A VARIATIONAL COMPLEX FOR DIFFERENCE EQUATIONS

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Abstract

An analogue of the Poincaré lemma for exact forms on a lattice is stated and proved. Using this result as a starting-point, a variational complex for difference equations is constructed and is proved to be locally exact. The proof uses homotopy maps, which enable one to calculate Lagrangians for discrete Euler-Lagrange systems. Furthermore, such maps lead to a systematic technique for deriving conservation laws of a given system of difference equations (whether or not it is an Euler-Lagrange system).

1 Introduction

Most partial differential equations (PDEs) that are used as mathematical models have some interesting geometrical features, the most common of which are symmetries and conservation laws. If the PDE comes from a variational principle, its variational symmetries and nontrivial conservation laws are related to one another by Noether's Theorem [34, 4]. There are variants of Noether's Theorem for Hamiltonian and multisymplectic systems, which have a structural conservation law for the (multi-)symplectic two-form [5, 6].

Much information about a given PDE can be gleaned from its geometrical structures, which act as constraints on the behaviour of the solutions. Therefore it is reasonable to try to incorporate these structures into numerical methods for solving the PDE. Various approaches have been developed, each of which preserves a particular geometric feature; together, these approaches form the discipline of *geometric integration*. For example, low-order conservation laws of hyperbolic PDEs can be preserved by using a Lax-Wendroff scheme [17]. For some integrable PDEs, there are methods that that preserve one or two conservation laws [15]. If a differential equation is derived from a known variational problem, and if variational symmetries are preserved by the discretiztion, then a discrete version of Noether's Theorem [24] can be used to obtain difference analogues of the corresponding conservation laws [31, 38, 30]. At present, there is no systematic approach to preserving arbitrary conservation laws of PDEs that do not fall into any of the above categories.

It is sometimes desirable to preserve symmetries that are not variational. In particular, if the intermediate asymptotic behaviour of a given PDE is scaling-invariant, discretizations that preserve such invariance are highly effective. They can reproduce the correct asymptotic behaviour, even as singularities are approached [8]. For more general symmetries, various adaptive methods have been proposed that retain Lie symmetries exactly [10].

Throughout this paper, we focus on conservation laws and (to a lesser extent) symmetries. However there is a rapidly-growing body of work on numerical methods that preserve other geometric structures. These methods include symplectic and multisymplectic integrators [36, 7, 30], discrete gradients [32], and Lie group solvers [21, 22, 33]. (A good starting-point for the newcomer is the overview by Budd & Iserles [9]).

There is a well-developed set of mathematical tools for finding symmetries and conservation laws of a given PDE. Generators of one-parameter Lie groups of symmetries can be determined straightforwardly from the linearized symmetry condition, a technique that can also be used to obtain some generalized symmetries [34]. (For an elementary introduction to symmetry methods, consult [18, 37].) There are many computer algebra packages that will help with the work of finding these symmetries (see Hereman's review [16] for further information). However, it is not usually possible to find all generalized symmetries of a given PDE. If the PDE is the Euler-Lagrange equation for a known variational problem, and if some generalized symmetries can be found, it is easy to check which of these are variational symmetries. Then Noether's Theorem can be used to construct the corresponding conservation law.

These well-known methods lead to several questions. Is it possible to find out whether a given PDE is an Euler-Lagrange equation? If so, is there a technique for determining a corresponding variational problem? Is it possible to construct conservation laws systematically if the PDE has no variational, Hamiltonian, or multisymplectic formulation? Even if there is such a formulation, can one determine conservation laws if no variational symmetries are known (so that Noether's Theorem cannot be used)? These questions, which are related to one another, can be answered with the aid of the variational complex.

$$
\begin{array}{lll}
0 & & \\
\downarrow & \text{inclusion} & \\
\mathbb{R} & & c \qquad \text{(constant)} \\
\downarrow & \text{inclusion} & \\
\Lambda^0 & & f(\mathbf{x}, u^{(N)}) \\
\downarrow & \text{Grad} & \\
\Lambda^1 & & f_i(\mathbf{x}, u^{(N)})\,\mathrm{d}x_i \\
\downarrow & \text{Curl} & \\
\Lambda^2 & & f_{ij}(\mathbf{x}, u^{(N)})\,\mathrm{d}x_i \wedge \mathrm{d}x_j\,, \qquad (i<j) \\
\downarrow & \text{Div} & \\
\Lambda^3 & & f(\mathbf{x}, u^{(N)})\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\
\downarrow & \text{Euler} & \\
\Lambda^1_* & & \int \{F(\mathbf{x}, u^{(N)}) \cdot \mathrm{d}u\}\,\mathrm{d}\mathbf{x} \\
\downarrow & \text{Helmholtz} & \\
\Lambda^2_* & & \int \{\mathrm{d}u \wedge \mathcal{D}(\mathrm{d}u)\}\,\mathrm{d}\mathbf{x} \\
\downarrow & & \\
\Lambda^3_* & & \\
\downarrow & & \\
\vdots & &
\end{array}
$$

Figure 1: The three-dimensional continuous variational complex

The continuous variational sequence for a three-dimensional base space is given in Figure 1. This sequence has two types of vector space: the total

r-forms, Λ^r, and the functional r-forms, Λ^r_* (see Olver [34] for details). A typical element of a space in the sequence appears on the right. Functions may depend on the dependent variables u and their derivatives up to some finite order N; these are denoted collectively by $u^{(N)}$. We adopt the summation convention: where an index occurs twice in a term, summation over all possible values of the index is implied. The integral

$$\int \{\cdot\}\,\mathrm{d}\mathbf{x} \equiv \int \{\cdot\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

denotes an element of an equivalence class; two coefficient functions are equivalent if they differ by a total divergence. In Λ^2_*, the matrix-valued differential operator $\mathcal{D}$ is skew-adjoint with respect to the L_2 inner product.

If the functions depend on the independent variables only then the first part of the sequence is well-known (this consists of the vector spaces down to Λ^3). Provided the domain of definition of functions is diffeomorphic to $\mathbb{R}^3$ (or $\mathbb{C}^3$), the gradient of a function is zero if and only if it is constant. Under the same condition, the curl of a vector field $\mathbf{f}(\mathbf{x})$ is zero if and only if $\mathbf{f}(\mathbf{x})$ is a gradient, and the divergence of $\mathbf{f}(\mathbf{x})$ is zero if and only if $\mathbf{f}(\mathbf{x})$ is a curl. It is less well known that such vector fields may also depend on arbitrary functions of the dependent variables and their derivatives; these extra variables are regarded as functions of the independent variables in each of the vector spaces Λ^r. (Consequently, total derivatives are used in the operators Grad, Curl, and Div.) This result is proved constructively, using homotopy operators that take an element of the kernel of one map to an element of the pre-image.

The above sequence and its maps form a *complex*, because the image of one map is contained in the kernel of the next (irrespective of the domain). For example, Div(Curl($\cdot$)) is identically zero. More generally, a complex is a sequence of vector spaces with maps between consecutive elements in the sequence, such that the composition of any two successive maps is identically zero. A complex is *exact* if the image of each map is precisely the kernel of the following map. Vinogradov [39] was the first to prove that the variational complex is formally *locally exact*. *Local* means that the domains of definition of coefficient functions are restricted to be open balls, or (more generally) to be "totally star-shaped." A much simpler (constructive) proof due to Ian Anderson is detailed in Olver [34]; again, this uses homotopy operators. The variational complex is a formal construction, with an underlying assumption that there are never any contributions from the boundary of the domain; we shall use this assumption henceforth.

The fact that the variational complex is locally exact has several uses. For example, if the Euler operator acting on a function is zero, that function is necessarily a total divergence. This result can be used to construct scalar conservation laws systematically [34, 1, 2]. (Form-valued conservation laws can be calculated with the aid of the *variational bicomplex*, which may be regarded as an extension of the variational complex; see Anderson [3] for details.) Similarly, if a PDE $\mathcal{P} = 0$ satisfies the Helmholtz condition then $\mathcal{P} = 0$ is necessarily an Euler-Lagrange equation. Moreover, the Lagrangian can be calculated with the aid of a homotopy operator.

The above techniques can be used to find many of the geometric features of a given PDE. This raises a key question: to what extent can one do the same thing for partial difference equations (PΔEs)? To date, most of the various different approaches to geometric integration have focused on preserving a single geometric feature. Given a class of methods that preserve a particular feature, is it possible to determine which (if any) of the methods preserve other geometric features? For example, given a class of multisymplectic integrators, can one find out which symmetries or conservation laws are preserved by each discretization [29]?

Systematic techniques for determining some of the geometric structures of a given PΔE have been developed recently, although much work still needs to be done. There are various techniques for finding Lie symmetries; they depend upon the type of symmetry that is sought. The development of such techniques was initiated by Maeda [27], who showed that autonomous systems of first-order ordinary difference equations (OΔEs) can be simplified or solved with the aid of Lie point symmetries. Maeda also showed that the linearized symmetry condition for such OΔEs amounts to a set of functional equations. In general, these are hard to solve, but Maeda described two examples for which a very restrictive ansatz yields Lie symmetries. Gaeta [13] used formal series expansions to derive some symmetries of those systems of OΔEs that are discretizations of continuous systems. Maeda's ideas have been extended in various ways. Series-based methods have been developed for obtaining some solutions of the linearized symmetry condition for OΔEs [35, 25] and PΔEs [12, 25]. Series expansions can be calculated if the symmetry condition has a fixed point. However, it may not be possible to sum the series and obtain solutions in closed form, in which case the symmetries are non-local. Another approach uses differential elimination to determine local symmetries in closed form [19]. This is a systematic method, but it is also computationally intensive. As with symmetry methods for PDEs, it is limited by the generality of the class of symmetries being sought.

To obtain conservation laws of an arbitrary PΔE, irrespective of whether or not there is an underlying variational structure, a difference analogue of the variational complex is needed. The purpose of the current paper is to introduce such a complex, and to show that it is exact on topologically trivial domains. We present a homotopy operator that can be used systematically to construct conservation laws. This is not a trivial generalization of the continuous homotopy operator, because the independent variables lie on a lattice rather than a continuous space. Furthermore, the difference operator does not act like a derivative: there is no analogue of the Leibniz product rule.

Although the independent variables are discrete, the dependent variables are continuous (just as for PDEs). Consequently, parts of the new complex that involve the dependent variables correspond closely to parts of the continuous complex; wherever it is possible, we develop analogues of the structures described in §5.4 of Olver [34]. As early as 1985, Kupershmidt [23] formulated the difference analogue of Λ^1_*, in order to provide an algebraic basis for the study of integrable difference equations. However, this investigation was not extended to the rest of the complex.

We introduce the complex in three stages. In §2 we construct the difference complex, whose elements depend only on the lattice points. The dependent variables are added in §3, which completes the first (or 'horizontal') part of the complex; the analogues of Λ^r_* are also described. In §4 the whole complex is spliced together. Some applications are presented in §5; others can be found in [20, 28].

2 The difference complex

2.1 Lattice coordinates and the shift maps

Throughout this section, we retrict attention to spaces that involve only the lattice coordinates $\mathbf{n} = (n^1, \ldots, n^p) \in \mathbb{Z}^p$. These coordinates will be the independent variables when we consider difference equations (from §3 onwards). The algebra of functions depending only on the lattice variables $\mathbf{n}$ is written as $\mathcal{B}$. Unlike $\mathbb{R}^p$, the lattice does not have a differentiable structure; instead it has an ordering on each n^i. Consequently, the natural operators on the lattice are the *shift maps*,

$$S_k : n^i \mapsto n^i + \delta^i_k, \qquad k = 1, \ldots, p\,;$$

here δ^k_i is the Kronecker delta. To simplify things, let $\mathbf{1}_k$ be the p-tuple whose only nonzero entry is in the k^{th} place; this entry is 1. Then the k^{th} shift map is

$$S_k : \mathbf{n} \mapsto \mathbf{n} + \mathbf{1}_k\,.$$

The action of each shift map extends naturally to any function $f(\mathbf{n}) \in \mathcal{B}$ as follows:

$$S_k : f(\mathbf{n}) \mapsto f(\mathbf{n} + \mathbf{1}_k). \tag{1}$$

Note that the shift maps commute (i.e. $S_k S_j = S_j S_k$), and each shift map is a homomorphism on $\mathcal{B}$:

$$S_k\{f(\mathbf{n})g(\mathbf{n})\} = f(\mathbf{n}+\mathbf{1}_k)g(\mathbf{n}+\mathbf{1}_k) = S_k\{f(\mathbf{n})\}\,S_k\{g(\mathbf{n})\}\,, \qquad \forall f(\mathbf{n}), g(\mathbf{n}) \in \mathcal{B}.$$

2.2 The difference map

Let $\textsf{Ex}(p)$ be the exterior algebra on p symbols $\Delta_1, \ldots \Delta_p$, so that

$$\Delta_i^2 = 0, \qquad \Delta_i \Delta_j = -\Delta_i \Delta_j.$$

Definition 2.2.1. We define the algebra of *difference forms* to be

$${}^p\textsf{Ex} = \bigcup_{\mathbf{n} \in \mathbb{Z}^p} \textsf{Ex}(p)$$

with coefficients in $\mathcal{B}$ and pointwise multiplication and addition. A typical element of ${}^p\textsf{Ex}$ takes the form

$$\omega = P_0(\mathbf{n}) + \sum_{r=1}^{p} \sum_{i_1<\ldots<i_r} P_{i_1 i_2 \ldots i_r}(\mathbf{n}) \Delta_{i_1} \Delta_{i_2} \ldots \Delta_{i_r} \qquad (2)$$

where $P_0(\mathbf{n}), P_{i_1 i_2 \ldots i_r}(\mathbf{n}) \in \mathcal{B}$.

Definition 2.2.2. The action of each shift map on elements of ${}^p\textsf{Ex}$ is defined by (1), together with $S_k(\Delta_i) = \Delta_i$ and

$$S_k(\eta\omega) = S_k(\eta) S_k(\omega), \qquad \eta, \omega \in {}^p\textsf{Ex},$$

so that the action of S_k on the typical element (2) is

$$S_k\omega = S_k(P_0(\mathbf{n})) + \sum_{r=1}^{p} \sum_{i_1<\ldots<i_r} S_k(P_{i_1 i_2 \ldots i_r}(\mathbf{n})) \Delta_{i_1} \Delta_{i_2} \ldots \Delta_{i_r}.$$

There is a natural grading of ${}^p\textsf{Ex}$. We say that ω is a *difference r-form* and write $\omega \in {}^p\textsf{Ex}^r$ if

$$\omega = \sum_{i_1<\ldots<i_r} P_{i_1 i_2 \ldots i_r}(\mathbf{n}) \Delta_{i_1} \Delta_{i_2} \ldots \Delta_{i_r}.$$

The difference 0-forms are defined to be functions of $\mathbf{n}$, so that ${}^p\textsf{Ex}^0 = \mathcal{B}$.

Definition 2.2.3. We define the *difference map* $\Delta : {}^p\textsf{Ex}^r \longrightarrow {}^p\textsf{Ex}^{r+1}$ to be

$$\Delta(\omega) = \sum_{k=1}^{p} \Delta_k \cdot (S_k - \mathrm{id})\omega \qquad (3)$$

for $r = 0, \ldots, p-1$; here id denotes the identity map. On the typical element (2), we have

$$\Delta(\omega) = \sum_{k=1}^{p} (S_k - \mathrm{id})(P_0(\mathbf{n}))\Delta_k$$

$$+\sum_{k=1}^{p}\sum_{r=1}^{p-1}\sum_{i_1<\ldots<i_r}(S_k-\mathrm{id})(P_{i_1\ldots i_r}(\mathbf{n}))\Delta_k\Delta_{i_1}\ldots\Delta_{i_r}.$$

In particular,

$$\Delta(n^i)=\Delta_i\,.$$

Example 2.2.4. If $\omega=(n^1+(n^2)^2)\Delta_1+n^1\Delta_2$ then

$$\Delta(\omega)=\left[\Delta_1(\Delta_1+\Delta_2)+\left((n^2+1)^2-(n^2)^2\right)\Delta_2\Delta_1\right]=-2n^2\Delta_1\Delta_2.$$

The maps $(S_k-\mathrm{id})$ commute pairwise, whereas the symbols Δ_i anticommute pairwise; it follows that $\Delta^2=0$.

Definition 2.2.5. The *difference complex* is

$$0\longrightarrow\mathbb{R}\xrightarrow{\imath}{}^p\mathsf{Ex}^0\xrightarrow{\Delta}{}^p\mathsf{Ex}^1\xrightarrow{\Delta}\cdots\xrightarrow{\Delta}{}^p\mathsf{Ex}^p\xrightarrow{\Delta}0 \tag{4}$$

where $\imath$ is the inclusion map.
It is important to note that Δ is *not* a derivation, that is,

$$\Delta(\omega\eta)\neq(\Delta\omega)\eta\pm\omega(\Delta\eta).$$

Furthermore, although the difference complex mimics the de Rham complex, there is no obvious sense in which the space ${}^p\mathsf{Ex}^1|_{\mathbf{n}}$ can be considered as the dual space to some tangential object. The geometric meaning of the difference complex is still being investigated.

Shortly, we shall show that the difference complex is locally exact. First though, we make some general remarks on complexes and the use of homotopy maps to prove exactness.

2.3 Homotopy maps

A complex is a sequence of linear spaces $\{A_i\}$ together with a collection of linear maps $\delta_i : A_i \longrightarrow A_{i+1}$ such that

$$\delta_{i+1} \circ \delta_i = 0, \qquad \text{for all } i.$$

This means that

$$\operatorname{im} \delta_i \subseteq \ker \delta_{i+1} \tag{5}$$

for all i. Often a complex is written as

$$\ldots \longrightarrow A_{i-1} \xrightarrow{\delta_{i-1}} A_i \xrightarrow{\delta_i} A_{i+1} \xrightarrow{\delta_{i+1}} \ldots$$

One way to show that the complex is exact, i.e. that

$$\ker \delta_{i+1} = \operatorname{im} \delta_i \tag{6}$$

for each i, is to construct a sequence of so-called *homotopy* maps

$$H_i : A_i \longrightarrow A_{i-1}$$

such that

$$H_{i+1}\delta_i + \delta_{i-1}H_i = \mathrm{id}, \qquad \text{for all } i. \tag{7}$$

In pictures, one has

$$\ldots \; A_{i-1} \underset{H_i}{\overset{\delta_{i-1}}{\rightleftarrows}} A_i \underset{H_{i+1}}{\overset{\delta_i}{\rightleftarrows}} A_{i+1} \; \ldots$$

If $\omega \in \ker \delta_i$, then evaluating equation (7) on ω yields

$$\delta_{i-1} H_i \omega = \omega.$$

Thus $\omega \in \operatorname{im} \delta_{i-1}$, and we have

$$\ker \delta_i \subseteq \operatorname{im} \delta_{i-1} \tag{8}$$

Putting (8) and (5) together for all i yields the result (6). What is more, $H_i(\omega)$ is a pre-image (under δ_{i-1}) of ω, so the proof of exactness is constructive.

2.4 Local exactness of the difference complex

The Poincaré Lemma states that the de Rham complex is exact over star-shaped domains; this means that there is a point in the domain that can be reached from any other point in the domain along a straight path. Indeed, the homotopy map uses an integral along such a path. The lattice is not a continuous space, so we should not expect to be able to use the same construction. However, we can make use of the ordering on $\mathbb{Z}$ to construct a homotopy map.

Definition 2.4.6. A *cube-shaped domain* in $\mathbb{Z}^p$ is a set of all lattice points $\mathbf{k} = (k^1, \ldots, k^p)$ such that

$$n_0^i \leq k^i \leq n^i, \qquad i = 1, \ldots p,$$

for some integers n_0^i and n^i. In other words, a cube-shaped domain consists of all points that lie within a cube with opposite vertices $\mathbf{n}_0 = (n_0^1, \ldots, n_0^p)$ and $\mathbf{n} = (n^1, \ldots, n^p)$.

Definition 2.4.7. Given a cube-shaped domain, a *path* from $\mathbf{n}_0$ to $\mathbf{n}$ is an ordered set of lattice points $\mathbf{k}_j$, $j = 0, \ldots, j_n$, that has the following properties: $\mathbf{k}_0 = \mathbf{n}_0$, $\mathbf{k}_{j_n} = \mathbf{n}$, and for each $j < j_n$

$$\mathbf{k}_{j+1} - \mathbf{k}_j = \mathbf{1}_i,$$

for some i. In other words, a path is a sequence of lattice points such that each successive point is one step closer to $\mathbf{n}$ than its predecessor. Note that although there may be many paths from $\mathbf{n}_0$ to $\mathbf{n}$, they all have the same number of points. Paths between any pair of opposite vertices of the cube are defined similarly. An *edge path* is a path whose points all lie on the edges of the cube.

We can now state the difference analogue of the Poincaré Lemma.

Lemma 2.4.8. The difference complex (4) is exact on cube-shaped domains; hence

$$\ker\Delta|_{{}^p\mathbf{Ex}^r} = \mathrm{im}\Delta|_{{}^p\mathbf{Ex}^{r-1}}, \qquad r = 1, \dots p-1,$$

and $\ker\Delta|_{{}^p\mathbf{Ex}^0} = \mathbb{R}$.

To prove this result, we need to construct a set of homotopy maps for the difference complex. For simplicity, we choose our lattice coordinates so that $\mathbf{n}_0 = \mathbf{0}$ and all coordinates of each point in the domain are non-negative. We also choose the edge path consisting of the points

$$\begin{array}{cc} (k, 0, \dots, 0), & k = 0, 1, \dots, n^1, \\ (n^1, k, 0, \dots, 0), & k = 1, \dots, n^2, \\ \vdots & \\ (n^1, n^2, \dots, n^{p-1}, k), & k = 1, \dots, n^p. \end{array}$$

This allows us to construct the homotopy map by induction on the number of edges needed to get to $\mathbf{n}$ (that is, on the dimension of the lattice). However, it is easy to modify the formulae and arguments that follow, to allow for other edge paths from an arbitrary origin to a generic point $\mathbf{n}$ in the domain.

Define the projection maps

$$\Pi_j : {}^j\mathbf{Ex}^r \longrightarrow {}^{j-1}\mathbf{Ex}^r, \qquad \Pi_j(\omega) = \omega|_{n^j=0, \Delta_j=0} \tag{9}$$

and note that

$$\Pi_r \circ \Pi_{r+1} \circ \cdots \circ \Pi_p \omega = 0 \tag{10}$$

for all $\omega \in {}^p\mathbf{Ex}^r$, $r \geq 1$.

We introduce a formal analogue of the interior product of a vector field and a differential form, by defining the derivations $\partial_{n^i} \lrcorner : {}^j\mathbf{Ex}^r \longrightarrow {}^j\mathbf{Ex}^{r-1}$ generated by $\partial_{n^i} \lrcorner\, \Delta_k = \delta^i_k$, where δ is the Kronecker symbol. This is extended to all difference forms by linearity and the product rule. Let $h_i : {}^j\mathbf{Ex}^r \longrightarrow {}^j\mathbf{Ex}^{r-1}$ be defined by

$$h_i(\omega) = \sum_{k=0}^{n^i-1} (\partial_{n^i} \lrcorner\, \omega)\,|_{n^i=k} \tag{11}$$

For example,

$$h_3(\alpha(n^1,n^2,n^3)\Delta_1\Delta_3 + \beta(n^1,n^2,n^3)\Delta_1\Delta_2) = \sum_{k=0}^{n^3-1} -\alpha(n^1,n^2,k)\Delta_1.$$

Notes: If ω is a 0-form then $h_i(\omega) = 0$, $i = 1,\ldots,p$. We also use the convention that any sum whose lower limit exceeds its upper limit is assigned the value zero, so $h_i(\omega) = 0$ if $n^i = 0$.

Theorem 2.4.9. Under the above assumptions on the domain of definition of $\omega \in {}^p\mathbf{Ex}^r/\mathcal{B}$, let

$$h(\omega) = h_p(\omega) + \sum_{i=1}^{p-1} h_i\left(\Pi_{i+1} \circ \Pi_{i+2} \circ \cdots \circ \Pi_p \omega\right). \tag{12}$$

Then

$$H_{\mathcal{B}}(\omega) = \begin{cases} h(\omega) & \omega \in {}^p\mathbf{Ex}^r,\ r > 0 \\ \omega|_{n^1=\cdots=n^p=0} & \omega \in {}^p\mathbf{Ex}^0 \end{cases} \tag{13}$$

is a homotopy operator for the complex ${}^p\mathbf{Ex}$ over $\mathcal{B}$.

Example 2.4.10. If $p = 2$ then for 1-forms $\omega = \alpha(n^1,n^2)\Delta_1 + \beta(n^1,n^2)\Delta_2$ the homotopy map is

$$h(\omega) = \sum_{k=0}^{n^2-1} \beta(n^1,k) + \sum_{k=0}^{n^1-1} \alpha(k,0)$$

whereas for 2-forms $\omega = f(n^1,n^2)\Delta_1\Delta_2$ the homotopy map is

$$h(\omega) = -\sum_{k=0}^{n^2-1} f(n^1,k)\Delta_1.$$

Notes:

1. If $p = 1$ then $h(\omega) = h_1(\omega)$ is the operator that is used to solve the ordinary difference equation $(S_1 - \mathrm{id})\, g(n^1) = f(n^1)$ for g given f.

2. If $r \geq 1$ then, from (10), the sum in (12) need only be taken from $i = r$ to $i = p - 1$.

3. If $\tilde{\omega} = \Pi_p \omega$ then
$$h(\omega) = h_p(\omega) + h(\tilde{\omega}) \tag{14}$$

Proof: It is sufficient to prove that

$$h(\Delta\omega) + \Delta h(\omega) = \omega - \Pi_1 \circ \cdots \circ \Pi_p \omega. \tag{15}$$

To see this, note that if $\omega \in {}^p\mathbf{Ex}^r$ and $r \geq 1$ then, by (10), $\Pi_1 \circ \cdots \circ \Pi_p \omega = 0$ and thus $H_{\mathcal{B}} = h$ is a homotopy map. To show exactness at ${}^p\mathbf{Ex}^0$, we need to show that

$$h\Delta(\omega) + \omega|_{n^1 = \cdots = n^p = 0} = \omega$$

for $\omega \in {}^p\mathbf{Ex}^0$. But this is precisely (15), since

$$\omega \in {}^p\mathbf{Ex}^0 \Longrightarrow h(\omega) = 0, \qquad \Pi_1 \circ \cdots \circ \Pi_p \omega = \omega|_{n^1 = \cdots = n^p = 0}.$$

The proof of (15) is by induction on p. First note that if $\omega \in {}^1\mathbf{Ex}^0$ then $\omega = f(n^1)$ for some function f, and therefore

$$\begin{aligned} h\Delta(\omega) + \Delta h(\omega) &= h(\{f(n^1+1) - f(n^1)\}\Delta_1) \\ &= \textstyle\sum_{k=0}^{n^1-1} (f(k+1) - f(k)) \\ &= f(n^1) - f(0) \\ &= \omega - \omega|_{n^1=0} \\ &= \omega - \Pi_1 \omega. \end{aligned} \tag{16}$$

Also if $\omega \in {}^p\mathbf{Ex}^p$ then ω is a multiple of the p-form $\Delta_1\Delta_2\cdots\Delta_p$ and so $\Pi_p\omega = 0$ and $\Delta\omega = 0$. Hence

$$\begin{aligned} h\Delta(\omega) + \Delta h(\omega) &= \Delta h_p(\omega) \\ &= \Delta\left(\textstyle\sum_{k=0}^{n^p-1} (\partial_{n^p} \lrcorner\, \omega)\,|_{n^p=k}\right) \\ &= (S_p - \mathrm{id})\left(\textstyle\sum_{k=0}^{n^p-1} \omega|_{n^p=k}\right) \\ &= \textstyle\sum_{k=0}^{n^p} \omega|_{n^p=k} - \sum_{k=0}^{n^p-1} \omega|_{n^p=k} \\ &= \omega \\ &= \omega - \Pi_1 \circ \cdots \circ \Pi_p\omega. \end{aligned} \tag{17}$$

Now fix $r < p$ and suppose that $H_{\mathcal{B}}$ is a homotopy operator for all $p' < p$. We set $\tilde{\omega} = \Pi_p(\omega)$ and observe that $\tilde{\omega} \in {}^{p-1}\mathbf{Ex}^r$; the induction hypothesis implies that

$$h\Delta\tilde{\omega} + \Delta h\tilde{\omega} = \tilde{\omega} - \Pi_1 \circ \cdots \circ \Pi_{p-1}\tilde{\omega} \tag{18}$$

The last term is nonzero only if $r = 0$. Note that

$$\begin{aligned} \Pi_p(\Delta\omega) &= \Pi_p(\textstyle\sum_{j=1}^{p} \Delta_j(S_j - \mathrm{id})\omega) \\ &= \textstyle\sum_{j=1}^{p-1} \Delta_j(S_j - \mathrm{id})(\Pi_p\omega) \\ &= \Delta\tilde{\omega} \end{aligned}$$

and so, from (14),

$$\begin{aligned} h(\Delta\omega) &= h_p(\Delta\omega) + h(\Pi_p\Delta\omega) \\ &= h_p(\Delta\omega) + h(\Delta\tilde{\omega}). \end{aligned}$$

Also from (14),

$$\Delta h(\omega) = \Delta h_p(\omega) + \Delta h(\tilde{\omega}),$$

and therefore, using (18),

$$\begin{aligned} h\Delta(\omega) + \Delta h(\omega) &= h_p(\Delta\omega) + \Delta h_p(\omega) + \tilde{\omega} - \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_{p-1}\tilde{\omega} \\ &= h_p(\Delta\omega) + \Delta h_p(\omega) + \Pi_p(\omega) - \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_p\omega. \end{aligned}$$

So to prove the correctness of the homotopy formula, we need only show that

$$h_p(\Delta\omega) + \Delta h_p(\omega) = \omega - \Pi_p\omega.$$

But this can be verified by direct calculation, as follows.

$$\begin{aligned}
h_p(\Delta\omega) & + \Delta h_p(\omega) \\
& = \sum_{k=0}^{n^p-1}\left(\partial_{n^p}\lrcorner\left(\sum_{j=1}^{p}\Delta_j(S_j-\mathrm{id})\omega\right)\right)\Big|_{n^p=k} \\
& \quad + \sum_{j=1}^{p}\Delta_j(S_j-\mathrm{id})\left(\sum_{k=0}^{n^p-1}(\partial_{n^p}\lrcorner\,\omega)\Big|_{n^p=k}\right) \\
& = \sum_{j=1}^{p-1}\sum_{k=0}^{n^p-1}\left(\partial_{n^p}\lrcorner\left(\Delta_j(S_j-\mathrm{id})\omega\right)+\Delta_j\left(\partial_{n^p}\lrcorner(S_j-\mathrm{id})\omega\right)\right)\Big|_{n^p=k} \\
& \quad + \textstyle\sum_{k=0}^{n^p-1}\left(\partial_{n^p}\lrcorner\left(\Delta_p(S_p-\mathrm{id})\omega\right)\right)\Big|_{n^p=k} + \Delta_p(\partial_{n^p}\lrcorner\,\omega) \\
& = \partial_{n^p}\lrcorner(\Delta_p\omega) - \left(\partial_{n^p}\lrcorner(\Delta_p\omega)\right)\big|_{n^p=0} + \Delta_p(\partial_{n^p}\lrcorner\,\omega) \\
& = \omega - \omega\big|_{n^p=0,\Delta_p=0} \\
& = \omega - \Pi_p\omega
\end{aligned}$$

as required, where we have used the identity $\partial_{n^p}\lrcorner\Delta_j\eta + \Delta_j(\partial_{n^p}\lrcorner\eta) = \delta_j^p\eta$. Equations (16) and (17) show that (15) holds for $p = r$ if $r \geq 1$ and for $p = 1$ if $r = 0$. By induction, (15) holds for all p, r, as required. □

One question remains: why did we require the domain to be cube-shaped? It turns out that for each path along the edges of a cube-shaped domain there is an operator $\tilde{h}$ which plays the role of h. After a long calculation, it can be shown that $\tilde{h}(\omega) = h(\omega)$, whichever edge path is taken between $\mathbf{0}$ and $\mathbf{n}$. Now consider the two-dimensional lattice with the point $(1, 1)$ removed, which is not a cube-shaped domain. The difference 1-form

$$\omega(n^1, n^2) = \begin{cases} \Delta_1 & , \quad n^1 = 1,\ n^2 \geq 2, \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{19}$$

is closed, but it cannot be obtained by applying the difference operator to any singled-valued function on the punctured lattice. There are exactly two paths between (0,0) and (2,2). Our homotopy operator takes the path passing below the deleted point (1,1) and gives the result $h(\omega(2,2)) = 0$. The homotopy operator that takes the path passing over (1,1) is

$$\tilde{h}(\omega) = h_1(\omega) + h_2\left(\Pi_1\, \omega\right);$$

this gives the result $\tilde{h}(\omega(2,2)) = 1$. As the result is path-dependent, the 1-form ω is not exact. This imitates what happens in the continuous case: on the punctured plane, there are 1-forms that are closed but not exact.

In fact, the requirement that the domain is cube-shaped is slightly more restrictive than is necessary. Suppose that the domain contains a point $\mathbf{0}$ such that every other point $\mathbf{n}$ in the domain can be reached by a path $\mathcal{P}$ that is a sequence of consecutive edge paths along cube-shaped subdomains. (By consecutive, we mean that the last point of the edge path in one subdomain coincides with the first point of the edge path in the next subdomain.) Then our homotopy operator can be modified so that it follows the path $\mathcal{P}$, which proves that the difference complex is exact on any such domain. However, the details are too lengthy to be included in this paper.

3 The Horizontal, Vertical and Vertical Functional Complexes

For difference equations on a p-dimensional lattice, the independent variables are $\mathbf{n} = (n^1, \ldots, n^p)$. The dependent variables $\mathbf{u_n} = (u^1_{\mathbf{n}}, \ldots, u^q_{\mathbf{n}})$ are assumed to vary continuously and to take values in $\mathbb{R}$. A smooth function depending on $\mathbf{n}$, $u^\alpha_\mathbf{n}$ and finitely many iterates of $u^\alpha_\mathbf{n}$ is written as $P[u]$. The algebra of such functions is denoted by $\mathcal{A}$.

If we regard $\mathbf{u_n}$ as a function of $\mathbf{n}$, the shift map S_k acts on the dependent variables as follows:

$$S_k u^\alpha_\mathbf{n} = u^\alpha_{\mathbf{n}+\mathbf{1}_k}\,. \tag{20}$$

We write the composite of shifts using multi-index notation as

$$S^\mathbf{m} = S_1^{m_1}\dots S_p^{m_p} \tag{21}$$

so that $u^\alpha_{\mathbf{n}+\mathbf{m}} = S^\mathbf{m}u^\alpha_\mathbf{n}$. The action of S_k on a typical element of $\mathcal{A}$ is given by

$$S_k P(\mathbf{n},\dots,u^\alpha_{\mathbf{n}+\mathbf{m}},\dots) = P(\mathbf{n}+\mathbf{1}_k,\dots,u^\alpha_{\mathbf{n}+\mathbf{m}+\mathbf{1}_k},\dots) \tag{22}$$

3.1 The Horizontal Complex

Definition 3.1.11. We define the algebra of horizontal forms to be

$$\mathsf{Ex} = \bigcup_{\mathbf{n}\in\mathbb{Z}^p} \mathsf{Ex}(p)$$

with coefficients in $\mathcal{A}$ and pointwise multiplication and addition. This is like ${}^p\mathsf{Ex}$, except that the coefficients now involve the dependent variables. We call ω a *total difference r-form* and write $\omega \in \mathsf{Ex}^r$ if

$$\omega = \sum_{i_1<\dots<i_r} P_{i_1 i_2 \dots i_r}[u]\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_r}.$$

Also $\mathsf{Ex}^0 = \mathcal{A}$.

Definition 3.1.12. We define the *total difference* map $\boldsymbol{\Delta} : \mathsf{Ex}^j \longrightarrow \mathsf{Ex}^{j+1}$ to be

$$\boldsymbol{\Delta}(\omega) = \sum_{k=1}^{p} \Delta_k \cdot (S_k - \mathrm{id})\omega \tag{23}$$

for $j = 0, \dots, p-1$.

Example 3.1.13. If $\omega = n^1 u_{n^1,n^2} u_{n^1+1,n^2} \Delta_2$ then

$$\boldsymbol{\Delta}(\omega) = \left[(n^1+1) u_{n^1+1,n^2} u_{n^1+2,n^2} - n^1 u_{n^1,n^2} u_{n^1+1,n^2} \right] \Delta_1 \Delta_2.$$

The proof that $\boldsymbol{\Delta}^2 = 0$ duplicates the proof that $\Delta^2 = 0$. Note that if ω is a function of the independent variables only then $\boldsymbol{\Delta}\omega = \Delta\omega$.

Theorem 3.1.14. The *horizontal complex*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\imath} \mathbf{Ex}^0 \xrightarrow{\boldsymbol{\Delta}} \cdots \xrightarrow{\boldsymbol{\Delta}} \mathbf{Ex}^p \qquad (24)$$

is exact.
The proof will be given after we have introduced the analogue of the higher Euler operators.

3.2 Vertical forms

Definition 3.2.15. A *vertical r-form* is a finite sum

$$\widehat{w} = \sum_{\alpha, \mathbf{m}^1 \dots \mathbf{m}^r} P^{\alpha}_{\mathbf{m}^1 \dots \mathbf{m}^r}[u] \mathrm{d}u^{\alpha_1}_{\mathbf{n}+\mathbf{m}^1} \wedge \dots \wedge \mathrm{d}u^{\alpha_r}_{\mathbf{n}+\mathbf{m}^r}$$

where $P^{\alpha}_{\mathbf{m}^1 \dots \mathbf{m}^r}[u] \in \mathcal{A}$. We define the differential $\widehat{\mathrm{d}}$ to be

$$\widehat{\mathrm{d}}(\widehat{w}) = \sum_{\beta, \mathbf{m}^j} \sum_{\alpha, \mathbf{m}^1 \dots \mathbf{m}^r} \frac{\partial}{\partial u^{\beta}_{\mathbf{n}+\mathbf{m}^j}} P^{\alpha}_{\mathbf{m}^1 \dots \mathbf{m}^r}[u] \mathrm{d}u^{\beta}_{\mathbf{n}+\mathbf{m}^j} \wedge \mathrm{d}u^{\alpha_1}_{\mathbf{n}+\mathbf{m}^1} \wedge \dots \wedge \mathrm{d}u^{\alpha_r}_{\mathbf{n}+\mathbf{m}^r}.$$

Example 3.2.16. If $\widehat{w} = n u_n \mathrm{d}u_{n+1} - u^2_{n+2} \mathrm{d}u_{n+2}$ then $\widehat{\mathrm{d}}\widehat{w} = n \mathrm{d}u_n \mathrm{d}u_{n+1}$.

Any given vertical form can depend on only finitely many of the iterates; therefore the $\widehat{\Lambda}$ complex with differential $\widehat{\mathrm{d}}$ is an extension of the well-known de Rham complex, with independent variables $u^\alpha_{\mathbf{n}+\mathbf{m}}$; the n_i play the role of parameters. Indeed, $\widehat{\mathrm{d}}$ is bilinear, is a derivation,

$$\widehat{\mathrm{d}}(\widehat{w}\wedge\widehat{\eta}) = \mathrm{d}\widehat{w}\wedge\widehat{\eta} + (-1)^r\widehat{w}\wedge\widehat{\mathrm{d}}\,\widehat{\eta}$$

and satisfies $\widehat{\mathrm{d}}^{\,2} = 0$. The Poincaré lemma for the continuous vertical complex extends immediately to yield the following result.

Theorem 3.2.17. The *vertical complex*

$$\widehat{\Lambda}^0 \xrightarrow{\widehat{\mathrm{d}}} \widehat{\Lambda}^1 \xrightarrow{\widehat{\mathrm{d}}} \widehat{\Lambda}^2 \xrightarrow{\widehat{\mathrm{d}}} \dots \tag{25}$$

is exact.

Proof: Define the vector field

$$\mathbf{v}_u = \sum_{\alpha,\mathbf{m}} u^\alpha_{\mathbf{n}+\mathbf{m}} \frac{\partial}{\partial u^\alpha_{\mathbf{n}+\mathbf{m}}}$$

and the homotopy map

$$\widehat{h} : \widehat{\Lambda}^{r+1} \longrightarrow \widehat{\Lambda}^r$$

by

$$\widehat{h}(\widehat{\omega}) = \int_0^1 \mathbf{v}_u \lrcorner\, \widehat{\omega}[\lambda u]\, \frac{\mathrm{d}\lambda}{\lambda} \tag{26}$$

where in $\widehat{\omega}[\lambda u]$ each $u^\alpha_{\mathbf{n}+\mathbf{m}}$ is replaced by $\lambda u^\alpha_{\mathbf{n}+\mathbf{m}}$ and each $\mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}}$ is replaced by $\lambda \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}}$. By the definition of functions in $\mathcal{A}$, the number of terms in $\mathbf{v}_u \lrcorner\, \widehat{\omega}$ is finite.

It is a standard calculation that

$$\widehat{h}\,\widehat{\mathrm{d}} + \widehat{\mathrm{d}}\,\widehat{h} = \mathrm{id}.$$

Hence if $\widehat{\mathrm{d}}\,\widehat{\omega} = 0$ then $\widehat{\mathrm{d}}\,\widehat{h}\widehat{\omega} = \widehat{\omega}$, showing that $\ker \widehat{\mathrm{d}}|_{\Lambda^{j+1}} \subseteq \widehat{\mathrm{d}}(\widehat{\Lambda}^j)$. The reverse inclusion follows from $\widehat{\mathrm{d}}^{\,2} = 0$. □

Example 3.2.18. If $\widehat{\omega} = nu_{n+1}\mathrm{d}u_n\mathrm{d}u_{n+1}$ then $\widehat{\omega}[\lambda u] = \lambda^3 nu_{n+1}\mathrm{d}u_n\mathrm{d}u_{n+1}$ and

$$\begin{aligned}
\frac{1}{\lambda}\mathbf{v}_u \lrcorner \widehat{\omega}[\lambda u] &= \lambda^2 \left[nu_n u_{n+1}\mathrm{d}u_{n+1} - nu_{n+1}^2\mathrm{d}u_n\right] \\
\widehat{h}(\widehat{\omega}) &= \frac{1}{3}(nu_n u_{n+1}\mathrm{d}u_{n+1} - nu_{n+1}^2\mathrm{d}u_n) \\
(\widehat{\mathrm{d}}\,\widehat{h} + \widehat{h}\widehat{\mathrm{d}})(\widehat{\omega}) &= \widehat{\mathrm{d}}\,\widehat{h}(\widehat{\omega}) \qquad \text{as } \widehat{\mathrm{d}}(\widehat{\omega}) = 0 \\
&= \frac{1}{3}(nu_{n+1}\mathrm{d}u_n\mathrm{d}u_{n+1} + 2nu_{n+1}\mathrm{d}u_n\mathrm{d}u_{n+1}) \\
&= \widehat{\omega}
\end{aligned}$$

Definition 3.2.19. The action of each shift map on vertical forms is defined by

$$\begin{aligned}
S_k(\widehat{w} + \widehat{\eta}) &= S_k\widehat{w} + S_k\widehat{\eta}, \\
S_k(c\widehat{w}) &= cS_k\widehat{w}, \qquad c \in \mathbb{R}, \\
S_k(\widehat{\mathrm{d}}\widehat{w}) &= \widehat{\mathrm{d}}(S_k\widehat{w}), \\
S_k(\widehat{\omega} \wedge \widehat{\eta}) &= S_k(\widehat{\omega}) \wedge S_k(\widehat{\eta}),
\end{aligned} \tag{27}$$

together with the standard action on the coefficients given in (22). Hence $S_k\mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}} = \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}+\mathbf{1}_k}$ and $S^{\mathbf{m}}\mathrm{d}u^\alpha_{\mathbf{n}} = \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}}$.

3.3 Functional forms

The second part of the discrete variational complex is a quotient of the vertical complex described above under an equivalence relation. In the continuous case, two functions are equivalent if they differ by a total divergence. Here we say that two functions of the iterates are equivalent if they differ by the *total discrete divergence* Div_Δ of a vector $\mathbf{g} \in \mathcal{A}^p$, which is defined as follows:

$$\mathrm{Div}_\Delta(\mathbf{g}) = \sum_{k=1}^{p}(S_k - \mathrm{id})g_k[u]. \tag{28}$$

Note that every total difference p-form ω which belongs to $\mathrm{im}(\boldsymbol{\Delta})$ is of the form

$$\omega = \mathrm{Div}_{\Delta}(\mathbf{F})\Delta_{i_1}\Delta_{i_2}\ldots\Delta_{i_p},$$

for some $\mathbf{F} \in \mathcal{A}^p$.

Definition 3.3.20. We define an equivalence class on $\mathcal{A}$ by

$$f_1 \sim f_2 \quad \Longleftrightarrow \quad f_1 - f_2 = \mathrm{Div}_{\Delta}(\mathbf{g})$$

for some functions $\mathbf{g} \in \mathcal{A}^p$. The set of *functionals* $\mathcal{F}$ is defined to be the set of equivalence classes,

$$\mathcal{F} = \mathcal{A}/\sim .$$

We denote the equivalence class of $f \in \mathcal{A}$ by $\sum f$. The notation reflects the (formal) identity that

$$\sum_{\mathbf{n}} f[u] = 0$$

if f is a total discrete divergence. Note that $\mathcal{F}$ is not an algebra, that is, products of functionals are not functionals.

An equivalence relation on $\widehat{\Lambda}^r$ of vertical r-forms can be defined similarly,

$$\widehat{w} \sim \widehat{w}^1 \qquad \Longleftrightarrow \qquad \widehat{w} = \widehat{w}^1 + \sum_{k=1}^{p}(S_k - \mathrm{id})\widehat{\eta}_k \qquad \widehat{\eta}_k \in \widehat{\Lambda}^r$$

for some $\widehat{\eta}_k$, $k = 1, \ldots, p$, where S_k acts on $\widehat{\eta}_k$ according to the formulae (27). Again, we denote the equivalence class of $\widehat{w}$ by

$$\sum \widehat{w}.$$

The equivalence classes are called *functional forms*, and the set of equivalence classes $\widehat{\Lambda}^r/\sim$ is denoted by Λ^r_*

3.3.1 Analogue of integration by parts

In the continuous variational complex, much use is made of the product rule of differential calculus, and the consequent integration by parts, not only in the study of canonical forms of equivalence classes but at every stage in the proof of exactness of the continuous variational complex. In the discrete case studied here, neither the shift maps S_k nor the difference maps $S_k - \mathrm{id}$ obey the Leibniz product rule. However, there is a formula which plays the role of integration by parts in our variational complex for discrete systems.

To motivate this formula, we first consider functions of a single lattice coordinate, n; we write the corresponding shift operator as S. Given any two (square-summable) sequences $\{f_n\}$, $\{g_n\}$, we have the identity

$$\sum_{n=-\infty}^{\infty} (Sf)_n \, g_n = \sum_{n=-\infty}^{\infty} f_{n+1} \, g_n = \sum_{n=-\infty}^{\infty} f_n \, g_{n-1} = \sum_{n=-\infty}^{\infty} f_n (S^{-1} g)_n$$

by a change of dummy variable. This result is easily extended to an arbitrary number of lattice coordinates. Moreover, given $f,\, g \in \mathcal{A}$,

$$(S_k f) g - f(S_k^{-1} g) = (S_k - \mathrm{id})(f S_k^{-1} g)$$

for each k. Hence

$$\textstyle\sum (S_k f) g = \sum f(S_k^{-1} g) \tag{29}$$

using both the definition and the natural interpretation. Equation (29) is the analogue of "integration by parts". To prove that the variational complex is locally exact, we need analogues of the higher Euler operators, which we derive below. Our analogues are obtained by replacing $(-D)^{\mathbf{m}}$ by $S^{-\mathbf{m}}$. This "replacement rule" follows from using (29) rather than the usual integration by parts formula.

3.3.2 The vertical functional complex

In this section we take the vertical complex defined earlier and project it to the equivalence classes of functional forms. The result is again an exact complex, which will form the right hand side of the variational complex we are developing.

Definition 3.3.21. Let $w = \sum \widehat{w}$ be the functional r-form corresponding to the vertical r-form $\widehat{w}$. Then the *variational derivative* of w is defined to be

$$\delta w = \sum \widehat{\mathrm{d}}\widehat{w}.$$

Lemma 3.3.22. δ is well-defined.

Proof: We need to show that $\widehat{\mathrm{d}}$ is identically zero on total differences, or equivalently that $\widehat{\mathrm{d}}\, S_k = \widehat{\mathrm{d}}$, for any k. Let f be an arbitrary function in $\mathcal{A}$. Then

$$\begin{aligned}
\widehat{\mathrm{d}}(S_k f) &= \sum_{\alpha,\mathbf{m}} \frac{\partial S_k f}{\partial u^\alpha_{\mathbf{n}+\mathbf{m}}} \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}} \\
&= \sum_{\alpha,\mathbf{m}} S_k \left(\frac{\partial f}{\partial u^\alpha_{\mathbf{n}+\mathbf{m}-\mathbf{1}_k}} \right) \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}} \\
&= \sum_{\alpha,\mathbf{m}} \frac{\partial f}{\partial u^\alpha_{\mathbf{n}+\mathbf{m}-\mathbf{1}_k}} S_k^{-1} \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}} \\
&= \sum_{\alpha,\mathbf{m}} \frac{\partial f}{\partial u^\alpha_{\mathbf{n}+\mathbf{m}-\mathbf{1}_k}} \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}-\mathbf{1}_k} \\
&= \widehat{\mathrm{d}} f \qquad \square
\end{aligned}$$

As $\widehat{\mathrm{d}}^{\,2} = 0$, it follows immediately that $\delta^2 = 0$.

Theorem 3.3.23. The *vertical functional complex*

$$0 \longrightarrow \Lambda^0_* \xrightarrow{\delta} \Lambda^1_* \xrightarrow{\delta} \cdots \tag{30}$$

is exact.

Proof: We show first that the homotopy operator $\widehat{h}$ (26) is well-defined on equivalence classes. Let S denote S_k for some k. From the identities

$$S\mathbf{v}_u = \mathbf{v}_u, \qquad S(\omega[\lambda u]) = S(\omega)[\lambda u],$$

it is simple to see that

$$\widehat{h}S = S\widehat{h}.$$

Thus if $f_1 - f_2 = Sg - g$ then $\widehat{h}f_1 - \widehat{h}f_2 = (S - \mathrm{id})\widehat{h}g$, and so

$$f_1 \sim f_2 \Longrightarrow \widehat{h}f_1 \sim \widehat{h}f_2.$$

Hence

$$(\delta\widehat{h} + \widehat{h}\delta)\sum\omega = \sum(\widehat{\mathrm{d}}\,\widehat{h} + \widehat{h}\,\widehat{\mathrm{d}})\omega = \sum\omega$$

showing that

$$\Lambda^0_* \xrightarrow{\delta} \Lambda^1_* \xrightarrow{\delta} \cdots$$

is exact.

The kernel of $\widehat{\mathrm{d}}\Big|_{\widehat{\Lambda}^0}$ consists of functions which do not depend on the $u^\alpha_{\mathbf{n}+\mathbf{m}}$, that is, are functions of the n_i alone. All such functions are equivalent to zero, as we will now show, completing the proof of exactness of (30).

To show that $f = f(\mathbf{n}) \Longrightarrow f \sim 0$, it is enough to show that the equation

$$f(\mathbf{n}) = S_p g - g \tag{31}$$

has a solution g for any given f. We may take as the initial condition that g is zero on the hyperplane $\{(n^1, \ldots n^{p-1}, n^p) \mid n^p \equiv n^0\}$. Then, for any given f, the values of $g(n^1, \ldots n^{p-1}, n^0 \pm 1)$ can be obtained from (31). The process repeats to give g on all of $\mathbb{Z}^p$. $\square$

4 The discrete variational complex

To complete the construction of a variational complex for difference equations, we must patch together the horizontal and vertical functional complexes. This is accomplished with the Euler operator, which has been studied by Kupershmidt [23]. It turns out that the Euler operator E can be defined in the same way for both differential and difference equations, by using the Fréchet derivative as follows.

The Fréchet derivative of an r-tuple $P[u] \in \mathcal{A}^r$ is the differential operator $\mathbf{D}_P : \mathcal{A}^q \to \mathcal{A}^r$ defined by

$$\mathbf{D}_P(Q) = \lim_{\epsilon \to 0} \left\{ \frac{P[u + \epsilon Q[u]] - P[u]}{\epsilon} \right\} \tag{32}$$

where $Q[u]$ is an arbitrary element of $\mathcal{A}^q$. (Note that the independent variables act as parameters here.) Then the Euler operator is defined by its action on arbitrary elements $P[u] \in \mathcal{A}^r$, as follows:

$$E(P[u]) = \mathbf{D}_P^*(1), \tag{33}$$

where $\mathbf{D}_P^*$ is the adjoint of $\mathbf{D}_P$ with respect to the appropriate inner product. For difference equations, this is the ℓ_2 inner product. The component of the Euler operator that corresponds to the dependent variable u^α is

$$E_\alpha(f) = \sum_{\mathbf{m}} S^{-\mathbf{m}} \frac{\partial f[u]}{\partial u^\alpha_{\mathbf{n}+\mathbf{m}}}\,.$$

The Euler operator induces the following action on elements of $\mathbf{Ex}^p$:

$$E(f \Delta_1 \cdots \Delta_p) = \sum_{\alpha=1}^{q} E_\alpha(f)\, \mathrm{d}u^\alpha.$$

We let $\pi : \widehat{\Lambda}^r \longrightarrow \Lambda^r_*$ denote the projection which takes a vertical form to its equivalence class,

$$\pi(\widehat{\omega}) = \sum \widehat{\omega}. \tag{34}$$

Note that π is a surjection. Then the discrete variational complex is, writing $\pi \circ E$ as $\mathbf{E}$

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbf{Ex}^0 \xrightarrow{\boldsymbol{\Delta}} \mathbf{Ex}^1 \longrightarrow \cdots \xrightarrow{\boldsymbol{\Delta}} \mathbf{Ex}^p \xrightarrow{\mathbf{E}} \Lambda^1_* \xrightarrow{\delta} \Lambda^2_* \xrightarrow{\delta} \cdots \tag{35}$$

In §3.3 we showed exactness of the complex to the right of Λ^1_*. In this section we complete the proof that this complex is exact. First we look in detail at the point where the horizontal complex and the vertical functional complex are spliced together using the Euler operator.

4.1 Exactness around E

Consider Figure 2, where π is given by (34), and where we have written $E(f)\mathrm{d}u$ for $\sum_\alpha E_\alpha(f)\mathrm{d}u^\alpha_{\mathbf{n}}$ for simplicity.

$$\begin{array}{ccccccc} \xrightarrow{\boldsymbol{\Delta}} \mathbf{Ex}^{p-1} \xrightarrow{\boldsymbol{\Delta}} & \mathbf{Ex}^p & \xrightarrow{E} & \widehat{\wedge}^1 & & \\ & \pi\downarrow & & \downarrow\pi & & \\ 0 \longrightarrow & \wedge^0_* & \xrightarrow{\delta} & \wedge^1_* & \xrightarrow{\delta} & \wedge^2_* \longrightarrow \end{array}$$

$$\begin{array}{ccc} f\ \Delta_1 \cdots \Delta_p & \xrightarrow{E} & E(f)du \\ \pi\downarrow & & \downarrow\pi \\ \sum f & \xrightarrow{\delta} & \sum E(f)du \end{array}$$

Figure 2: Splicing the horizontal and vertical functional complexes

Lemma 4.1.24. $\delta \circ \pi = \pi \circ E$.

Proof:

$$\pi \circ E(f) \;=\; \sum \sum_\alpha E_\alpha(f) \mathrm{d}u^\alpha_{\mathbf{n}}$$

$$= \sum \left(\sum_{\alpha,\mathbf{m}} (S^{-\mathbf{m}} \partial_{u^\alpha_{\mathbf{n}+\mathbf{m}}} f) \mathrm{d}u^\alpha_{\mathbf{n}}\right)$$

$$= \sum \left(\sum_{\alpha,\mathbf{m}} (\partial_{u^\alpha_{\mathbf{n}+\mathbf{m}}} f)(S^{\mathbf{m}} \mathrm{d}u^\alpha_{\mathbf{n}})\right)$$

$$= \sum \left(\sum_{\alpha,\mathbf{m}} \partial_{u^\alpha_{\mathbf{n}+\mathbf{m}}} f \mathrm{d}u^\alpha_{\mathbf{n}+\mathbf{m}}\right)$$

$$= \sum \widehat{\mathrm{d}} f$$

$$= \delta \circ \pi(f)$$

where $S^{\mathbf{m}}$ is defined in (21) and we have used the action of the shift maps on vertical forms given in (27). $\square$

Theorem 4.1.25. The variational complex is exact at $\mathbf{Ex}^p$ and at Λ^1_*. Specifically,

A. The Euler-Lagrange operator has for its kernel precisely those functions in $\mathcal{A}$ that are total discrete divergences, that is,

$$\sum E(f)\mathrm{d}u = 0 \Leftrightarrow \sum f = 0$$

B. The variational derivative $\delta|_{\Lambda^1_*}$ has for its kernel precisely those expressions which are Euler-Lagrange equations, that is,

$$\delta(\sum \sum_\alpha f_\alpha \mathrm{d}u^\alpha) = 0 \Leftrightarrow f_\alpha = E_\alpha(\mathcal{L}) \text{ for some } \mathcal{L}$$

Proof: To show Part A,

$$
\begin{aligned}
& \sum E(f)\mathrm{d}u = 0 & \\
\iff\ & \pi \circ E(f) = 0 & \\
\iff\ & \delta \circ \pi(f) = 0 & \text{by preceeding Lemma} \\
\iff\ & \pi(f) = 0 & \text{by exactness of (30)} \\
\iff\ & \sum f = 0, &
\end{aligned}
$$

which means that f is a total discrete divergence.

To show Part B, if $\omega^* \in \Lambda^1_*$ is such that $\delta\omega^* = 0$, then by exactness of the vertical functional complex, there exists $\eta^* \in \Lambda^0_*$ such that $\delta\eta^* = \omega^*$. But $\eta^* = \sum \eta = \pi(\eta)$ for some $\eta \in \widehat{\Lambda}^0$ because π is surjective. Hence $\omega^* = \delta\pi(\eta) = \pi E(\eta)$, showing that ω^* is in the image of πE as required. Since $\eta \in \mathsf{Ex}^p$ it is of the form $f\Delta_1 \cdots \Delta_p$ for some $f \in \mathcal{A}$. Then the desired Lagrangian $\mathcal{L}$ is f. $\square$

4.2 Exactness of the horizontal complex

4.2.1 The Higher Euler operators and the Total Homotopy Operator

Definition 4.2.26. Given two multi-indices $\mathbf{m} = (m_1, m_2, \ldots, m_p)$ and $\mathbf{l} = (l_1, l_2, \ldots, l_p)$ we say $\mathbf{m} \supset \mathbf{l}$ if $m_i \geq l_i$ for all $i = 1, \ldots, p$. We further define the order of the multi-index, $\#\mathbf{m} = m_1 + \ldots + m_p$.

Definition 4.2.27. We define the higher Euler operators

$$E^J_\alpha = \sum_{I \supset J} \binom{I}{J} S^{-I} \partial_{u^\alpha_{\mathbf{n}+I}},$$

where $S^{-I} = S_1^{-i_1} S_2^{-i_2} \cdots S_p^{-i_p}$.

Definition 4.2.28. The *total interior product* for the q-tuple $\mathbf{Q} \in \mathcal{A}^q$ is

$$\mathbf{I}_{\mathbf{Q}}(\omega) = \sum_I \sum_{\alpha=1}^{q} \sum_{k=1}^{p} \frac{i_k + 1}{p - r + \#I + 1} (S - id)^I \left(Q_\alpha E_\alpha^{I+1_k} \left(\frac{\partial}{\partial_{n^k}} \lrcorner\, \omega \right) \right),$$

for $\omega \in \mathbf{Ex}^r$.

Definition 4.2.29. Given $\omega = \omega[n, \mathbf{u}]$, let $\omega[n, 0]$ be the projection of ω obtained by setting $u_{\mathbf{m}}^\alpha = 0$ for all α, $\mathbf{m}$. Then $\omega[n, 0]$ is in the difference complex, that is, has coefficients in $\mathcal{B}$ (recall a function is in $\mathcal{B}$ if it depends on the n^j only).

Example 4.2.30. If $\omega = u_{n^1, n^2+1}\Delta_1 + n^1\Delta_2$ then $\omega[n, 0] = n^1\Delta_2$.

Theorem 4.2.31. For $\omega \in \mathbf{Ex}^r$ with $r > 0$ define the operator

$$H(\omega) = \int_{\lambda=0}^{1} \mathbf{I}_{\mathbf{u}}(\omega)[\lambda u_{\mathbf{n}}] \frac{\mathrm{d}\lambda}{\lambda}$$

where $\mathbf{u} = (u^1, \ldots, u^q)$, and set

$$H|_{\mathbf{Ex}^0}(\omega) = \omega[n, 0].$$

Then the operator H satisfies

$$\begin{aligned} \omega[n, \mathbf{u}] - \omega[n, 0] &= \boldsymbol{\Delta} H(\omega) + H(\boldsymbol{\Delta}\omega) \qquad \omega \in \mathbf{Ex}^r,\ r > 0 \\ \omega[n, \mathbf{u}] - \omega[n, 0] &= H(\omega) + H(\boldsymbol{\Delta}\omega) \qquad \omega \in \mathbf{Ex}^0 \end{aligned}$$

The form $\omega[n, 0]$ is in the difference complex. That is, the coefficients are in $\mathcal{B}$. In order to obtain the *total homotopy operator*, we need to add to H the homotopy operator $H_{\mathcal{B}}$ for the difference complex.

Theorem 4.2.32. Let $H_{\mathcal{B}}$ be the homotopy operator (13) for the difference complex. Then the *total homotopy operator*

$$H_T(\omega) = H(\omega) + H_{\mathcal{B}}(\omega[n, 0])$$

satisfies

$$\begin{aligned} \omega &= \boldsymbol{\Delta} H_T(\omega) + H_T(\boldsymbol{\Delta}\omega) \qquad \omega \in \mathbf{Ex}^r,\ r > 0 \\ \omega &= H_T(\omega) + H_T(\boldsymbol{\Delta}\omega) \qquad\qquad \omega \in \mathbf{Ex}^0 \end{aligned}$$

Example 4.2.33. As an example of the use of the total homotopy operator, consider the difference equation,

$$w[n, \mathbf{u}] \equiv ((n+1)u_{n+1}u_{n+2} - nu_n u_{n+1} - (n+1)) = 0.$$

This is the OΔE

$$u_{n+2} - \frac{n}{n+1}u_n - \frac{1}{u_{n+1}} = 0$$

multiplied by the characteristic $P = S(nu_n) = (n+1)u_{n+1}$. As $E(w) = 0$, w must be a divergence. In the one dimensional case, this means that we may reconstruct the first integral by using the homotopy map!

To do this, we write the equation as an element of $\mathbf{Ex}^p$, (in this case, $p = 1$),

$$\omega = w\Delta_1.$$

The higher order Euler operators in one dimension are

$$E^{(1)} = S^{-1}\partial_{u_{n+1}} + 2S^{-2}\partial_{u_{n+2}} + \cdots, \quad E^{(2)} = S^{-2}\partial_{u_{n+2}} + \cdots$$

and the total interior product is

$$\mathbf{I}_u(\omega) = \sum_{I=0}^{\infty}(S - \mathrm{id})^I(u_n E^{(I+1)}((n+1)u_{n+1}u_{n+2} - nu_n u_{n+1} - (n+1)))$$

Therefore

$$\begin{aligned} \mathbf{I}_u(\omega) = u_n[S^{-1}(n+1)u_{n+2} - nu_n] + 2S^{-2}((n+1)u_{n+1}) \\ +(S - \mathrm{id})(u_n S^{-2}((n+1)u_{n+1})) \\ = 2nu_n u_{n+1}. \end{aligned}$$

Now $\omega[n,0] = -(n+1)\Delta_1$. Using $\partial_n \lrcorner \Delta_1 = 1$ in the formula for $H_{\mathcal{B}}$,

$$H_{\mathcal{B}}(\omega[n,0]) = -\sum_{k=0}^{n-1} (\partial_n \lrcorner (n+1)\Delta_1)\,|_{n=k} = -\sum_{k=0}^{n-1} (n+1)|_{n=k} = -\sum_{k=1}^{n} k.$$

Finally $\omega = \boldsymbol{\Delta}\theta$, where

$$\begin{aligned} \theta = H_T(\omega) &= \int_{\lambda=0}^{1} 2\lambda n u_n u_{n+1} \mathrm{d}\lambda - \sum_{k=1}^{n} k \\ &= n u_n u_{n+1} - \frac{1}{2} n(n+1). \end{aligned}$$

5 Some applications of the discrete variational complex

5.1 How to obtain Lagrangians

What do the above results mean for the study of systems of partial difference equations? Given a system of PΔEs $P_1 = 0$, $P_2 = 0$, $\ldots$, $P_q = 0$, we write down an element of Λ_1^*, namely

$$\mathbf{P} = \sum P_1 \mathrm{d}u_{\mathbf{n}}^1 + P_2 \mathrm{d}u_{\mathbf{n}}^2 + \cdots + P_q \mathrm{d}u_{\mathbf{n}}^q.$$

This involves deciding which equation belongs to which dependent variable, that is, for which j is $P_i = E_j(\mathcal{L})$ for some (as yet unspecified) $\mathcal{L}$. Assigning the wrong P_i to each $\mathrm{d}u_{\mathbf{n}}^j$ may cause the discrete Helmholtz condition to fail, i.e. $\delta\mathbf{P} \neq 0$, so the fact that the system is an Euler-Lagrange system may be missed. Worse is if the system is only *equivalent* to an Euler-Lagrange system, for example, if $E_1(\mathcal{L}) = g_1$, $E_2(\mathcal{L}) = g_2$, and $P_1 = g_1 + g_2$, $P_2 = g_1 - Sg_2$. Even for the continuous case, the general equivalence problem, of detecting when a system is equivalent to an Euler-Lagrange system, is open (see p.

355 of Olver [34]). However, given that $\delta\mathbf{P} = 0$, we may use the homotopy operator $\widehat{h}$ for the vertical functional complex to find $\mathcal{L}$.

Example 5.1.34. The first discrete Painlevé equation is

$$P \equiv u_{n+1} + u_n + u_{n-1} + \frac{\alpha n + \beta}{1 + u_n} + \mu = 0. \tag{36}$$

(See equation (3.3.1) in [14]; we have set a third constant γ to be zero and translated u_n to $u_n + 1$ to simplify our calculations here). This equation satisfies $\mathbf{D}_P = \mathbf{D}_P^*$ and hence a Lagrangian exists for this equation. Calculating the homotopy yields

$$\begin{aligned} L &= \int_0^1 P[\lambda u] u_n \,\mathrm{d}\lambda \\ &= \int_0^1 \left[\mu u_n + \lambda(u_{n+1} + u_n + u_{n-1}) u_n + \frac{\alpha n + \beta}{1 + \lambda u_n} u_n \right] \mathrm{d}\lambda \\ &= \mu u_n + \tfrac{1}{2}(u_{n+1} + u_n + u_{n-1}) u_n + (\alpha n + \beta) \log(1 + u_n) \end{aligned}$$

This is equivalent to

$$\tilde{L} = \mu u_n + \tfrac{1}{2}(u_n)^2 + u_n u_{n+1} + (\alpha n + \beta) \log(1 + u_n).$$

It is straightforward to check that $E(L) = P$.

Example 5.1.35. As a more substantial example, consider the following system of PΔEs:

$$P_1 \equiv u^2_{n^1, n^2-1} - u^2_{n^1+1, n^2} + \alpha(u^1_{n^1, n^2}) = 0, \tag{37}$$

$$P_2 \equiv u^1_{n^1, n^2+1} - u^1_{n^1-1, n^2} - \alpha(u^2_{n^1, n^2}) = 0. \tag{38}$$

It is easy to check that $\delta\mathbf{P} = 0$. Then, provided that $t\alpha(t) \to 0$ as $t \to 0$, the homotopy map yields (up to equivalence)

$$L = u^1_{n^1, n^2+1} u^2_{n^1, n^2} - u^1_{n^1, n^2} u^2_{n^1+1, n^2} + \int_{t = u^2_{n^1, n^2}}^{u^1_{n^1, n^2}} \alpha(t) \,\mathrm{d}t.$$

(For problems in which the homotopy operator is singular, consult [2].)

5.2 How to obtain conservation laws

We give an application of the higher Euler operators to calculating first integrals of equations which are Euler-Lagrange equations of a Lagrangian. The Lagrangian is assumed to have a variational symmetry, which we define below. Thus we have a discrete analogue of Noether's theorem.

The multi-dimensional case is discussed in [20]; here we show the definitions, calculations and an example for ordinary difference equations.

Definition 5.2.36. The first-order partial differential operator

$$X = Q\partial_{u_n} + (SQ)\partial_{u_{n+1}} + \dots$$

generates a variational symmetry of a Lagrangian $L(n, u_n, \dots u_{n+m})$ if

$$XL = 0.$$

The function Q is called the characteristic of the symmetry.

Theorem 5.2.37. If X is a variational symmetry of a Lagrangian L with characteristic Q, then

$$\varphi = \sum_{k=0}^{\infty}(S - \mathrm{id})^k(QE^{(k+1)}L)$$

is a first integral of

$$E(L) = 0,$$

where the $E^{(k)}$ are the higher Euler operators

$$E^{(k)} = \sum_{\ell=0}^{\infty} \binom{k+\ell}{k} S^{-(k+\ell)}\,\partial_{u_{n+k+\ell}}$$

Indeed, it can be shown by direct calculation that

$$(S\varphi - \varphi)|_{E(L)=0} = 0.$$

Example 5.2.38. Take

$$L = \frac{u_n}{u_{n+1}} + n\left(\log|u_{n+1}| - \log|u_n|\right)$$

which has a scaling symmetry. Indeed,

$$X = u_n \partial_{u_n} + u_{n+1}\partial_{u_{n+1}} + \dots$$

is the variational symmetry, with characteristic $Q = u_n$. The Euler-Lagrange equation can be written, after some simplification, as

$$u_{n+1} = \frac{u_n^2}{u_n + u_{n-1}}$$

and thus

$$\begin{aligned} \varphi &= QE^{(1)}\,L \\ &= QS^{-1}\left(\partial_{u_{n+1}}L\right) \\ &= n - 1 - \tfrac{u_{n-1}}{u_n} \end{aligned}$$

is a first integral, that is, is constant on solutions. Indeed it is simple to verify that $(S\varphi - \varphi)|_{E(L)=0} = 0$. Finally, it is a simple matter to solve the equation $\varphi - c_1 = 0$ to obtain the solution,

$$u_n = c_2/\Gamma(n - c_1).$$

Although the above example is fairly simple, it is possible to use the variational complex to construct conservation laws of arbitrary systems of difference equations, whether or not Noether's Theorem is valid. For details of how to do this, see [20].

5.3 Continuum Limits

Several authors have developed discretized versions of particular continuous variational calculations [10, 31]. Such developments, by construction, have an inbuilt continuum limit. The variational complex developed in this article has no implicit continuum limit. This is important because difference equations arise naturally in applications such as quantum gravity, and should therefore be regarded as objects in their own right. Indeed, difference equations can have several continuum limits, or no continuum limit at all, and they may have important solutions which are not approximations to solutions of a continuum limit.

Nevertheless, given the use of difference equations as approximations to continuous models especially for numerical calculation, it is important to consider how our results behave under limiting processes. Here we consider only an example.

Recall the first discrete Painlevé equation (cf. 36),

$$P \equiv u_{n+1} + u_n + u_{n-1} + \frac{a+bn}{1+u_n} - 3. \tag{39}$$

A continuum limit is given by

$$\begin{aligned} u_n &\sim \epsilon^2 w(t) \\ u_{n\pm 1} &\sim \epsilon^2 w(t \pm \epsilon) \\ a + bn &\sim 3 + \epsilon^4 t \end{aligned} \tag{40}$$

which when inserted into (39) yields

$$P \sim \epsilon^4 (w'' + 3w^2 + t) + \mathcal{O}(\epsilon^6). \tag{41}$$

The discrete Lagrangian for (39) (calculated in §4.2) is

$$\begin{aligned} L &= \tfrac{1}{2}(u_{n+1} + u_n + u_{n-1})u_n - 3u_n + (a+bn)\log(1+u_n) \\ &\sim \epsilon^6 (\tfrac{1}{2} w w'' + wt + w^3) + \mathcal{O}(\epsilon^8). \end{aligned} \tag{42}$$

Now, $\mathcal{L} = \frac{1}{2} w w'' + wt + w^3$ is in fact a Lagrangian for (39), being equivalent to the usual Lagrangian $-\frac{1}{2}(w')^2 + wt + w^3$ (recall two Lagrangians are equivalent if they differ by a total divergence).

Next, we consider the continuum limit of the discrete Euler-Lagrange operator. It is helpful to introduce the forward difference operator $\Delta = S - \mathrm{id}$ (which should not be confused with the difference operator of §2). Suppose that $L = L(n, u_n, \ldots u_{n+m})$. Then, using $L_{u_{n+m}}$ to denote $\partial L/\partial u_{n+m}$, we obtain

$$\begin{aligned}
E(L) &= L_{u_n} + S^{-1} L_{u_{n+1}} + S^{-2} L_{u_{n+2}} + \cdots \\
&= L_{u_n} + (S^{-1} - \mathrm{id}) L_{(S-\mathrm{id})u_n} + (S^{-1} - \mathrm{id})^2 L_{(S-\mathrm{id})^2 u_n} + \cdots \\
&= L_{u_n} - \Delta S^{-1} L_{\Delta u_n} + \Delta^2 S^{-2} L_{\Delta^2 u_n} \mp \cdots \\
&= L_{u_n} - \tfrac{\Delta}{\epsilon} S^{-1} L_{\Delta u_n/\epsilon} + \tfrac{\Delta^2}{\epsilon^2} S^{-2} L_{\Delta^2 u_n/\epsilon^2} \mp \cdots
\end{aligned}$$

where to obtain the final line we have used the dummy scalar ϵ together with the identity $\partial/\partial u = (1/\epsilon)\, \partial/\partial(u/\epsilon)$.
If we now use the continuum limit (40) given above, with $u_{n\pm k} = \epsilon^2 w(t \pm k\epsilon)$, we obtain

$$\Delta u_n \sim \epsilon^3 w', \qquad \Delta^2 u_n \sim \epsilon^4 w'', \qquad \frac{\Delta}{\epsilon} \sim \frac{\mathrm{d}}{\mathrm{d}t}$$

and thus

$$E(L) \sim \frac{1}{\epsilon^2} \left(\mathcal{L}_w - \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{w'} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{w''} \mp \cdots \right) \tag{43}$$

The first point to note is that the continuum limit of the Euler-Lagrange operator is the continuous Euler-Lagrange operator. Further, it can be seen that the powers of ϵ are consistent, so that taking the separate continuum limits of the Lagrangian and the Euler-Lagrange operator is consistent with taking the continuum limit of $E(L)$, to obtain the continuum limit of P.

6 Conclusion

In this paper, we have derived a discrete analogue of the variational complex. All of the constructions that involve the (continuous) dependent variables are analogous to the continuous case, although the formulae are modified somewhat. However, the proof of local exactness of the difference complex is not at all similar to the proof of the Poincaré Lemma. Now that the homotopy maps for the discrete variational complex are known, it is simple (in principle) to construct conservation laws of PΔEs, without reference to Noether's Theorem. The main difficulty lies in the complexity of the calculations (see [20] for further details).

Throughout the paper we have restricted attention to difference equations with real-valued coefficients. However, all results are also true if $\mathbb{R}$ is replaced by $\mathbb{C}$.

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References

[1] Anco, S. C. & Bluman, G. 2002 Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications. *Eur. J. Appl. Math.* (in press).

[2] Anco, S. C. & Bluman, G. 2002 Direct construction method for conservation laws of partial differential equations. Part II: General treatment. *Eur. J. Appl. Math.* (in press).

[3] Anderson, I. M. 1992 Introduction to the variational bicomplex. *Contemp. Math.* **132** Amer. Math. Soc., Providence, RI, pp 51–73.

[4] Bluman, G. W. & Kumei, S. 1989 *Symmetries and differential equations.* New York: Springer.

[5] Bridges, T.J. 1997 Multi-symplectic structures and wave propagation. *Math. Proc. Camb. Phil. Soc.* **121** 147–190.

[6] Bridges, T.J. 1997 A geometric formulation of the conservation of wave action and its implications for signature and the classification of instabilities, *Proc. Roy. Soc. Lond.* A **453** 1365–1395.

[7] Bridges, T.J. & Reich, S. 2001 Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity. *Phys. Lett. A* **284** 184–193.

[8] Budd, C. J. & Collins, G. J. 1998 An invariant moving mesh scheme for the nonlinear diffusion equation. *Appl. Num. Math.* **26**: 23–39.

[9] Budd, C. J. & Iserles, A. 1999 Geometric integration: Numerical solution of differential equations on manifolds. *Phil. Trans. Roy. Soc. Lond.* A **357**: 945–956.

[10] Dorodnitsyn, V. A. 1994 Symmetry of finite-difference equations. In *CRC handbook of Lie group analysis of differential equations, vol. 1.* (ed. N. H. Ibragimov), pp. 365–403. Boca Raton: CRC Press.

[11] Dorodnitsyn, V. A., Kozlov, R. & Winternitz, P. 2000 Lie group classification of second-order ordinary difference equations. *J. Math. Phys.* **41**, 480–504.

[12] Floreanini, R. & Vinet, L. 1995 Lie symmetries of finite-difference equations. *J. Math. Phys.* **36**, 7024–7042.

[13] Gaeta, G. 1993 Lie-point symmetries of discrete versus continuous dynamical systems. *Phys. Lett.* **178A**, 376–384.

[14] Grammaticos, B., Nijhoff, F. W. & Ramani, A. 1996 *Discrete Painlevé Equations.* Lecture Notes for the Cargèse School, 1996

[15] Gu C. H. 1995 *Soliton Theory and its Applications.* Springer-Verlag, Berlin.

[16] Hereman, W. 1996 Symbolic software for Lie symmetry analysis. In *CRC handbook of Lie group analysis of differential equations, vol. 3.* (ed. N. H. Ibragimov), pp. 367–413. Boca Raton: CRC Press.

[17] Hirsch, C. 1988 *Numerical Conservation of Internal and External Flows* (Vol. 1). Wiley, Chichester.

[18] Hydon, P. E. 2000 *Symmetry methods for differential equations.* Cambridge: Cambridge University Press.

[19] Hydon, P. E. 2000 Symmetries and first integrals of ordinary difference equations. *Proc. Roy. Soc. Lond.* A **456**: 2835–2855.

[20] Hydon, P. E. 2001 Conservation laws of partial difference equations with two independent variables. *J. Phys. A: Math. Gen.* **34**: 10347–10355.

[21] Iserles, A., Munthe-Kaas, H., Nørsett, S. P. & Zanna, A. 2000 Lie-group methods. *Acta Numerica* **9**: 215–365.

[22] Iserles, A. & Nørsett, S. P. 1999 On the solution of linear differential equations in Lie groups. *Phil. Trans. Roy. Soc. Lond.* A **357**: 983–1019.

[23] Kupershmidt, B. 1985 Discrete Lax equations and differential-difference calculus. *Astérisque* **123**, SMF, Paris.

[24] Lee, T. D. 1987 Difference equations and conservation laws. *J. Stat. Phys.* **46**, 843–860.

[25] Levi, D., Vinet, L. & Winternitz, P. 1997 Lie group formalism for difference equations. *J. Phys. A: Math. Gen.* **30**, 633–649.

[26] Maeda, S. 1980 Canonical structure and symmetries for discrete systems. *Math. Jap.* **4**, 405–420.

[27] Maeda, S. 1987 The similarity method for difference equations. *IMA J. Appl. Math.* **38**, 129–134.

[28] Mansfield, E. L. & Hydon P. E. 2001 On a variational complex for difference equations. In: *The Geometrical Study of Differential Equations, eds Joshua A. Leslie and Thierry P. Robart*, American Mathematical Society, Providence, RI, 121–129.

[29] Mansfield E. L. & Hydon P. E. 2001 Towards approximations of difference equations which preserve first integrals. In: *Proc. of the 2001 International Symposium on Symbolic and Algebraic Computation (ISSAC 2001), ed. B. Mourrain*, ACM, New York, 217–222.

[30] Marsden, J. E., Patrick, G. W. & Shkoller, S. 1998 Multisymplectic geometry, variational integrators, and nonlinear PDEs. *Comm. Math. Phys.* **199**, 351–395.

[31] Marsden, J. E. & West, M. 2001 Discrete mechanics and variational integrators. *Acta Numerica* **10**: 357–514.

[32] McLachlan, R. L., Quispel, G. R. W. & Robidoux, N. 1999 Geometric integration using discrete gradients. *Phil. Trans. Roy. Soc. Lond.* A **357**: 1021–1045.

[33] Munthe-Kaas, H. 1998 Runge-Kutta methods on Lie groups. *BIT* **38**: 92–111.

[34] Olver, P. J. 1993 *Applications of Lie groups to differential equations*, 2nd edn. New York: Springer.

[35] Quispel, G. R. W. & Sahadevan, R. 1993 Lie symmetries and the integration of difference equations. *Phys. Lett.* **184A**: 64–70.

[36] Sanz-Serna, J. M. & Calvo, M. P. 1994 *Numerical Hamiltonian problems.* Chapman & Hall, London.

[37] Stephani, H. 1989 *Differential equations: their solution using symmetries.* Cambridge: Cambridge University Press.

[38] Veselov, A. P. 1988 Integrable discrete-time systems and difference operators. *Funct. Anal. Appl.* **22**, 83–93.

[39] Vinogradov, A. M. 1977 On the algebra-geometric foundations of Lagrangian field theory. *Sov. Math. Dokl.* **18**: 1200–1204.