Lecture 2: How to find Lie symmetries

Symmetry Methods for Differential and Difference Equations

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Reduction of order for ODEs

A given ODE of order p may be written in canonical coordinates as

$$s^{(p)} = \Omega\big(r, \dot{s}, \ddot{s}, \ldots, s^{(p-1)}\big), \qquad s^{(k)} = \frac{\mathrm{d}^k s}{\mathrm{d}r^k}\,. \tag{1}$$

The variable s is missing, because the ODE has the Lie symmetries $(\hat{r}, \hat{s}) = (r, s+\varepsilon)$, which give $\hat{s}^{(k)} = s^{(k)}$ for each $k \geq 1$.

In other words, the ODE can be written entirely in terms of differential functions that are invariant under the Lie symmetries.

Consequently, (1) reduces to

$$v^{(p-1)} = \Omega\big(r, v, \dot{v}, \ldots, v^{(p-2)}\big), \qquad \text{where} \quad v = \dot{s}. \tag{2}$$

If (2) can be solved, the solution of (1) is

$$s = \int v(r; c_1, \ldots c_{p-1})\,\mathrm{d}r + c_p.$$

Note Sometimes, it is more convenient to choose v to be a different function of r and $\dot{s}$. Any choice will do, provided that v depends locally diffeomorphically on $\dot{s}$. This is to ensure that $(r, \dot{s})$ can be obtained from (r, v) and vice versa, and that the local smooth structure is preserved.

Whichever choice of v is made, the reduced ODE will always be of the form

$$v^{(p-1)} = \Omega\big(r, v, \dot{v}, \dots, v^{(p-2)}\big).$$

Example The ODE

$$y'' = \frac{1}{xy^2} \tag{3}$$

has Lie symmetries with $\xi(x,y) = x^2,\ \eta(x,y) = xy$. The canonical coordinates

$$(r,s) = \left(\frac{y}{x}, -\frac{1}{x}\right) \quad \text{give} \quad (\dot{s}, \ddot{s}) = \left(\frac{1}{xy'-y}, \frac{x^3 y''}{(xy'-y)^3}\right).$$

With $v = \dot{s}$, the ODE reduces to

$$\dot{v} = \frac{v^3}{r^2},$$

which happens to be separable. Therefore, the general solution of (3) can be obtained by integration.

Reduction of order for OΔEs

A given OΔE of order p may be written in canonical coordinates as

$$s_p - s_{p-1} = \Omega\big(n,\, s_1 - s,\, s_2 - s_1, \dots, s_{p-1} - s_{p-2}\big). \tag{4}$$

The variable s is missing, because the ODE has the Lie symmetries $(\hat{n}, \hat{s}) = (n, s + \varepsilon)$, so $\hat{s}_{k+1} - \hat{s}_k = s_{k+1} - s_k$ for each $k \geq 0$.

In other words, the OΔE can be written entirely in terms of functions that are invariant under the Lie symmetries.

Consequently, (1) reduces to

$$v_{p-1} = \Omega\big(n, v, v_1, \dots, v_{p-2}\big), \qquad \text{where} \quad v = s_1 - s. \tag{5}$$

If (5) can be solved, the solution of (4) is

$$s = \sum v(n; c_1, \dots c_{p-1}) + c_p.$$

Example The OΔE

$$u_2 = \frac{u_1(uu_1 + u_1 - u)}{(u_1)^2 + u_1 - u}$$

has Lie symmetries whose characteristic is $Q(n, u) = u^2$. Thus

$$s_2 = \frac{(s_1)^2 - ss_1 + s}{s_1 - s + 1}, \quad \text{where} \quad s = -\frac{1}{u}.$$

With the substitution $v = s_1 - s$, this reduces to

$$v_1 = -\frac{v}{v+1},$$

whose general solution is $v = 2\{1 + c_1(-1)^n\}/\{(c_1)^2 - 1\}$. Then

$$s = \sum v(n) + c_2 = \frac{2n + c_1\{1 - (-1)^n\}}{(c_1)^2 - 1} + c_2,$$

which is easily inverted to obtain u.

The infinitesimal generator

Question How does an arbitrary locally-smooth function $F(x, y)$ vary along the orbit of a 1-parameter local Lie group?

Answer Replace (x, y) by $(\hat{x}, \hat{y})$ and expand in powers of ε:

$$F(\hat{x}, \hat{y}) = F(x, y) + \varepsilon\big\{\xi(x, y)F_{,x} + \eta(x, y)F_{,y}\big\} + O(\varepsilon^2),$$

where the subscripts indicate the partial derivative with respect to each variable that follows the comma.

Equivalently, $F(\hat{x}, \hat{y}) = F(x, y) + \varepsilon X\{F(x, y)\} + O(\varepsilon^2)$, where

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

is the *infinitesimal generator* of the local Lie group.

(We use the shorthand ∂_a for $\frac{\partial}{\partial a}$ here and from now on.)

The infinitesimal generator turns up in the formal solution of the equations that define the tangent vector to the orbit at $(\hat{x}, \hat{y})$:

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = \xi(\hat{x}, \hat{y}), \quad \frac{\mathrm{d}\hat{y}}{\mathrm{d}\varepsilon} = \eta(\hat{x}, \hat{y}), \qquad (\hat{x}, \hat{y})\Big|_{\varepsilon=0} = (x, y).$$

The formal solution is

$$\hat{x} = \exp\{\varepsilon X\}x, \quad \hat{y} = \exp\{\varepsilon X\}y, \quad \text{where} \quad \exp\{\varepsilon X\} = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} X^k.$$

The infinitesimal generator determines $(\hat{x}, \hat{y})$ locally, to all orders in ε. Indeed, wherever the above series converges,

$$F(\hat{x}, \hat{y}) = \exp\{\varepsilon X\}F(x, y).$$

Theorem $F(x, y)$ is invariant if and only if $X\{F(x, y)\} = 0$.

More generally, if a 1-parameter local Lie group acts on any space with coordinates x^α, such that

$$\hat{x}^\alpha = x^\alpha + \varepsilon\,\zeta^\alpha(\mathbf{x}) + O(\varepsilon^2),$$

the infinitesimal generator is

$$X = \zeta^\alpha\,\partial_{x^\alpha} \qquad \text{(summed over } \alpha\text{)}.$$

A locally-smooth function $F(\mathbf{x})$ is invariant if and only if

$$X\{F(\mathbf{x})\} = 0.$$

The change-of-variables formula is straightforward for infinitesimal generators. Given local coordinates (a, b) on the (x, y)-plane,

$$X = (Xa)\,\partial_a + (Xb)\,\partial_b.$$

In particular, canonical coordinates (r, s) satisfy

$$Xr = 0, \quad Xs = 1, \qquad \text{so} \;\; X = \partial_s.$$

[By the above theorem, r is invariant under the action of the 1-parameter Lie group.]

More generally, in a space with coordinates x^α,

$$X = (Xx^\alpha)\,\partial_{x^\alpha}.$$

The symmetry condition for a p^{th}-order ODE,

$$y^{(p)} = \omega(x, y, y', \dots, y^{(p-1)}), \tag{6}$$

is

$$\hat{y}^{(p)} = \omega\big(\hat{x}, \hat{y}, \hat{y}^{(1)}, \dots, \hat{y}^{(p-1)}\big) \qquad \text{when } \ (6) \text{ holds.}$$

(Remember: solutions must be mapped to solutions.)

This condition involves not only $\hat{x}$ and $\hat{y}$, but also the derivatives $\hat{y}^{(k)}$ for k from 1 to p. Consequently, we must prolong the infinitesimal generator to act on the p^{th} jet space, with local coordinates $(x, y, y', \dots, y^{(p)})$. To deal with all p at once, write

$$X = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \eta^{(2)}\partial_{y''} + \cdots .$$

We will use (6) to evaluate the output on solutions of the ODE.

The symmetry condition gives, to first-order in ε,

$$X\Big(y^{(p)}-\omega(x,y,\dots,y^{(p-1)})\Big)=0 \quad \text{when} \quad y^{(p)}=\omega(x,y,\dots,y^{(p-1)}).$$

Explicitly, this *linearized symmetry condition* (LSC) amounts to

$$\eta^{(p)}-\xi\,\omega_{,x}-\eta\,\omega_{,y}-\eta^{(1)}\omega_{,y'}-\cdots-\eta^{(p-1)}\omega_{,y^{(p-1)}}=0, \qquad (7)$$

when (6) holds. [Here ω is shorthand for $\omega(x,y,\dots,y^{(p-1)})$.]

The LSC holds as an identity in $x,y,\dots,y^{(p-2)}$ and $y^{(p-1)}$.

Example For a given first-order ODE, $y'=\omega(x,y)$, the LSC is

$$\eta^{(1)}-\xi\,\omega_{,x}-\eta\,\omega_{,y}=0 \quad \text{when} \quad y'=\omega(x,y);$$

this holds as an identity in x and y (because the ODE eliminates derivatives of y).

The coefficients $\eta^{(k)}$ are defined by the expansion

$$\hat{y}^{(k)} = y^{(k)} + \varepsilon\eta^{(k)} + O(\varepsilon^2).$$

To calculate $\eta^{(1)}$, use the *total derivative* operator,

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + y'''\partial_{y''} \cdots ,$$

to expand $\hat{y}^{(1)}$ as follows:

$$\hat{y}^{(1)} = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{y' + \varepsilon D_x\eta + O(\varepsilon^2)}{1 + \varepsilon D_x\xi + O(\varepsilon^2)} = y' + \varepsilon\left(D_x\eta - y' D_x\xi\right) + O(\varepsilon^2).$$

Similarly, each $\eta^{(k)}$ is calculated recursively from

$$\hat{y}^{(k)} = \frac{y^{(k)} + \varepsilon D_x\eta^{(k-1)} + O(\varepsilon^2)}{1 + \varepsilon D_x\xi + O(\varepsilon^2)} = y^{(k)} + \varepsilon\left(D_x\eta^{(k-1)} - y^{(k)} D_x\xi\right) + O(\varepsilon^2).$$

Examples Assuming that ξ and η depend only on x and y, the prolongation formulae yield:

$$\eta^{(1)} = \eta_{,x} + \left(\eta_{,y} - \xi_{,x}\right)y' - \xi_{,y}y'^2,$$

$$\begin{aligned}\eta^{(2)} = {}& \eta_{,xx} + (2\eta_{,xy} - \xi_{,xx})y' + (\eta_{,yy} - 2\xi_{,xy})y'^2 - \xi_{,yy}y'^3 \\ & + \{\eta_{,y} - 2\xi_{,x} - 3\xi_{,y}y'\}y'',\end{aligned}$$

$$\begin{aligned}\eta^{(3)} = {}& \eta_{,xxx} + (3\eta_{,xxy} - \xi_{,xxx})y' + 3(\eta_{,xyy} - \xi_{,xxy})y'^2 \\ & + (\eta_{,yyy} - 3\xi_{,xyy})y'^3 - \xi_{,yyy}y'^4 \\ & + 3\{\eta_{,xy} - \xi_{,xx} + (\eta_{,yy} - 3\xi_{,xy})y' - 2\xi_{,yy}y'^2\}y'' \\ & - 3\xi_{,y}y''^2 + \{\eta_{,y} - 3\xi_{,x} - 4\xi_{,y}y'\}y'''.\end{aligned}$$

How to solve the LSC

We are now able to write out the LSC as a set of partial differential equations (PDEs) for $\xi(x, y)$ and $\eta(x, y)$. Usually, for ODEs of order $p \geq 2$, one can solve these equations.

This is because the LSC can be split according to the dependence of each term on $y', \ldots, y^{(p-1)}$.

The easiest way to understand how this works in practice is to study some examples.

Example The LSC for the simplest second-order ODE, y”=0, is

$$\eta^{(2)} = 0 \quad \text{when} \quad y'' = 0,$$

which amounts to

$$\eta_{,xx} + (2\eta_{,xy} - \xi_{,xx})y' + (\eta_{,yy} - 2\xi_{,xy})y'^2 - \xi_{,yy}y'^3 = 0.$$

As ξ and η are independent of y', the LSC splits into

$$\eta_{xx} = 0, \quad 2\eta_{xy} - \xi_{xx} = 0, \quad \eta_{yy} - 2\xi_{xy} = 0, \quad \xi_{yy} = 0. \qquad (8)$$

The general solution of the last two of (8) is

$$\xi(x,y) = A(x)y + B(x), \qquad \eta(x,y) = A'(x)y^2 + C(x)y + D(x),$$

where A, B, C and D are arbitrary locally-smooth functions.

The remaining equations in (8) amount to

$$A'''(x)y^2 + C''(x)y + D''(x) = 0, \qquad 3A''(x)y + 2C'(x) - B''(x) = 0,$$

which are split (by comparing powers of y) into

$$A''(x) = 0, \qquad C''(x) = 0, \qquad D''(x) = 0, \qquad B''(x) = 2C'(x).$$

All that remains is to solve this simple system of ODEs (in terms of some arbitrary constants, c_i) and reconstruct ξ and η:

$$\xi(x,y) = c_1 + c_3 x + c_5 y + c_7 x^2 + c_8 xy,$$
$$\eta(x,y) = c_2 + c_4 y + c_6 x + c_7 xy + c_8 y^2.$$

Therefore, every symmetry generator X is a linear combination of

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x, \quad X_4 = y\partial_y, \quad X_5 = y\partial_x,$$
$$X_6 = x\partial_y, \quad X_7 = x^2\partial_x + xy\partial_y, \quad X_8 = xy\partial_x + y^2\partial_y.$$

Example The ODE $y''' = y^{-3}$ arises in models of thin-film flow. The LSC is

$$\eta^{(3)} + 3y^{-4}\eta = 0 \quad \text{when} \quad y''' = y^{-3},$$

which amounts to

$$\begin{aligned} 0 = {} & \eta_{,xxx} + (3\eta_{,xxy} - \xi_{,xxx})y' + 3(\eta_{,xyy} - \xi_{,xxy})y'^2 \\ & + (\eta_{,yyy} - 3\xi_{,xyy})y'^3 - \xi_{,yyy}y'^4 \\ & + 3\{\eta_{,xy} - \xi_{,xx} + (\eta_{,yy} - 3\xi_{,xy})y' - 2\xi_{,yy}y'^2\}y'' \\ & -3\xi_{,y}y''^2 + \{\eta_{,y} - 3\xi_{,x} - 4\xi_{,y}y'\}y^{-3} + 3y^{-4}\eta. \end{aligned}$$

Although this looks horrible, only the red term is quadratic in y''. Consequently, $\xi = A(x)$.

Taking this into account, the LSC reduces to

$$0 = \eta_{,xxx} + \left(3\eta_{,xxy} - A'''(x)\right)y' + 3\eta_{,xyy}y'^2 + \eta_{,yyy}y'^3$$
$$+3\{\eta_{,xy} - A''(x) + \eta_{,yy}y'\}y'' + \{\eta_{,y} - 3a'(x)\}y^{-3} + 3y^{-4}\eta.$$

Only the blue terms are linear in y'', so (equating powers of y' in such terms)

$$\eta_{,xy} - A''(x) = 0, \quad \eta_{,yy} = 0 \quad \rightarrow \quad \eta = \left(A'(x) + c_1\right)y + B(x).$$

Substitute this into the remaining terms of the LSC and compare powers of y', then y, to obtain the general solution of the LSC:

$$\xi(x, y) = -4c_1x + c_2, \qquad \eta(x, y) = -3c_1y.$$

Thus every symmetry generator X is a linear combination of

$$X_1 = -4x\partial_x - 3y\partial_y, \qquad X_2 = \partial_x.$$

For a given first-order ODE, the LSC has infinitely many linearly independent solutions. However, the LSC cannot be split with respect to derivatives of y, so one cannot find its general solution.

The best that can be done is to restrict ξ and η and hope that this yields some symmetries.

Example The LSC for the ODE $y' = (1-y^2)/(xy) + 1$ is

$$\begin{aligned} 0 =& \eta_{,x} + \left(\frac{1-y^2}{xy} + 1\right)(\eta_{,y} - \xi_{,x}) - \left(\frac{1-y^2}{xy} + 1\right)^2 \xi_{,y} \\ &+ \left(\frac{1-y^2}{x^2 y}\right)\xi + \left(\frac{1+y^2}{xy^2}\right)\eta. \end{aligned}$$

Let us see whether there are any symmetry generators for which

$$\xi = \alpha(x), \quad \eta = \beta(x)y + \gamma(x). \quad \text{(This is a common restriction.)}$$

The LSC amounts to

$$0 = \beta' y + \gamma' + (\beta - \alpha')\left(\frac{1-y^2}{xy} + 1\right) + \alpha\left(\frac{1-y^2}{x^2 y}\right) + (\beta y + \gamma)\left(\frac{1+y^2}{xy^2}\right).$$

Now one can split the (restricted) LSC in powers of y, to get (after some simplification)

$$\gamma = 0, \qquad \beta - \alpha' = 0, \qquad \alpha' + \alpha/x = 0.$$

There are no further constraints, so $\alpha = c_1 x^{-1}$ and $\beta = -c_1 x^{-2}$. Consequently, every generator of the restricted form is a multiple of

$$X_1 = x^{-1}\partial_x - x^{-2} y \partial_y.$$

Exercise Use this symmetry generator to construct canonical coordinates and solve the ODE.

Beyond point symmetries

A nice version of the prolongation formula (given ξ and η) is

$$\eta^{(k)} = D_x^k Q + y^{(k+1)}\xi,$$

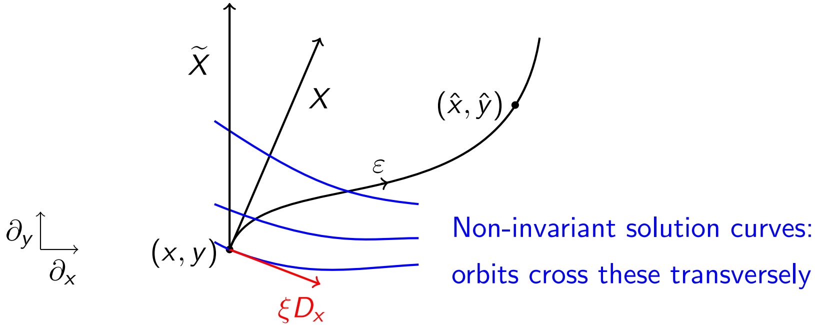
where $Q = \eta - y'\xi$ is the characteristic; this is proved recursively.

It leads to a very useful splitting of the infinitesimal generator:

$$X = \xi D_x + Q\partial_y + (D_x Q)\partial_{y'} + (D_x^2 Q)\partial_{y''} + \cdots .$$

The red term generates trivial Lie symmetries (for arbitrary ξ), because its restriction to solutions maps points along solution curves. So X is equivalent (up to a trivial symmetry generator) to

$$\widetilde{X} = Q\partial_y + (D_x Q)\partial_{y'} + (D_x^2 Q)\partial_{y''} + \cdots .$$



For first-order ODEs, the geometrical picture is shown above: the generator X splits into a trivial tangential component, ξD_x, and a nontrivial vertical component, $\widetilde{X}$. Here, 'vertical' means that there is no component in the direction of the independent variable x.

The same picture holds (in the appropriate jet space) for higher order ODEs. The LSCs with X and $\widetilde{X}$ give the same characteristics.

So far, we have restricted attention to *point symmetries*, which act as local diffeomorphisms on the (x, y)-plane. More symmetries are available, which act as local diffeomorphisms on various jet spaces.

Given an ODE of order p, the types of symmetries are as follows.

- *Point symmetries* have a characteristic that depends only on x, y and y', and is linear in y', so $Q = \eta(x, y) - \xi(x, y)y'$.

- *Contact symmetries* have a characteristic $Q(x, y, y')$, with

$$\xi = -Q_{,y'}, \quad \eta = Q - y'Q_{,y'}, \quad \eta^{(1)} = Q_{,x} + y'Q_{,y}.$$

- *Dynamical symmetries* have a characteristic that depends on some or all of $x, y, y', \dots y^{(p-1)}$. It is usual to set $\xi = 0$, so that $\hat{x} = x$.

To find any of these characteristics, first pick a suitable restriction on Q. Then solve the LSC in the form

$$\widetilde{X}\left(y^{(p)} - \omega(x, y, \ldots, y^{(p-1)})\right) = 0 \quad \text{on solutions,}$$

using $y^{(p)} = \omega(x, y, \ldots, y^{(p-1)})$ to eliminate $y^{(p+k)}$ for all $k \geq 0$.

Example Every characteristic of Lie contact symmetries for

$$y''' = x(x-1)y''^3 - 2xy''^2 + y''$$

is a linear combination of

$$Q_1 = 1, \quad Q_2 = x, \qquad \text{(point symmetry characteristics)}$$
$$Q_3 = e^{y'}, \quad Q_4 = (xy' - y - x)e^{y'}, \quad Q_5 = e^{y - xy' + y' + x}.$$

These are obtained by splitting the LSC in powers of y''.

Just as for first-order ODEs, there are infinitely many dynamical symmetry characteristics $Q(x, y, \dots, y^{(p-1)})$. To find any of them by splitting the LSC, one must make a further restriction.

The most common restriction is to look for characteristics that are independent of $y^{(p-1)}$.

All nontrivial symmetries, dynamical or otherwise, act by changing the arbitrary constants in the general solution of the ODE. (One can regard them as local point transformations on the p-dimensional space of first integrals.)

The same idea applies to OΔEs.

The LSC for a given O∆E

The LSC for a given O∆E is written using the *forward shift operator*, S_n, which replaces n by $n+1$ in all functions of n:

$$\mathrm{S}_n\{f(n)\} = f(n+1), \qquad \mathrm{S}_n u = u_1, \qquad \mathrm{S}_n u_i = u_{i+1}.$$

The last two equations can be combined by using u_0 as another notation for u; we will use this alternative whenever it is helpful.

For simplicity, we will assume that u is defined for all $n \in \mathbb{Z}$, though what follows can be adapted to any domain.

For later use, we also introduce the identity operator I, which maps every function of n to itself.

For scalar OΔEs, Lie symmetries are of the form

$$\hat{n} = n, \qquad \hat{u} = u + \varepsilon Q + O(\varepsilon^2).$$

In particular, Lie point symmetries have $\hat{u}$ (and thus, Q) dependent on n and u only.

The prolongation formula is obtained by replacing n by $n + k$:

$$\hat{u}_k = u_k + \varepsilon\, \mathrm{S}_n^k Q + O(\varepsilon^2).$$

Therefore, the infinitesimal generator (prolonged to all orders) is

$$X = \sum_{k \in \mathbb{Z}} \left(\mathrm{S}_n^k Q \right) \frac{\partial}{\partial u_k}\,.$$

The symmetry condition for

$$u_p = \omega(n, u, \dots, u_{p-1}) \tag{9}$$

is

$$\hat{u}_p = \omega(\hat{n}, \hat{u}, \dots, \hat{u}_{p-1}) \qquad \text{when (9) holds.}$$

Just as for ODEs, the LSC for Lie symmetries is obtained by expanding the symmetry condition to first order in ε:

$$X\{u_p - \omega(n, u, \dots, u_{p-1})\} = 0 \qquad \text{when (9) holds.}$$

Using ω as shorthand for $\omega(n, u, \dots, u_{p-1})$, the LSC amounts to

$$\mathrm{S}_n^p Q - \omega_{,1} Q - \omega_{,2}\, \mathrm{S}_n Q - \cdots \omega_{,p}\, \mathrm{S}_n^{p-1} Q = 0 \qquad \text{when (9) holds,}$$

where the subscript $,i$ denotes the partial derivative with respect to the i^{th} *continuous* argument. For example, $\omega_{,p} = \partial\omega/\partial u^{(p-1)}$.

For most nonlinear OΔEs of order $p > 1$, one can find all Lie point symmetries from the LSC, without having to restrict Q. To see how, let us examine the LSC for a given second-order OΔE,

$$u_2 = \omega(n, u, u_1), \tag{10}$$

which is a *functional equation* (not an ODE or PDE) for Q:

$$Q(n+2, \omega) - \omega_{,2}\, Q(n+1, u_1) - \omega_{,1}\, Q(n, u) = 0. \tag{11}$$

Although functional equations are generally hard to solve, Lie symmetries are diffeomorphisms. Consequently, Q is a locally smooth function of its continuous argument and so the LSC can be solved by the method of *differential elimination*. Before discussing this method in general, we begin with a simple example.

Example The LSC for Lie point symmetries of $u_2 = (u_1)^2/u$ is

$$Q\left(n+2, (u_1)^2/u\right) - 2(u_1/u)\,Q(n+1, u_1) + (u_1/u)^2\,Q(n, u) = 0.$$

In each term, Q depends on only one continuous variable (and one discrete variable). We could differentiate the LSC in order to eliminate that term. In particular,

$$\left\{-\frac{u^2}{(u_1)^2}\frac{\partial}{\partial u} - \frac{u}{2u_1}\frac{\partial}{\partial u_1}\right\} Q\left(n+2, (u_1)^2/u\right) = 0.$$

By applying the differential operator in braces to the LSC, we eliminate the first term in the LSC and obtain

$$Q'(n+1, u_1) - \frac{1}{u_1}\,Q(n+1, u_1) - Q'(n, u) + \frac{1}{u}\,Q(n, u) = 0. \quad (12)$$

This has more terms than the LSC does, but Q and Q' only take two pairs of arguments, instead of three.

Continue the process, by differentiating (12) with respect to u, in order to eliminate $Q'(n+1, u_1)$. This gives

$$\left[-Q'(n,u) + \frac{1}{u}\,Q(n,u)\right]' = 0. \tag{13}$$

Having found a (parametrized) differential equation for Q, we can start to solve the system. Integrate (13) *once*, to obtain

$$Q'(n,u) - \frac{1}{u}\,Q(n,u) = \alpha(n), \tag{14}$$

where $\alpha(n)$ is to be determined. Now substitute (14) into the intermediate equation (12), which gives the OΔE

$$\alpha_1 - \alpha = 0.$$

Consequently $\alpha = c_1$, so the general solution of (12) is

$$Q(n,u) = c_1\,u\ln|u| + \beta(n)\,u. \tag{15}$$

Finally, substitute (15) into the LSC to obtain (after cancellation)

$$\beta_2 - 2\beta_1 + \beta = 0.$$

The general solution of this linear OΔE is

$$\beta(n) = c_2 n + c_3,$$

and hence the general solution of the LSC is

$$Q(n, u) = c_1 u \ln|u| + c_2 nu + c_3 u.$$

In other words, every characteristic is a linear combination of

$$Q_1 = u \ln|u|, \quad Q_2 = nu, \quad Q_3 = u.$$

This example illustrates the general technique for obtaining Lie point of any difference equation of order $p \geq 2$:

1. Write down the LSC.
2. Apply appropriate differential operators to reduce the number of unknown functions.
3. Having reached a differential equation, back-substitute and solve the resulting linear difference equations.
4. Iterate, if necessary.

Exactly the same approach may be used to obtain dynamical symmetries, with Q dependent on $n, u, u_1, \ldots, u_{p-1}$, provided that Q is restricted is some way.

For first-order O∆Es, it is necessary to assume that Q has a particular dependence on u. A common assumption is

$$Q(n, u) = \alpha(n)\, u^2 + \beta(n)\, u + \gamma(n). \tag{16}$$

Example The LSC for the OΔE $u_1 = u/(1+nu)$ is

$$Q\big(n+1,\ u/(1+nu)\big) - Q(n,u)/(1+nu)^2 = 0.$$

With the restriction (16), the LSC reduces to

$$u^2\alpha_1 + u(1+nu)\,\beta_1 + (1+nu)^2\gamma_1 - u^2\alpha - u\,\beta - \gamma = 0,$$

which we split into a system of OΔEs by comparing powers of u:

$$\begin{aligned} u^2 \text{ terms}: &\qquad \alpha_1 + n\,\beta_1 + n^2\gamma_1 - \alpha = 0,\\ u \text{ terms}: &\qquad \beta_1 + 2n\gamma_1 - \beta = 0,\\ \text{other terms}: &\qquad \gamma_1 - \gamma = 0. \end{aligned}$$

Systems in this 'triangular' form are solved by starting at the bottom and working up. In this case, each of the scalar equations is easy to solve, so there is no need to restrict α, β or γ. We obtain

$$\begin{aligned}\gamma &= c_1,\\ \beta &= c_2 - c_1 n(n-1),\\ \alpha &= c_3 - c_2 n(n-1)/2 + c_1 n^2(n-1)^2/4.\end{aligned}$$

Therefore every characteristic that is quadratic in u is of the form

$$Q(n,u) = c_1\left(1 - \frac{n(n-1)}{2}\,u\right)^2 + c_2 u\left(1 - \frac{n(n-1)}{2}\,u\right) + c_3 u^2.$$

Note For systems not in triangular form, one may need to combine the OΔEs to solve them. If necessary, restrict α, β and γ.

Potential problems with differential elimination

1. Generally, each elimination increases the number of terms, a phenomenon known as *expression swell*. This makes it difficult to find dynamical symmetries of high-order OΔEs.

2. In view of the above, it is wise to use computer algebra to keep track of the calculations. However, beware of computer algebra solvers for differential and difference equations. They may overlook some solutions!

3. It may be difficult (or even impossible) to solve some of the linear difference equations in Step 3. However, this is unusual for nonlinear OΔEs.

Inherited symmetries

For ODEs and OΔEs, the set of Lie symmetry characteristics that satisfy a particular restriction is a vector space. However, for the set of all Lie point symmetries, there is an extra structure.

Theorem If X_i and X_j generate Lie symmetries (of any type), so does $[X_i, X_j] = X_iX_j - X_jX_i$.

Corollary The space $\mathcal{L}$ of all Lie point symmetry generators is a Lie algebra under the commutator $[\cdot,\cdot]$.

Note The space of all dynamical symmetry generators is also a Lie algebra, but it is easier to find the set of all Lie point symmetries (when $p \geq 2$) and to construct their canonical coordinates!

An R-dimensional Lie algebra $\mathcal{L}$ is *solvable* if the derived series,

$$\mathcal{L}^{(0)} = \mathcal{L}, \qquad \mathcal{L}^{(i+1)} = [\mathcal{L}^{(i)}, \mathcal{L}^{(i)}],$$

stops at $\{0\}$. If $\mathcal{L}$ is solvable, one can use the derived series to construct a chain of subalgebras,

$$\{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_R = \mathcal{L},$$

such that each $\mathcal{L}_k$ is a k-dimensional ideal of $\mathcal{L}_{k+1}$. Given such a chain, a *canonical basis* for $\mathcal{L}$ satisfies $\mathcal{L}_k = \mathrm{Span}(X_1, \ldots, X_k)$, for $k = 1, \ldots, R$.

Theorem Given an ODE or OΔE of order $p \geq 2$, suppose that $\mathcal{L}$ is solvable and that $X_1, \ldots, X_R$ is a canonical basis. When the ODE or OΔE is reduced using canonical coordinates corresponding to X_1, the reduced equation inherits the symmetries generated by $X_2, \ldots, X_R$.

Example The Lie algebra $\mathcal{L}$ of point symmetry generators for

$$u_3 = u_2 + \frac{(u_2 - u_1)^2}{u_1 - u}$$

has a canonical basis

$$X_1 = \partial_u, \qquad X_2 = u\partial_u.$$

Reduce the order by using X_1, whose canonical coordinate is $s = u$:

$$v_2 = v_1^2/v \qquad v = u_1 - u.$$

The reduced OΔE inherits the symmetries generated by X_2, as

$$X_2 v = u\,\frac{\partial v}{\partial u} + u_1\frac{\partial v}{\partial u_1} = u_1 - u = v.$$

Therefore X_2 reappears as the scaling symmetry generator

$$\tilde{X}_2 = v\partial_v\,,$$

which enables one to carry out a second reduction of order.

By contrast, let us use X_2 first, with $\tilde{s} = \ln|u|$ and $\tilde{v} = u_1/u$ for simplicity. (For the same reason as for ODEs, we can take $\tilde{v}$ to be a locally invertible function of $\tilde{s}_1 - \tilde{s}$.)

This reduction of order works (as it is guaranteed to do), yielding

$$\tilde{v}_2 = 1 + \frac{\tilde{v}\,(\tilde{v}_1 - 1)^2}{\tilde{v}_1(\tilde{v} - 1)}\,. \tag{17}$$

However, the reduced OΔE (17) has no Lie point symmetries. What has happened to the symmetries generated by X_1? As before, apply X_1 (prolonged) to $\tilde{v}$, to get

$$X_1\,\tilde{v} = \frac{\partial \tilde{v}}{\partial u} + \frac{\partial \tilde{v}}{\partial u_1} = \frac{1}{u} - \frac{u_1}{u^2} = \frac{1 - \tilde{v}}{u}\,.$$

This is not a function of $\tilde{v}$ only, so it cannot be a generator of Lie point symmetries for the reduced OΔE (17).

Summary: the main results in Lecture 2

- Given an ODE or OΔE of order $p \geq 2$, canonical coordinates reduce the order by one. If the reduced equation can be solved, the original equation is solved by one more integration or summation.
- The infinitesimal generator X determines the local behaviour of Lie symmetries.
- A function is invariant if and only if it is in the kernel of X.
- The prolonged infinitesimal generator yields the LSC. For an ODE, this is a differential equation; for an OΔE, the LSC is a functional equation.
- By restricting the characteristic, Q, one can seek all Lie symmetries that satisfy the restriction.
- Point symmetries act on the space of independent and dependent variables; dynamical symmetries act on the space of first integrals (or arbitrary constants).

Conclusion: There are substantial differences between OΔE and ODE symmetry methods!