Lecture 1: From symmetries to solutions

Symmetry Methods for Differential and Difference Equations

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Notation for scalar ordinary differential equations (ODEs)

Independent variable: $x \in \mathbb{R}$;   dependent variable: $y \in \mathbb{R}$.

First-order ODEs (solved for highest derivative):

$$y' = \omega(x, y), \quad \text{where} \quad y = y(x), \quad y' = \frac{\mathrm{d}y(x)}{\mathrm{d}x}\,.$$

ODEs of order p (solved for highest derivative):

$$y^{(p)} = \omega(x, y, y', \ldots, y^{(p-1)}), \quad \text{where} \;\; y^{(k)} = \frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k}\,.$$

Assumption: ω is locally smooth in each argument.

The general solution has p arbitrary constants.

Some elementary solution methods for first-order ODEs

Linear ODEs: To solve the linear ODE

$$y' + a(x)\,y = b(x),$$

use an integrating factor. Equivalently, introduce new coordinates

$$r = x, \quad s = y\exp\left\{\int a(x)\,\mathrm{d}x\right\} \quad \longrightarrow \quad \frac{\mathrm{d}s}{\mathrm{d}r} = b(r)\exp\left\{\int a(r)\,\mathrm{d}r\right\}.$$

Homogeneous ODEs: To solve the homogeneous ODE

$$y' = F(y/x), \qquad F(z) \neq z,$$

introduce new *local* coordinates

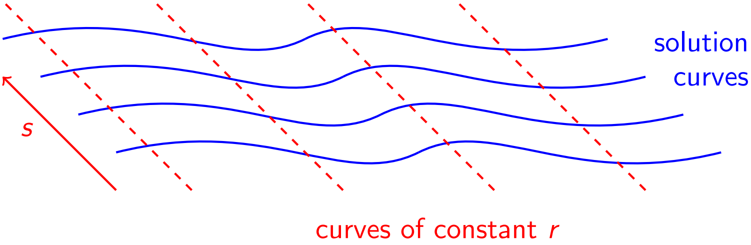
$$r = y/x, \quad s = \ln|x| \quad \longrightarrow \quad \frac{\mathrm{d}s}{\mathrm{d}r} = \frac{1}{F(r) - r}\,.$$

Common aspects of these methods

In each case, local coordinates (r, s) put the ODE in the form

$$\frac{\mathrm{d}s}{\mathrm{d}r} = f(r) \;\longrightarrow\; s = \int f(r)\,\mathrm{d}r + c;$$

any local coordinates that do this are called *canonical*.
Locally, s parametrizes the solutions.



Some questions

- The solution curves of a first-order ODE foliate regions of the (x, y) plane. Does a set of canonical coordinates always exist?
- If so, how can one find them?
- How can one solve an ODE of unfamiliar type?

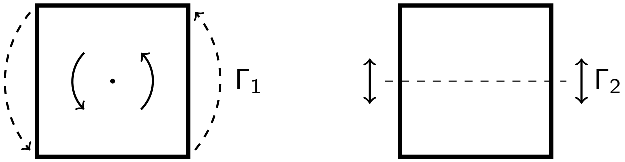
$$\text{Example}: \qquad y' = \frac{1 - y^2}{xy} + 1.$$

- Are canonical coordinates useful for higher-order ODEs?

Sophus Lie's **symmetry methods** answer these questions.

Definition A *symmetry* of a geometrical object is an invertible transformation that maps the object to itself. Individual points of an object may be mapped to different points, but the object as a whole is unchanged by any symmetry.

Example Some symmetries of a square:

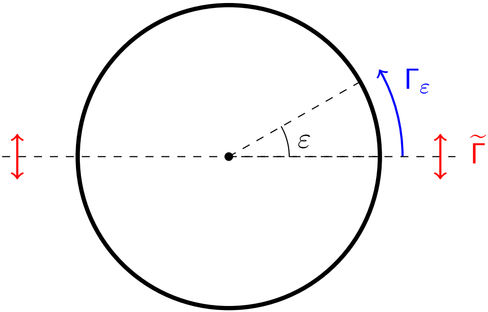


If the object has some associated structure, every symmetry must preserve this structure. (Otherwise, the object would change.) Examples include rigidity and smoothness.

The set of symmetries of an object is a group under composition of transformations, which is an associative operation.

The identity (id) maps each point of the object to itself.

The group may be finite or infinite.

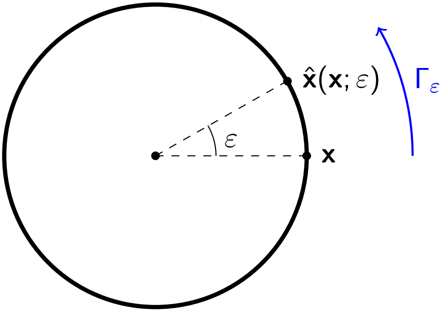


Symmetries of the circle:

All rotations about centre

Reflection in each diagonal

A closer look at rotations of the circle



In cartesian coordinates:

$$\mathbf{x} = (x, y) = (\cos\theta, \sin\theta),$$

$$\hat{\mathbf{x}} = \big(\cos(\theta+\varepsilon), \sin(\theta+\varepsilon)\big).$$

Note: $\Gamma_0 = \mathrm{id},$

$$\Gamma_\delta\,\Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}.$$

For sufficiently small $|\varepsilon|$, the Taylor expansion about $\varepsilon = 0$ gives

$$\hat{\mathbf{x}} = \big(x\cos\varepsilon - y\sin\varepsilon,\ x\sin\varepsilon + y\cos\varepsilon\big) = (x, y) + \varepsilon\,(-y, x) + O(\varepsilon^2).$$

The set of rotations is a one-parameter local Lie group.

Definition A parametrized set of transformations,

$$\Gamma_\varepsilon : \mathbf{x} \mapsto \hat{\mathbf{x}}(\mathbf{x};\varepsilon), \qquad \varepsilon \in (\varepsilon_0, \varepsilon_1),$$

where $\varepsilon_0 < 0 < \varepsilon_1$, is a *one-parameter local Lie group* if:

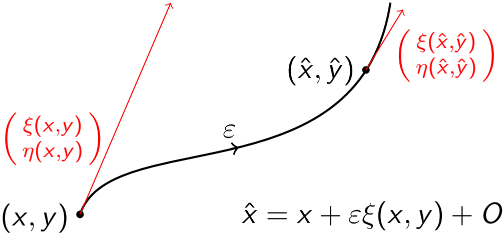
1. Γ_0 is the identity map, so that $\hat{\mathbf{x}} = \mathbf{x}$ when $\varepsilon = 0$.
2. $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ for every δ, ε sufficiently close to zero.
3. Each $\hat{x}^\alpha$ can be represented as a Taylor series in ε (in a neighbourhood of $\varepsilon = 0$ that is determined by $\mathbf{x}$), and so

$$\hat{x}^\alpha(\mathbf{x};\varepsilon) = x^\alpha + \varepsilon\,\zeta^\alpha(\mathbf{x}) + O(\varepsilon^2), \qquad \alpha = 1,\ldots,N.$$

1 and 2 imply that $\Gamma_\varepsilon^{-1} = \Gamma_{-\varepsilon}$ when $|\varepsilon|$ is sufficiently small.

Note: A local Lie group may not be a group; it need only satisfy the group axioms for sufficiently small parameter values.

Orbits of a one-parameter local Lie group acting on the plane



$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = \xi(\hat{x}, \hat{y})$$
$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}\varepsilon} = \eta(\hat{x}, \hat{y})$$

$$\hat{x} = x + \varepsilon\xi(x, y) + O(\varepsilon^2)$$
$$\hat{y} = y + \varepsilon\eta(x, y) + O(\varepsilon^2)$$

The black curve is part of the *orbit* through (x, y).

Tangent vectors to the orbit are shown in red.

A point (x, y) is *invariant* if and only if $\xi(x, y) = \eta(x, y) = 0$.

Symmetries of a given ODE

An ODE (of any order) may be represented by the set of its solutions. For ODEs with a locally-smooth structure, symmetries are defined as follows.

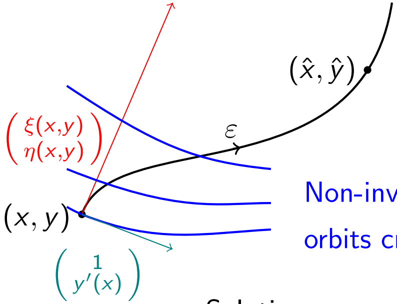
Definition A symmetry of a given ODE is a locally-defined diffeomorphism, Γ, that maps the set of all solutions to itself. (Consequently, every solution is mapped to a solution.)

If Γ maps a solution to itself, that solution is *invariant*.

If every solution is invariant, Γ is said to be *trivial*.

In effect, the solutions are 'points' of the ODE; trivial symmetries act like the identity transformation.

Action of Lie transformations on solution curves



Non-invariant solution curves:

orbits cross these transversely

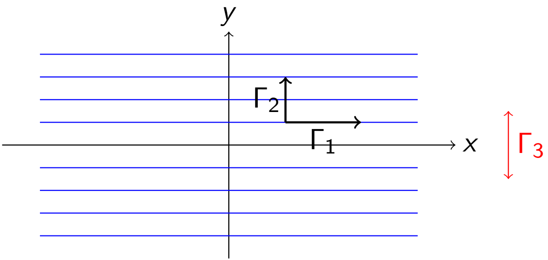
Solution curve coincides with orbit at (x, y) if

$$Q(x, y, y') \equiv \begin{vmatrix} 1 & \xi(x,y) \\ y' & \eta(x,y) \end{vmatrix} = \eta(x, y) - y'\xi(x, y)$$

is zero on curve; then the solution is invariant.

$Q = 0$ on all solutions $\Leftrightarrow$ all solutions invariant $\Leftrightarrow$ trivial symmetries

Symmetries of $y' = 0$



$\Gamma_1 : (x,\ y) \to (x + \varepsilon_1,\ y)$ (Lie, trivial)

$\Gamma_2 : (x,\ y) \to (x,\ y + \varepsilon_2)$ (Lie, nontrivial)

$\Gamma_3 : (x,\ y) \to (x,\ -y)$ (discrete, nontrivial)

Which ODEs have vertical translations?

The Symmetry Condition (SC) for $\frac{\mathrm{d}y}{\mathrm{d}x} = \omega(x, y)$:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} = \omega(\hat{x}, \hat{y}) \quad \text{when} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \omega(x, y)$$

A first-order ODE $y' = \omega(x, y)$ admits all vertical translations,

$$(\hat{x}, \hat{y}) = (x, y + \varepsilon), \qquad \varepsilon \in \mathbb{R},$$

iff ω is a function of x only.

$$\text{Proof}: \ \frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{x}} = \frac{\mathrm{d}y}{\mathrm{d}x}\,; \ \text{SC} \rightarrow \ \omega(x, y{+}\varepsilon) = \omega(x, y) \ \rightarrow \ \omega(x, y) = f(x).$$

These ODEs are easily solved: $y = \int f(x)\, \mathrm{d}x + c.\quad (c \mapsto c + \varepsilon)$

Canonical coordinates

Idea Introduce local coordinates (r, s) in which Lie symmetries look (locally) like vertical translations:

$$r(\hat{x}, \hat{y}) = r(x, y); \qquad s(\hat{x}, \hat{y}) = s(x, y) + \varepsilon.$$

Note Any ODE with these symmetries can be written in terms of the invariant functions $r, \dot{s}, \ddot{s}, \ldots$, where the derivative with respect to r is denoted by a dot.

Method Solve

$$\frac{\mathrm{d}\hat{x}}{\xi(\hat{x}, \hat{y})} = \frac{\mathrm{d}\hat{y}}{\eta(\hat{x}, \hat{y})} = \mathrm{d}\varepsilon, \qquad (\hat{x}, \hat{y})\big|_{\varepsilon=0} = (x, y).$$

Potential problems: (a) invariant points (orbit is zero-dimensional), (b) patched solutions, (c) system too hard to solve (rare).

Example: The ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x} + \frac{y^2}{x^3}$$

has Lie symmetries with $\xi(x,y) = x^2, \quad \eta(x,y) = xy.$

Canonical coordinates are obtained from

$$\frac{\mathrm{d}\hat{x}}{\hat{x}^2} = \frac{\mathrm{d}\hat{y}}{\hat{x}\hat{y}} = \mathrm{d}\varepsilon, \qquad (\hat{x},\hat{y})\big|_{\varepsilon=0} = (x,y).$$

Simple solution: $r(x,y) = y/x, \quad s(x,y) = -1/x.$

The ODE reduces to $\dot{s} = \dfrac{1}{1+r^2}\,.$ Solution: $y = -x\tan\left(\dfrac{1}{x}+c\right).$

Notation for scalar ordinary difference equations (O∆Es)

Independent variable: $n \in \mathbb{Z}$; dependent variable: $u \in \mathbb{R}$.

First-order O∆Es (forward form):

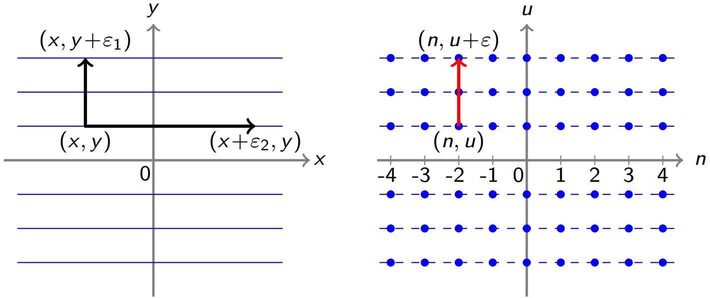
$$u_1 = \omega(n, u), \quad \text{where} \quad u = u(n), \quad u_1 = u(n+1).$$

O∆Es of order p (in forward form):

$$u_p = \omega(n, u, u_1, \ldots, u_{p-1}), \quad \text{where} \quad u_k = u(n+k).$$

Assumptions: ω is locally smooth in each continuous argument;
$\partial\omega/\partial u \neq 0$. (The O∆E is *exactly* p^{th}-order.)

The general solution has p arbitrary constants.



The simplest OΔE is $u_1 - u = 0$; its general solution is $u = c$.

Unlike $y' = 0$, it has no trivial Lie symmetries, because the independent variable is discrete. However, the vertical translation $(\hat{n}, \hat{u}) = (n, u + \varepsilon)$ is a symmetry for each $\varepsilon \in \mathbb{R}$.

More generally, every OΔE of the form

$$u_1 - u = f(n), \tag{1}$$

has the one-parameter Lie group of symmetries

$$(\hat{n}, \hat{u}) = (n, u + \varepsilon) \qquad \varepsilon \in \mathbb{R}.$$

$$\text{Proof:} \quad \hat{u}_1 - \hat{u} = (u_1 + \varepsilon) - (u + \varepsilon) = u_1 - u = f(n) = f(\hat{n}).$$

No other first-order OΔE $u_1 = \omega(n, u)$ has these Lie symmetries.

Just as $y' = f(x)$ is solved by integration, (1) is solved by summation:

$$u = \sum f(n) + c.$$

The summation operator

For convenience, we use the following shorthand for indefinite sums:

$$\sum f(n) = \begin{cases} \displaystyle\sum_{k=n_0}^{n-1} f(k), & n > n_0, \\ 0, & n = n_0, \\ \displaystyle -\sum_{k=n}^{n_0-1} f(k), & n < n_0, \end{cases}$$

where n_0 is an arbitrary fixed integer in the domain.

An OΔE will be regarded as solved if all that remains is to carry out summations, whether or not we can evaluate the sums in closed form.

The simplest Lie symmetries of are of the form

$$\hat{n} = n, \qquad \hat{u} = u + \varepsilon Q(n, u) + O(\varepsilon^2);$$

here $Q(n, u)$ is the *characteristic* with respect to (n, u).

[Vertical translations, $(\hat{n}, \hat{u}) = (n, u + \varepsilon)$, have $Q(n, u) = 1$.]

To see how these Lie symmetries transform the shifted variables u_k, simply replace the free variable n by $n + k$:

$$\hat{u}_k = u_k + \varepsilon Q(n + k, u_k) + O(\varepsilon^2).$$

This is the *prolongation formula* for OΔEs. It is much simpler than the corresponding formula for ODEs (see Part 2 of this course).

The change-of-variables formula

Now consider the effect of changing coordinates from (n,u) to (n,v), where $v'(n,u)\neq 0$. (Here v' is shorthand for $\partial v/\partial u$.)

Apply Taylor's Theorem to obtain

$$\hat{v}\equiv v(n,\hat{u})=v\big(n,u+\varepsilon Q(n,u)+O(\varepsilon^2)\big)=v+\varepsilon v'(n,u)\,Q(n,u)+O(\varepsilon^2).$$

Therefore the characteristic with respect to (n,v) is $\tilde{Q}(n,v)$, where

$$\tilde{Q}\big(n,v(n,u)\big)=v'(n,u)\,Q(n,u).$$

A canonical coordinate for O∆Es

Just as for ODEs, we seek a local *canonical coordinate*, s, such that the symmetries amount to translations in s:

$$(\hat{n}, \hat{s}) = (n, s + \varepsilon).$$

The characteristic with respect to (n, s) is $\tilde{Q}(n, s) = 1$; so, by the change-of-variables formula,

$$s(n, u) = \int \frac{\mathrm{d}u}{Q(n, u)},$$

in any neighbourhood in which $Q(n, u) \neq 0$.

Example The OΔE

$$u_1 = \frac{u}{1+nu}, \qquad n \geq 1,$$

has the Lie symmetries $(\hat{n}, \hat{u}) = \left(n,\ \frac{u}{1-\varepsilon u}\right)$, whose characteristic is $Q(n,u) = u^2$. The canonical coordinate

$$s(n,u) = \int u^{-2}\,\mathrm{d}u = -u^{-1}$$

transforms the OΔE to

$$s_1 - s = -(u_1)^{-1} + u^{-1} = -n,$$

and hence

$$s = c_1 - \sum_{k=1}^{n-1} k = c_1 - n(n-1)/2.$$

Consequently, the general solution of

$$u_1 = \frac{u}{1+nu}\,, \qquad n \geq 1,$$

is

$$u = \frac{2}{n(n-1) - 2c_1}\,.$$

One cannot define a canonical coordinate s at $u = 0$, because $Q(n, 0) = 0$. Consequently, the points $(n, 0)$ are invariant.

Clearly, $u = 0$ is a solution, although it is not part of the general solution. Any solution that consists entirely of invariant points is called an *invariant solution*.

Compatibility

Canonical coordinates for a given OΔE must satisfy an extra condition: locally, one must be able to write the whole OΔE in terms of s. Unlike ODEs, OΔEs involve several different base points; s must be consistent across all of them for the OΔE to be transformed correctly.

Any canonical coordinate s that meets this condition will be called *compatible* with the OΔE.

Problem: It is not always possible to find a real-valued compatible canonical coordinate!

Solution: Use a complex-valued canonical coordinate instead.

Example The OΔE

$$u_1 = \frac{u-n}{nu-1}, \qquad n \geq 2,$$

has a characteristic $Q(n,u) = (-1)^n(u^2-1)$. It is easy to show that $|u_1|$ is greater (less) than 1 whenever $|u|$ is less (greater) than 1 and $u \neq 1/n$.

No real-valued compatible s exists, so use

$$s(n,u) = \frac{(-1)^n}{2} \operatorname{Log}\left(\frac{u-1}{u+1}\right).$$

Here Log is the principal value of the complex logarithm:

$$\operatorname{Log}(z) = \ln(|z|) + i\operatorname{Arg}(z), \qquad \operatorname{Arg}(z) \in (-\pi, \pi].$$

Whether $|u|$ is greater or less than 1, the OΔE amounts to

$$s_1 - s = \tfrac{1}{2}\,(-1)^{n+1}\big\{\ln\big((n-1)/(n+1)\big) + i\pi\big\},$$

whose general solution is

$$s = \tfrac{1}{2}\,(-1)^{n}\big\{\ln\big((n-1)/n\big) + i\pi/2\big\} + c_1,$$

where c_1 is a complex-valued constant that can be written in terms of $u(2)$.A routine calculation yields the general solution of the original OΔE; here $c = u(2)$.

$$u = \begin{cases} \dfrac{(c+1)n\; +\; 2(c-1)(n-1)}{(c+1)n\; -\; 2(c-1)(n-1)}, & n \text{ even;} \\[2ex] \dfrac{2(1-c)n\; +\; (c+1)(n-1)}{2(1-c)n\; -\; (c+1)(n-1)}, & n \text{ odd.} \end{cases}$$

Summary: the main results in Lecture 1

- Most well-known methods for solving a given first-order ODE or OΔE use canonical coordinates to transform the equation into a simple solvable form.
- Symmetries of a given ODE or OΔE map the set of solutions to itself (invertibly, preserving the locally-smooth dependence on arbitrary constants).
- Nontrivial one-parameter local Lie groups of symmetries yield useful local canonical coordinates.