,,\quad \bar{F}_2 = -\ln(u_{0,1} - u_{0,-1})\,,$ |
| | $\bar{G}_1 = \ln(u_{1,0} - u_{-1,0})\,,\quad \bar{G}_2 = \ln(u_{1,0} - u_{0,-1})$, |
| $\mathbf{H2}$ | $\bar{F}_1 = (-1)^{k+l}F_2 - S_k^{-1}(-1)^{k+l}G_2 + \ln((\beta-\alpha-u_{0,1}+u_{-1,0})(\beta-\alpha+u_{0,1}-u_{-1,0})^3)$, |
| | $\bar{G}_1 = (-1)^{k+l}G_2 + S_k^{-1}(-1)^{k+l}G_2 - 4\ln(u_{1,0} - u_{-1,0})$, |
| | $\bar{F}_2 = (-1)^{k+l}F_2 + S_l^{-1}(-1)^{k+l}F_2 + 4\ln(u_{0,1} - u_{0,-1})$, |
| | $\bar{G}_2 = (-1)^{k+l}G_2 - S_l^{-1}(-1)^{k+l}F_2 - \ln((\alpha-\beta-u_{1,0}+u_{0,-1})(\alpha-\beta+u_{1,0}-u_{0,-1})^3)$, |

*Continued on next page*

| Eq. | Generators |
|---|---|
| $\mathbf{H3}_{\delta=0}$ | $\bar{F}_1 = -\ln\left((\alpha u_{0,1} - \beta u_{-1,0}) u_{0,0}^{-1}\right), \quad \bar{F}_2 = -\ln\left((u_{0,1} - u_{0,-1}) u_{0,0}^{-1}\right),$<br>$\bar{G}_1 = \ln\left((u_{1,0} - u_{-1,0}) u_{0,0}^{-1}\right), \quad \bar{G}_2 = \ln\left((\beta u_{1,0} - \alpha u_{0,-1}) u_{0,0}^{-1}\right),$ |
| $\mathbf{H3}_{\delta=1}$ | $\bar{F}_1 = (-1)^{k+l} F_1 - S_k^{-1}(-1)^{k+l} G_1 + \ln((\beta u_{0,1} - \alpha u_{-1,0})(\alpha u_{0,1} - \beta u_{-1,0})^3),$<br>$\bar{G}_1 = (-1)^{k+l} G_1 + S_k^{-1}(-1)^{k+l} G_1 - 4\ln(u_{1,0} - u_{-1,0}),$<br>$\bar{F}_2 = (-1)^{k+l} F_1 + S_l^{-1}(-1)^{k+l} F_1 + 4\ln(u_{0,1} - u_{0,-1}),$<br>$\bar{G}_2 = (-1)^{k+l} G_1 - S_l^{-1}(-1)^{k+l} F_1 - \ln((\alpha u_{1,0} - \beta u_{0,-1})(\beta u_{1,0} - \alpha u_{0,-1})^3),$ |
| $\mathbf{A1}_{\delta=0}$ | $\bar{F}_1 = (-1)^{k+l} F_4 - S_k^{-1}(-1)^{k+l} G_4 + 2\ln((u_{0,1} - u_{-1,0})(\alpha(u_{0,1} + u_{0,0}) - \beta(u_{0,0} + u_{-1,0}))),$<br>$\bar{G}_1 = (-1)^{k+l} G_4 + S_k^{-1}(-1)^{k+l} G_4 - 4\ln(u_{1,0} - u_{-1,0}),$<br>$\bar{F}_2 = (-1)^{k+l} F_4 + S_l^{-1}(-1)^{k+l} F_4 + 4\ln(u_{0,1} - u_{0,-1}),$<br>$\bar{G}_2 = (-1)^{k+l} G_4 - S_l^{-1}(-1)^{k+l} F_4 - 2\ln((u_{1,0} - u_{0,-1})(\beta(u_{1,0} + u_{0,0}) - \alpha(u_{0,0} + u_{0,-1}))),$ |
| $\mathbf{A1}_{\delta=1}$ | $\bar{F}_1 = (-1)^{k+l}(F_1 + F_2) - S_k^{-1}(-1)^{k+l}(G_1 + G_2) + 2\ln(((\alpha - \beta)^2 - (u_{0,1} - u_{-1,0})^2)(\alpha(u_{0,1} + u_{0,0}) - \beta(u_{0,0} + u_{-1,0}))^2),$<br>$\bar{G}_1 = (-1)^{k+l}(G_1 + G_2) + S_k^{-1}(-1)^{k+l}(G_1 + G_2) - 8\ln(u_{1,0} - u_{-1,0}),$<br>$\bar{F}_2 = (-1)^{k+l}(F_1 + F_2) + S_l^{-1}(-1)^{k+l}(F_1 + F_2) + 8\ln(u_{0,1} - u_{0,-1}),$<br>$\bar{G}_2 = (-1)^{k+l}(G_1 + G_2) - S_l^{-1}(-1)^{k+l}(F_1 + F_2) - 2\ln(((\alpha - \beta)^2 - (u_{1,0} - u_{0,-1})^2)(\beta(u_{1,0} + u_{0,0}) - \alpha(u_{0,0} + u_{0,-1}))^2),$ |
| $\mathbf{A2}$ | $\bar{F}_1 = (-1)^{k+l}(F_1 + F_2) - S_k^{-1}(-1)^{k+l}(G_1 + G_2) + 2\ln((\alpha u_{0,1} - \beta u_{-1,0})(\beta u_{0,1} - \alpha u_{-1,0})(\alpha(1 - \beta^2) u_{0,1} - u_{-1,0}(\beta(1 - \alpha^2) + (\alpha^2 - \beta^2) u_{0,0} u_{0,1}))^2),$<br>$\bar{G}_1 = (-1)^{k+l}(G_1 + G_2) + S_k^{-1}(-1)^{k+l}(G_1 + G_2) - 8\ln(u_{1,0} - u_{-1,0}),$<br>$\bar{F}_2 = (-1)^{k+l}(F_1 + F_2) + S_l^{-1}(-1)^{k+l}(F_1 + F_2) + 8\ln(u_{0,1} - u_{0,-1}),$<br>$\bar{G}_2 = (-1)^{k+l}(G_1 + G_2) - S_l^{-1}(-1)^{k+l}(F_1 + F_2) - 2\ln((\beta u_{1,0} - \alpha u_{0,-1})(\alpha u_{1,0} - \beta u_{0,-1})(\beta(1 - \alpha^2) u_{1,0} - u_{0,-1}(\alpha(1 - \beta^2) + (\beta^2 - \alpha^2) u_{0,0} u_{1,0}))^2),$ |

## 5. High-order conservation laws

New conservation laws can be obtained by applying the generator of a five-point symmetry repeatedly to a known conservation law [6]. For instance, let us consider equation **H1**. By applying the infinitesimal generator [9]

$$X = \frac{k}{u_{1,0} - u_{-1,0}} \partial_{u_{0,0}} - \partial_\alpha$$

to the conservation law

$$F = -\ln\left(u_{0,1} - u_{-1,0}\right), \quad G = \ln\left(u_{1,0} - u_{-1,0}\right),$$

then adding a trivial conservation law, we obtain

$$F_n = -\{(u_{0,0} - u_{-2,0})(u_{0,1} - u_{-1,0})\}^{-1}, \quad G_n = \{(u_{0,0} - u_{-2,0})(u_{1,0} - u_{-1,0})\}^{-1}.$$

At present, there is no proof that this method will always yield a new conservation law (that cannot be reduced to a known or trivial one); however, we do not know of any counterexamples.

From [9] we know that each equation from the ABS classification has symmetries in the $k$ and $l$ directions. By applying symmetries in the $k$ direction to a conservation law with component $F$ in the $k$ direction we also obtain a conservation law with component $F$ in the $k$ direction. In this way we might construct an infinite hierarchy of conservation laws with component $F$ in the $k$ direction and another hierarchy of conservation laws with component $G$ in the $l$ direction.

## 6. Conclusion and some open problems

All three-point conservation laws for all equations from the ABS classification have been found. For each of these equations we found three five-point conservation laws. We have used the direct method [5, 7, 8] as far as possible to calculate conservation laws, as this guarantees that all conservation laws of a particular type have been found. However, for all but two of the ABS equations, it was necessary to supplement the direct method with extra hypotheses, based on the results that we had obtained so far. This hybrid approach led to the discovery that each of the ABS equations (for constant $\alpha$ and $\beta$) has three five-point conservation laws. If only one of $\alpha$ and $\beta$ is constant then we can find only one five-point conservation law. It seems likely that these are the only five-point conservation laws, but we cannot yet be certain that this is so.

The technique which generates a conservation law from a known one was shown. So far it is the only technique which may give an infinite number of conservation laws. This technique is easy to use, but it does not guarantee that new conservation law cannot be reduced to a known or trivial one. Therefore we cannot say that ABS equations have an infinite number of conservation laws.

Our work rises an important question about conservation laws for nonautonomous quad-graph equations. Why are there no five-point conservation laws with $G$ in the $k$ direction when $\alpha = \alpha(k)$ and five-point conservation laws with $F$ in the $l$ direction when $\beta = \beta(l)$? In case of symmetries we have a very similar situation [9]; there are no five-point symmetries in the $k$ direction when $\alpha = \alpha(k)$ or five-point symmetries in the $l$ direction when $\beta = \beta(l)$. We think it likely that the answer to one of these questions will lead to the answer to the other one.

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