Multisymplectic formulation of near-local Hamiltonian balanced models

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We transform near-local Hamiltonian balanced models describing nearly geostrophic fluid motion (with constant Coriolis parameter) into multisymplectic systems. This allows us to determine conservation of Lagrangian momentum, energy and potential vorticity for Salmon's L_1 dynamics; a similar approach works for other near-local balanced models (such as the $\sqrt{3}$-model). The multisymplectic approach also enables us to determine a class of systems that have a contact structure similar to that of the semigeostrophic model. The contact structure yields a contact transformation which makes the problem of front formation tractable. The new class includes the first local model with a variable Coriolis parameter that preserves all of the most useful geometric features of the semigeostrophic model.

Keywords: Multisymplectic system, balanced models, semigeostrophic theory

1. Introduction

The rotating shallow-water model, which describes the behaviour of a single layer of incompressible fluid with a free surface under gravity over a rotating bed, is frequently used in geophysical fluid dynamics as an approximation for the dynamics of the atmosphere. When the Rossby number is small ($\mathrm{Ro} \ll 1$) a further approximation is to filter out the fast motions, driven by inertia gravity waves, to create so-called 'balanced models'. Typically, these incorporate a balance between the pressure gradient and the Coriolis force.

Salmon (1983, 1985) showed how Hamilton's principle can be used to derive an important class of balanced models systematically. The idea is to define a constraint corresponding to a balance condition (the geostrophic approximation), and to incorporate this into the Lagrangian for the parent model, which is the two-dimensional shallow-water model. A variational principle is then used to obtain the dynamical equations. The accuracy of the resulting balanced model is judged against the parent dynamics, with the latter being considered as exact. The balanced model inherits a Hamiltonian structure and consequent conservation laws. By considering a class of balance conditions which included Salmon's L_1 dynamics and semigeostrophic theory, McIntyre & Roulstone (2002) derived balanced models which go beyond geostrophy: when the Froude number is small ($\mathrm{Fr} \ll 1$), gravity waves are fast compared with the flow velocity and the geostrophic approximation is not sufficient to yield a good approximation. One, called the $\sqrt{3}$-model, improves the accuracy of the approximation to the geostrophic flow (but not the advection velocity) by one order in Ro over semigeostrophic theory. The balance conditions considered in McIntyre & Roulstone (2002) and in this paper are referred to as near-local, because they depend on the geopotential and a finite number of its derivatives only.

Recently, Bridges *et al.* (2005) showed that the shallow-water model and the semigeostrophic model each have a multisymplectic formulation; moreover, these two formulations are very similar. Multisymplectic systems have a common geometric structure: a vector of closed two-forms is conserved by the flow. (This generalizes the conservation of a single two-form in canonical Hamiltonian dynamics.) The multisymplectic structural conservation law can be exploited to analyse stability (see Bridges (1997) for details), but in the context of balanced models we focus on the scalar conservation laws that arise from it, namely conservation of Lagrangian momentum, energy and potential vorticity. Another advantage of the multisymplectic structure is that when it is preserved by a discretization, the resulting finite difference methods normally have extremely good stability properties (see Bridges & Reich (2006) and Ascher & McLachlan (2005) for details). Until now, it has not been clear how to extend the shallow-water multisymplectic structure to other near-local balanced models. One purpose of this paper is address this problem. We derive a multisymplectic description of L_1 dynamics. McIntyre & Roulstone's other balanced models (including the $\sqrt{3}$-model) can be tackled by a similar approach, but as this adds considerable complexity without giving further illumination, we merely summarize the steps that lead to their multisymplectic formulation.

Although the semigeostrophic model is only first-order, from the geometric viewpoint it is the richest of the balanced models. It possesses both Hamiltonian and contact structures (see McIntyre & Roulstone (2002) for details), and recently Delahaies & Roulstone (2009) showed that the semigeostrophic model possesses a hyper-Kähler structure. The contact structure is particularly useful, because it yields a contact transformation that expands the singularities which occur at fronts, and it is therefore able to describe the formation and evolution of fronts. However, like the other balanced models cited above, the semigeostrophic model has a constant Coriolis parameter. So it is natural to try to generalize this model in a way that retains the contact structure; this is the second aim of our paper. There have been several attempts to find models that preserve the structure of semigeostrophic theory and allow the Coriolis parameter to vary. Using Hamilton's principle, Salmon (1985) made the Coriolis parameter a function of geostrophic coordinates, while Shutts (1989) modelled planetary flow, replacing the geostrophic momentum with its projection onto the equatorial plane. Here we build on the common features of the multisymplectic formulations for the shallow-water and semigeostrophic models, and we derive a class of multisymplectic systems with the same contact structure. One such system, which allows the Coriolis parameter to vary with position, has the same accuracy as the semigeostrophic model. By restricting this model to the β-plane, we show that this differs from the planetary model of Roulstone & Sewell (1997) by the inclusion of a single non-obvious term.

The plan of the paper is as follows. Section 2 describes the main features of near-local Hamiltonian balanced models with constant Coriolis parameter. A brief presentation of multisymplectic systems is given in §3, together with the multisymplectic formulations of the shallow-water and semigeostrophic models. In §4, Salmon's L_1 dynamics is written as a multisymplectic system; we also outline how to recast the other balanced models in McIntyre & Roulstone's class as multisymplectic systems. Section 5 generalizes semigeostrophic theory by describing a class of multisymplectic systems that preserve energy, potential vorticity and Lagrangian momentum, and which also have a contact structure that yields analogues of Hoskins' geostrophic coordinates. This class includes the first local semigeostrophic-type model that allows the Coriolis parameter to vary with latitude, while retaining such a contact structure. The paper concludes with a brief discussion in §6.

2. Review of shallow-water balanced models

Following Bridges *et al.* (2005), we present the shallow-water theory in a Lagrangian formulation. The position and velocity of fluid particles are denoted by $\mathbf{x} = (x^1, x^2)$ and $\mathbf{u} = (u^1, u^2)$ respectively, and we label each fluid particle by its position at $t = 0$, which is denoted by $\mathbf{m} = (m^1, m^2)$; then all variables are treated as functions of $\mathbf{m}$ and t. The total derivatives with respect to t and m^α are denoted by subscript t or α after a comma, for example

$$x^i_{,\alpha} = \frac{\partial x^i(\mathbf{m},t)}{\partial m^\alpha}\ ,\ x^i_{,\alpha t} = \frac{\partial^2 x^i(\mathbf{m},t)}{\partial m^\alpha \partial t}\ ,\ \text{etc.,} \tag{2.1}$$

and partial derivatives with respect to any other variable are written in full.

In shallow-water theory, the motion of a shallow layer of incompressible inviscid fluid over a rotating flat-bottomed bed is approximated by the horizontal momentum equations,

$$\mathbf{u}_{,t} + f\mathbf{k} \times \mathbf{x}_{,t} + g\boldsymbol{\nabla}_{\mathbf{x}}\eta = 0. \tag{2.2}$$

The Coriolis parameter is f (which is assumed constant except in §5), the gravitational acceleration is directed downwards with magnitude g, and $\eta(\mathbf{x}, t)$ is the fluid height. The common shorthand $\mathbf{k} \times (a, b) \equiv (-b, a)$ is used and as a consequence of the Lagrangian formulation we have

$$\mathbf{x}_{,t} = \mathbf{u}. \tag{2.3}$$

The incompressibility hypothesis requires η to satisfy the relation $\eta \mathrm{d}\mathbf{x} = \eta_0 \mathrm{d}\mathbf{m}$, where the initial height $\eta_0 = \eta(\mathbf{m}, 0)$ is assumed to be uniform. We choose coordinates in which $\eta_0 = 1$, so that this condition yields the following form of the continuity equation:

$$\eta = \left(\frac{\partial(x^1, x^2)}{\partial(m^1, m^2)} \right)^{-1}. \tag{2.4}$$

Finally, the gradient with respect to $\mathbf{x}$ is given in terms of the Lagrangian formulation by

$$\boldsymbol{\nabla}_{\mathbf{x}} = \begin{pmatrix} \partial_{x^1} \\ \partial_{x^2} \end{pmatrix} = \eta \begin{pmatrix} x^2_{,2} & -x^2_{,1} \\ -x^1_{,2} & x^1_{,1} \end{pmatrix} \begin{pmatrix} \partial_{m^1} \\ \partial_{m^2} \end{pmatrix}. \tag{2.5}$$

From (2.2)–(2.5), it is easy to show that the potential vorticity

$$\mathcal{Q} = \frac{1}{\eta} \left(f + \frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2} \right), \tag{2.6}$$

is conserved following particles (see Bîlă *et al.* (2006) for a thorough analysis). The total energy of a blob of fluid that occupies $\mathcal{D}$ at $t=0$ is

$$\mathcal{E} = \int_{\mathcal{D}} \tfrac{1}{2}\left((u^1)^2 + (u^2)^2 + g\eta\right)\,\mathrm{d}\mathbf{m}; \tag{2.7}$$

given suitable boundary conditions, $\mathcal{E}$ is also conserved.

Equations (2.2) and (2.3) are Euler–Lagrange equations for the variational principle

$$\delta \int L_{\mathtt{sw}}[\mathbf{x},\mathbf{u}]\,\mathrm{d}\mathbf{m}\,\mathrm{d}t = 0,$$

where square brackets enclose the quantities that are varied. (In other words, $[\mathbf{x},\mathbf{u}]$ denotes the functions $\mathbf{x}$, $\mathbf{u}$ and finitely many of their derivatives with respect to $\mathbf{m}$ and t.) The shallow-water Lagrangian is

$$L_{\mathtt{sw}}[\mathbf{x},\mathbf{u}] = \left(u^1 - \tfrac{1}{2}fx^2\right)x^1_{,t} + \left(u^2 + \tfrac{1}{2}fx^1\right)x^2_{,t} - H[\mathbf{x},\mathbf{u}]; \tag{2.8}$$

here H is the total energy density:

$$H[\mathbf{x},\mathbf{u}] = \tfrac{1}{2}\left(\mathbf{u}\cdot\mathbf{u} + g\eta\right). \tag{2.9}$$

Conservation of potential vorticity and energy are consequences (via Noether's Theorem) of the particle relabelling symmetry and the time-invariance of the variational problem (see Salmon (1983)).

The *semigeostrophic* approximation to the shallow-water equations replaces the acceleration $\mathbf{u}_{,t}$ in the momentum equations (2.2) by the Lagrangian time derivative of the geostrophic wind $\mathbf{u}_{\mathrm{g}} = (u^1_{\mathrm{g}}, u^2_{\mathrm{g}})$, which is defined by

$$u^1_{\mathrm{g}} = -f^{-1}\frac{\partial\phi}{\partial x^2}, \quad u^2_{\mathrm{g}} = f^{-1}\frac{\partial\phi}{\partial x^1}, \tag{2.10}$$

where $\phi = g\eta$ is the geopotential. So the shallow-water semigeostrophic equations consist of (2.10), the momentum equation,

$$\mathbf{u}_{\mathrm{g},t} + f\mathbf{k}\times\mathbf{x}_{,t} + g\boldsymbol{\nabla}_{\mathbf{x}}\eta = 0, \tag{2.11}$$

and the continuity equation (2.4). Equations (2.10) and (2.11) arise from a variational problem

$$\delta \int L_{\mathtt{sg}}[\mathbf{x},\mathbf{u}_{\mathrm{g}}]\,\mathrm{d}\mathbf{m}\,\mathrm{d}t = 0.$$

In this case, the Lagrangian is

$$L_{\mathtt{sg}}[\mathbf{x},\mathbf{u}_{\mathrm{g}}] = \left(u^1_{\mathrm{g}} - fx^2\right)\left(x^1 + u^2_{\mathrm{g}}/f\right)_{,t} - H[\mathbf{x},\mathbf{u}_{\mathrm{g}}], \tag{2.12}$$

where H is given by equation (2.9). Once again the particle relabelling symmetry leads to conservation of potential vorticity, which is now defined by

$$\mathcal{Q}_{\mathtt{sg}} = \frac{1}{\eta}\left(f + \frac{\partial u^2_{\mathrm{g}}}{\partial x^1} - \frac{\partial u^1_{\mathrm{g}}}{\partial x^2} + \frac{1}{f}\frac{\partial(u^1_{\mathrm{g}}, u^2_{\mathrm{g}})}{\partial(x^1, x^2)}\right). \tag{2.13}$$

Given suitable boundary conditions, the total energy of a fluid blob,

$$\mathcal{E}_{\mathtt{sg}} = \int_{\mathcal{D}} \tfrac{1}{2}\left(\mathbf{u}_{\mathtt{g}} \cdot \mathbf{u}_{\mathtt{g}} + g\eta\right)\, \mathrm{d}\mathbf{m}, \tag{2.14}$$

is also conserved, because the variational problem is invariant under time translations.

The semigeostrophic approximation is a balanced model with many useful analytic and geometrical properties, including Hamiltonian and contact structures (see McIntyre & Roulstone (2002) for details). However, as it is formally correct only to first order in the Rossby number, its usefulness is limited. The question of how to build more accurate Hamiltonian balanced models while retaining the essential mathematical features of semigeostrophic theory has been much-studied. McIntyre & Roulstone (2002) provided one answer to this question, using the framework of constrained Hamiltonian dynamics pioneered in Salmon (1983, 1985) and Allen & Holm (1996). The method is to construct a constrained Lagrangian $L_{\mathtt{c}}[\mathbf{x}]$ by replacing $\mathbf{u}$ in the shallow-water Lagrangian $L_{\mathtt{sw}}[\mathbf{x}, \mathbf{u}]$ by a constraint velocity field of the form

$$\mathbf{u}_{\mathtt{c}} = \mathbf{u}_{\mathtt{g}} + a\mathbf{k} \times \left(f^{-1}(\mathbf{u}_{\mathtt{g}} \cdot \boldsymbol{\nabla}_{\mathbf{x}})\mathbf{u}_{\mathtt{g}}\right), \quad a \in \mathbb{R}. \tag{2.15}$$

In this formulation, $\mathbf{u}_{\mathtt{c}}$ cannot be varied independently of $\mathbf{x}$; the Euler–Lagrange equation is the momentum equation, which must be supplemented by the constraint (2.15) and the continuity equation (2.4). For each $a \in \mathbb{R}$, the resulting model conserves the potential vorticity,

$$\mathcal{Q}_{\mathtt{c}} = \frac{1}{\eta}\left[f + \frac{\partial u_{\mathtt{c}}^2}{\partial x^1} - \frac{\partial u_{\mathtt{c}}^1}{\partial x^2}\right], \tag{2.16}$$

and (given suitable boundary conditions) the total energy in a fluid blob, which is

$$\mathcal{E}_{\mathtt{c}} = \int_{\mathcal{D}} \tfrac{1}{2}\left(\mathbf{u}_{\mathtt{c}} \cdot \mathbf{u}_{\mathtt{c}} + g\eta\right)\, \mathrm{d}\mathbf{m}. \tag{2.17}$$

Most attention has been focused on just three values of a. The semigeostrophic model corresponds to $a = -1/2$; it is a simple calculation to show that $\mathcal{Q}_{\mathtt{c}}$ amounts to (2.13). For $a = 0$, the constraint velocity reduces to the geostrophic wind, $\mathbf{u}_{\mathtt{c}} = \mathbf{u}_{\mathtt{g}}$; this gives Salmon's L_1 dynamics. Finally, $a = 1$ yields the so-called $\sqrt{3}$-model, which is the most accurate of the balanced models derived by McIntyre & Roulstone. In this case, the potential vorticity (2.16) agrees to second order with an asymptotic expansion of the shallow-water potential vorticity (2.6) in terms of the Rossby number. By contrast, L_1 dynamics and the semigeostrophic model are each accurate only to first order (see Snyder *et al.* (1991), Delahaies (2009)).

The constrained Lagrangian formulations are incomplete, because they do not contain the velocity constraint. In the next section, we show that by using a multisymplectic approach (which arises from a complete first-order Lagrangian), balanced models are given a common geometric structure. This structure provides local conservation laws, from which one can determine the necessary boundary conditions for a fluid blob to have a conserved quantity. It also reveals that conservation of potential vorticity is a differential consequence of two more fundamental conservation laws.

3. Multisymplectic systems and the shallow-water equations

Bridges (1997) called a system of first-order quasilinear PDEs *multisymplectic* if there is a (symplectic) closed 2-form associated with each independent variable, such that these 2-forms satisfy a structural conservation law (which generalizes conservation of symplecticity for Hamiltonian ODEs). For models based on the shallow-water approximation, we seek multisymplectic systems in a (local) Cartesian coordinate system with n dependent variables $z^1, \dots, z^n$ and three independent variables (t, m^1, m^2). The structural conservation law is of the form

$$\omega_{,t} + \kappa^{\alpha}_{,\alpha} = 0, \tag{3.1}$$

where

$$\omega = \tfrac{1}{2} W_{ij}(\mathbf{z})\, \mathrm{d}z^i \wedge \mathrm{d}z^j, \qquad \kappa^{\alpha} = \tfrac{1}{2} K^{\alpha}_{ij}(\mathbf{z})\, \mathrm{d}z^i \wedge \mathrm{d}z^j, \tag{3.2}$$

are closed. Here summation is from 1 to n for Latin indices and from 1 to 2 for Greek indices; the functions W_{ij} and K^{α}_{ij} are locally smooth. Hydon (2005) showed that the structural conservation law is a differential consequence of a 1-form quasi-conservation law of the form

$$\left(\mathtt{W}_j(\mathbf{z})\, \mathrm{d}z^j\right)_{,t} + \left(\mathtt{K}^{\alpha}_j(\mathbf{z})\, \mathrm{d}z^j\right)_{,\alpha} = \mathrm{d}\left(\mathtt{W}_j(\mathbf{z}) z^j_{,t} + \mathtt{K}^{\alpha}_j(\mathbf{z}) z^j_{,\alpha} - S(\mathbf{z})\right), \tag{3.3}$$

for some locally smooth functions $\mathtt{W}_j, \mathtt{K}^{\alpha}_j$ and S. In the above, the exterior derivative d acts only on the dependent variables z^i; it commutes with the total derivative operators (see Bridges *et al.* (2010) for details). By comparing the exterior derivative of (3.3) with (3.2), one finds that

$$W_{ij}(\mathbf{z}) = \frac{\partial \mathtt{W}_j(\mathbf{z})}{\partial z^i} - \frac{\partial \mathtt{W}_i(\mathbf{z})}{\partial z^j}, \qquad K^{\alpha}_{ij}(\mathbf{z}) = \frac{\partial \mathtt{K}^{\alpha}_j(\mathbf{z})}{\partial z^i} - \frac{\partial \mathtt{K}^{\alpha}_i(\mathbf{z})}{\partial z^j}.$$

Moreover, by expanding the quasi-conservation law and collecting the coefficients of $\mathrm{d}z^i$, one obtains the general form of a multisymplectic system (subject to the above restrictions):

$$W_{ij}(\mathbf{z}) z^j_{,t} + K^{\alpha}_{ij}(\mathbf{z}) z^j_{,\alpha} = \frac{\partial S(\mathbf{z})}{\partial z^i}. \tag{3.4}$$

A routine calculation shows that (3.4) is the set of Euler–Lagrange equations for the variational principle

$$\delta \int \tilde{L}[\mathbf{z}]\, \mathrm{d}\mathbf{m}\, \mathrm{d}t = 0, \tag{3.5}$$

where

$$\tilde{L}[\mathbf{z}] = \mathtt{W}_j(\mathbf{z}) z^j_{,t} + \mathtt{K}^{\alpha}_j(\mathbf{z}) z^j_{,\alpha} - S(\mathbf{z}). \tag{3.6}$$

So the set of multisymplectic systems is equivalent to the set of variational problems with a first-order Lagrangian that is affine linear in the derivatives of $\mathbf{z}$. (Note: the above construction may be extended to deal with systems for which $\mathtt{W}_j, \mathtt{K}^{\alpha}_j$ and S also depend upon t and $\mathbf{m}$, but this is not needed for our purposes – see Bridges *et al.* (2010) for details.)

Given the above equivalence, it is straightforward to state Noether's Theorem for the multisymplectic system (3.4), as follows. A vector field of the form

$$X = Q^i[\mathbf{z}]\, \partial_{z^i}$$

generates variational symmetries if, for each $\epsilon \in \mathbb{R}$ sufficiently close to zero, the map $z^j \mapsto \exp\{\epsilon X\}(z^j)$ leaves the variational problem (3.5) unchanged. Thus the criterion for X to generate variational symmetries is

$$X \rfloor \, \mathrm{d}\tilde{L} = a_{,t} + b^{\alpha}_{,\alpha}\,, \tag{3.7}$$

for some functions a and b^{α}, where $\rfloor$ denotes the inner derivative of the differential form $\mathrm{d}\tilde{L}$ by the vector field X. By taking the inner derivative with respect to X of the quasi-conservation law (3.3), we obtain the conservation law associated with X, namely

$$\left(\mathtt{W}_j\, Q^j - a\right)_{,t} + \left(\mathtt{K}^{\alpha}_j\, Q^j - b^{\alpha}\right)_{,\alpha} = 0. \tag{3.8}$$

In particular, for every multisymplectic system (3.4), translations in t, m^1 and m^2 are variational symmetries (with Q^i being $z^i_{,t}$, $z^i_{,1}$ and $z^i_{,2}$ respectively); they yield the following three conservation laws:

$$\left[\mathtt{W}_i z^i_{,t} - \left(\mathtt{W}_i z^i_{,t} + \mathtt{K}^{\alpha}_i z^i_{,\alpha} - S(\mathbf{z})\right)\right]_{,t} + \left(\mathtt{K}^1_i z^i_{,t}\right)_{,1} + \left(\mathtt{K}^2_i z^i_{,t}\right)_{,2} \;=\; 0, \tag{3.9}$$

$$\left(\mathtt{W}_i z^i_{,1}\right)_{,t} + \left[\mathtt{K}^1_i z^i_{,1} - \left(\mathtt{W}_i z^i_{,t} + \mathtt{K}^{\alpha}_i z^i_{,\alpha} - S(\mathbf{z})\right)\right]_{,1} + \left(\mathtt{K}^2_i z^i_{,1}\right)_{,2} \;=\; 0, \tag{3.10}$$

$$\left(\mathtt{W}_i z^i_{,2}\right)_{,t} + \left(\mathtt{K}^1_i z^i_{,2}\right)_{,1} + \left[\mathtt{K}^2_i z^i_{,2} - \left(\mathtt{W}_i z^i_{,t} + \mathtt{K}^{\alpha}_i z^i_{,\alpha} - S(\mathbf{z})\right)\right]_{,2} \;=\; 0. \tag{3.11}$$

Conservation of energy is represented by (3.9), whereas (3.10) and (3.11) describe conservation of Lagrangian momentum, that is, the quantity that is canonically conjugate to translations in label space. All three conservation laws may also be regarded as components of the pullback of the quasi-conservation law (3.3) to the space of independent variables. From this viewpoint, the dependent variables z^i and their derivatives are treated as functions of t, m^1 and m^2; then the coefficients of $\mathrm{d}t$, $\mathrm{d}m^1$ and $\mathrm{d}m^2$ are (3.9), (3.10) and (3.11) respectively. The same approach can be applied to the structural conservation law (3.1), yielding conservation laws that are differential consequences of (3.9)–(3.11). In particular, the coefficient of $\mathrm{d}m^1 \wedge \mathrm{d}m^2$ is the difference between the m^1-derivative of (3.11) and the m^2-derivative of (3.10), namely

$$\left(\mathtt{W}_i z^i_{,2}\right)_{,1t} - \left(\mathtt{W}_i z^i_{,1}\right)_{,2t} + \left(\mathtt{K}^1_i z^i_{,2}\right)_{,11} - \left(\mathtt{K}^2_i z^i_{,1}\right)_{,22} = 0. \tag{3.12}$$

This is the conservation law that corresponds (by Noether's Theorem) to the particle relabelling symmetry, which is the infinite-dimensional pseudogroup of volume-preserving diffeomorphisms of label space. For shallow-water theory and for the balanced models that approximate it, (3.12) describes conservation of potential vorticity in terms of the Lagrangian coordinates†. It is interesting that from the Lagrangian viewpoint, conservation of potential vorticity is merely a differential consequence of the (possibly more fundamental) conservation laws (3.10) and (3.11). However, those two conservation laws, unlike conservation of energy and potential vorticity, do not appear in the Eulerian viewpoint (because they cannot be written without reference to the particle labels).

We now review some relevant details of the multisymplectic version of the shallow-water model, which was derived in Bridges *et al.* (2005). The shallow-water

† The other components of the pullback of the structural conservation law merely yield label-space derivatives of the energy conservation law (3.9).

Lagrangian (2.8) is not affine linear as it stands, because it contains a multiple of

$$\eta = \frac{1}{x^1_{,1}x^2_{,2} - x^1_{,2}x^2_{,1}}\,.$$

However, an equivalent affine linear Lagrangian can be created by introducing new variables $x^\beta_\alpha = x^\beta_{,\alpha}$. It is also convenient to write the internal (potential) energy term $g\eta/2$ as

$$e(\tau) = \frac{g}{2\tau}\,, \qquad \text{where} \quad \tau = \eta^{-1} = x^1_1x^2_2 - x^1_2x^2_1. \tag{3.13}$$

Inserting these new elements into (2.8), using Lagrange multipliers w^α_β to enforce the constraints $x^\beta_\alpha = x^\beta_{,\alpha}$, we obtain the affine linear Lagrangian

$$\tilde{L}_{\texttt{sw}} = \left(u^1 - \tfrac{1}{2}fx^2\right)x^1_{,t} + \left(u^2 + \tfrac{1}{2}fx^1\right)x^2_{,t} + w^\alpha_\beta(x^\beta_{,\alpha} - x^\beta_\alpha) - \left[\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau)\right]. \tag{3.14}$$

Then the Euler–Lagrange equations obtained by varying δx^1, δx^2, δu^α, δw^α_β and δx^β_α in turn (with fixed endpoint conditions) are

$$fx^2_{,t} - u^1_{,t} - w^\alpha_{1,\alpha} = 0, \tag{3.15}$$
$$-fx^1_{,t} - u^2_{,t} - w^\alpha_{2,\alpha} = 0, \tag{3.16}$$
$$x^\alpha_{,t} = u^\alpha, \tag{3.17}$$
$$x^\beta_{,\alpha} = x^\beta_\alpha, \tag{3.18}$$
$$0 = \frac{\partial e(\tau)}{\partial x^\beta_\alpha} + w^\alpha_\beta. \tag{3.19}$$

The multisymplectic structure emerges when we reorganize this system of first-order partial differential equations as

$$\mathbf{W}\mathbf{z}_{,t} + \mathbf{K}^\alpha\mathbf{z}_{,\alpha} = \boldsymbol{\nabla}_{\mathbf{z}}S. \tag{3.20}$$

Here

$$\mathbf{z} = (x^1, x^2, u^1, u^2, w^1_1, w^2_1, w^1_2, w^2_2, x^1_1, x^1_2, x^2_1, x^2_2)^{\mathrm{T}}, \tag{3.21}$$

$\mathbf{W}$ is the 12×12 skew-symmetric matrix

$$\mathbf{W} = \begin{pmatrix} W & \mathbf{0}_{(4\times8)} \\ \mathbf{0}_{(8\times4)} & \mathbf{0}_{(8\times8)} \end{pmatrix},$$

where W is the 4×4 block

$$W = \begin{pmatrix} 0 & f & -1 & 0 \\ -f & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.22}$$

the skew-symmetric matrices $\mathbf{K}^\alpha$, $\alpha = 1, 2$, are defined by

$$\mathbf{K}^\alpha = \begin{pmatrix} \mathbf{0}_{(4\times4)} & N^\alpha & \mathbf{0}_{(4\times4)} \\ -(N^\alpha)^{\mathrm{T}} & \mathbf{0}_{(4\times4)} & \mathbf{0}_{(4\times4)} \\ \mathbf{0}_{(4\times4)} & \mathbf{0}_{(4\times4)} & \mathbf{0}_{(4\times4)} \end{pmatrix}, \tag{3.23}$$

with

$$N^1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$S(\mathbf{z}) = \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau) + w^\alpha_\beta x^\beta_\alpha. \tag{3.24}$$

Using (3.18) and (3.19) together with (2.5) we see that

$$w^\alpha_{1,\alpha} = -\eta^{-1}\frac{\partial e'(\tau)}{\partial x^1}, \quad w^\alpha_{2,\alpha} = -\eta^{-1}\frac{\partial e'(\tau)}{\partial x^2},$$

so (3.13) yields

$$w^\alpha_{1,\alpha} = g\frac{\partial \eta}{\partial x^1}, \quad w^\alpha_{2,\alpha} = g\frac{\partial \eta}{\partial x^2}. \tag{3.25}$$

Inserting these expressions in (3.15) and (3.16) leads to the horizontal momentum equations (2.2), which shows that the system (3.15)–(3.19) is equivalent to the shallow-water model.

The multisymplectic formulation of the semigeostrophic model is almost the same as the above (with $\mathbf{u}_{\mathtt{g}}$ replacing $\mathbf{u}$), except that the affine linear Lagrangian is

$$\tilde{L}_{\mathtt{sg}}[\mathbf{x},\mathbf{u}_{\mathtt{g}}] = \left(u^1_{\mathtt{g}} - fx^2\right)\left(x^1 + u^2_{\mathtt{g}}/f\right)_{,t} + w^\alpha_\beta(x^\beta_{,\alpha} - x^\beta_\alpha) - \left[\tfrac{1}{2}\mathbf{u}_{\mathtt{g}}\cdot\mathbf{u}_{\mathtt{g}} + e(\tau)\right], \tag{3.26}$$

and so (3.22) is replaced by

$$W = \begin{pmatrix} 0 & f & -1 & 0 \\ -f & 0 & 0 & -1 \\ 1 & 0 & 0 & f^{-1} \\ 0 & 1 & -f^{-1} & 0 \end{pmatrix}. \tag{3.27}$$

The structural conservation law and the 1-form quasi-conservation law for each of these models is obtained by substituting the components into the general forms (3.1)–(3.3); see Bridges *et al.* (2005) and Hydon (2005) for details.

4. Multisymplectic formulation of constrained dynamics

The purpose of this section is to derive a multisymplectic version of McIntyre & Roulstone balanced models. In §3 we derived a Lagrangian $\tilde{L}_{\mathtt{sw}}$ that enabled us to recast the shallow-water model into multisymplectic form. We now seek to insert the constraint $\mathbf{u} = \mathbf{u}_{\mathtt{c}}$ into this Lagrangian. The process is described in full for L_1 dynamics and is summarized for McIntyre & Roulstone's class of more general constraints (2.15).

(*a*) *L_1 dynamics in multisymplectic form*

To recast L_1 dynamics into a multisymplectic form we need to insert the constraint $\mathbf{u} = \mathbf{u}_{\mathtt{g}}$, written in terms of the multisymplectic variables, into the Lagrangian (3.14). For the shallow-water model, equation (3.25) gives

$$u_{\mathtt{g}} = -f^{-1}w^\alpha_{2,\alpha}, \quad v_{\mathtt{g}} = f^{-1}w^\alpha_{1,\alpha},$$

where

$$w^\alpha_\beta = -\frac{\partial e(\tau)}{\partial x^\beta_\alpha}\,.$$

Recall that in this setting the variables w^α_β are introduced as the Lagrange multipliers corresponding to the constraints $x^\beta_{,\alpha} = x^\beta_\alpha$. We want w^α_β to play the same role in the derivation of L_1 dynamics, so that they remain as Lagrange multipliers without being used in the constraint $\mathbf{u} = \mathbf{u}_g$. Therefore we introduce auxiliary functions e^α_β, which depend on the variables x^α_β only, as follows:

$$e^\alpha_\beta = -\frac{\partial e(\tau)}{\partial x^\beta_\alpha}.$$

The constraint $\mathbf{u} = \mathbf{u_g}$ can now be written as

$$u^1 = -f^{-1}e^\alpha_{2,\alpha}\ ,\ u^2 = f^{-1}e^\alpha_{1,\alpha}; \tag{4.1}$$

it depends only on u^α, x^α_β and first-order derivatives of x^α_β. Inserting this constraint with a vector of Lagrange multipliers $\mathbf{v} = (v^1, v^2)$ into the Lagrangian defining the parent dynamics leads to the affine linear Lagrangian

$$\tilde{L}_1(\mathbf{z}, \mathbf{z}^{(1)}) = \mathtt{W}_j z^j_{,t} + \mathtt{K}^\alpha_j z^j_{,\alpha} - S(\mathbf{z}), \tag{4.2}$$

where now

$$\mathbf{z} = (x^1, x^2, u^1, u^2, w^1_1, w^2_1, w^1_2, w^2_2, v^1, v^2, x^1_1, x^1_2, x^2_1, x^2_2).$$

Here

$$\mathtt{W} = \left(u^1 - \tfrac{1}{2}fx^2, u^2 + \tfrac{1}{2}fx^1, 0, \ldots, 0\right),$$

$$\begin{aligned}\mathtt{K}^\alpha \quad = \quad & \frac{1}{f}\Bigg(fw^\alpha_1, fw^\alpha_2, 0,0,0,0,0,0,0,0, v^2\frac{\partial e^\alpha_1}{\partial x^1_1} - v^1\frac{\partial e^\alpha_2}{\partial x^1_1}\,,\\ & v^2\frac{\partial e^\alpha_1}{\partial x^1_2} - v^1\frac{\partial e^\alpha_2}{\partial x^1_2}\,, v^2\frac{\partial e^\alpha_1}{\partial x^2_1} - v^1\frac{\partial e^\alpha_2}{\partial x^2_1}\,, v^2\frac{\partial e^\alpha_1}{\partial x^2_2} - v^1\frac{\partial e^\alpha_2}{\partial x^2_2}\Bigg),\end{aligned}$$

and

$$S(\mathbf{z}) = \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau) + w^\alpha_\beta x^\beta_\alpha + \mathbf{v}\cdot\mathbf{u}.$$

The Euler–Lagrange equations corresponding to the variations δx^β, δu^β, δw^α_β, δv^β and δx^β_α are

$$fx^2_{,t} - u^1_{,t} - w^\alpha_{1,\alpha} \quad = \quad 0\,, \tag{4.3}$$

$$-fx^1_{,t} - u^2_{,t} - w^\alpha_{2,\alpha} \quad = \quad 0\,, \tag{4.4}$$

$$x^1_{,t} \quad = \quad u^1 + v^1\,, \tag{4.5}$$

$$x^2_{,t} \quad = \quad u^2 + v^2\,, \tag{4.6}$$

$$x^\beta_{,\alpha} \quad = \quad x^\beta_\alpha\,, \tag{4.7}$$

$$-f^{-1}e^\alpha_{2,\alpha} \quad = \quad u^1\,, \tag{4.8}$$

$$f^{-1}e^\alpha_{1,\alpha} \quad = \quad u^2\,, \tag{4.9}$$

$$f^{-1}\frac{\partial e^\gamma_2}{\partial x^\beta_\alpha}v^1_{,\gamma} - f^{-1}\frac{\partial e^\gamma_1}{\partial x^\beta_\alpha}v^2_{,\gamma} \quad = \quad \frac{\partial e(\tau)}{\partial x^\beta_\alpha} + w^\alpha_\beta\,. \tag{4.10}$$

As shown in Delahaies (2009) this multisymplectic system is equivalent to Salmon's L_1 dynamics.

(*b*) *Conservation laws for the multisymplectic L_1 model*

Just as for the shallow-water model, the closed 2-forms (3.2) are

$$\begin{aligned}\omega &= f\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}u^1 \wedge \mathrm{d}x^1 + \mathrm{d}u^2 \wedge \mathrm{d}x^2, \\ \kappa^\alpha &= \mathrm{d}w^\alpha_\beta \wedge \mathrm{d}x^\beta + \frac{1}{f}\left(\frac{\partial e^\alpha_2}{\partial x^\beta_\gamma}\mathrm{d}x^\beta_\gamma \wedge \mathrm{d}v^1 - \frac{\partial e^\alpha_1}{\partial x^\beta_\gamma}\mathrm{d}x^\beta_\gamma \wedge \mathrm{d}v^2\right).\end{aligned}$$

The $\mathrm{d}m^1 \wedge \mathrm{d}m^2$ component of the pullback of the structural conservation law is $\mathcal{Q}_{\mathtt{g},t} = 0$, where $\mathcal{Q}_\mathtt{g}$ is the geostrophic potential vorticity (that is, the potential vorticity $\mathcal{Q}_\mathtt{c}$ defined by (2.16) with $\mathbf{u}_\mathtt{c} = \mathbf{u}_\mathtt{g}$). Similarly, the local energy conservation law, which is the $\mathrm{d}t$ component of the quasi-conservation law, is

$$\left(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau)\right)_{,t} + \{w^\alpha_\beta x^\beta_{,t} - \frac{1}{f}e^\alpha_{2,t}v^1 + \frac{1}{f}e^\alpha_{1,t}v^2\}_{,\alpha} = 0.$$

It implies that, provided suitable boundary conditions are given, the total energy $\mathcal{E}_\mathtt{g}$, defined by equation (2.17) with $\mathbf{u}_\mathtt{c} = \mathbf{u}_\mathtt{g}$, is conserved. The components of Lagrangian momentum are derived similarly.

(*c*) *Multisymplectification for McIntyre & Roulstone's balanced models*

To recast McIntyre & Roulstone's other balanced models into a multisymplectic form we apply the same technique as for L_1-dynamics, that is, we apply the constraint $\mathbf{u} = \mathbf{u}_\mathtt{c}$ to the Lagrangian density $L_\mathtt{sw}$. However, by contrast with L_1-dynamics, additional functions are needed for the constraint to fit into the multisymplectic structure. Recall that the constraint velocity is

$$\mathbf{u}_\mathtt{c} = \mathbf{u}_\mathtt{g} + af^{-1}\mathbf{k}\times\mathbf{u}_\mathtt{g}\cdot\nabla\mathbf{u}_\mathtt{g},\ a \in \mathbb{R}. \tag{4.11}$$

The $\sqrt{3}$-model is obtained when $a = \sqrt{3}$. Expanding (4.11) using (2.5) and (4.1), the constraint $\mathbf{u} = \mathbf{u}_\mathtt{c}$ can be written in components as

$$u^1 = -f^{-1}e^\alpha_{2,\alpha} + af^{-3}e^\alpha_{\beta,\alpha}\tau^{-1}x^\beta_2 e^\gamma_{1,\gamma 1} - af^{-3}e^\alpha_{\beta,\alpha}\tau^{-1}x^\beta_1 e^\gamma_{1,\gamma 2}, \tag{4.12}$$

$$u^2 = f^{-1}e^\alpha_{1,\alpha} + af^{-3}e^\alpha_{\beta,\alpha}\tau^{-1}x^\beta_2 e^\gamma_{2,\gamma 1} - af^{-3}e^\alpha_{\beta,\alpha}\tau^{-1}x^\beta_1 e^\gamma_{2,\gamma 2}. \tag{4.13}$$

The above expressions involve second-order derivatives $x^\beta_{\alpha,\gamma\delta}$ (via the second derivatives of e^β_α), so to create an affine linear first-order Lagrangian, we introduce new variables,

$$x^\beta_{\sigma\alpha} = x^\beta_{\sigma,\alpha}, \tag{4.14}$$

and new functions,

$$e^\alpha_{\gamma\alpha} = \frac{\partial e^\alpha_\gamma}{\partial x^\beta_\sigma}x^\beta_{\sigma\alpha}. \tag{4.15}$$

Then, as

$$e^\alpha_{\gamma\alpha,\delta} = \frac{\partial^2 e^\alpha_\gamma}{\partial x^\beta_\sigma \partial x^\nu_\mu}x^\beta_{\sigma\alpha}x^\nu_{\mu,\delta} + \frac{\partial e^\alpha_\gamma}{\partial x^\beta_\sigma}x^\beta_{\sigma\alpha,\delta}, \tag{4.16}$$

(4.12) and (4.13) amount to

$$u^1 = -f^{-1}e^{\alpha}_{2,\alpha} + af^{-3}e^{\alpha}_{\gamma\alpha}\tau^{-1}x^{\gamma}_2 e^{\beta}_{1\beta,1} - af^{-3}e^{\alpha}_{\gamma\alpha}\tau^{-1}x^{\gamma}_1 e^{\beta}_{1\beta,2}, \tag{4.17}$$

$$u^2 = f^{-1}e^{\alpha}_{1,\alpha} + af^{-3}e^{\alpha}_{\gamma\alpha}\tau^{-1}x^{\gamma}_2 e^{\beta}_{2\beta,1} - af^{-3}e^{\alpha}_{\gamma\alpha}\tau^{-1}x^{\gamma}_1 e^{\beta}_{2\beta,2}. \tag{4.18}$$

This ensures that the model obtained by adding the constraints (4.14), (4.17) and (4.18) to the Lagrangian (3.14), can be written in multisymplectic form†.

5. A contact-preserving generalisation of semigeostrophic theory

Despite its limited accuracy, the semigeostrophic theory has remarkable geometrical properties such as a contact structure, Legendre duality and a Monge Ampère structure. These properties have been used to show that solutions exist (see Benamou & Brenier (1998)). In this section, we derive a class of multisymplectic systems which generalizes semigeostrophic theory in a way that retains a contact structure and a modified version of the Hamiltonian structure that can be obtained by using Hoskins' geostrophic coordinates. By restricting attention to members of the class that have the same qualitative features as the semigeostrophic model, we find a new model that allows the Coriolis parameter f to vary with latitude. Throughout this section $\mathbf{u}$ denotes the constraint velocity (which is $\mathbf{u_g}$ when f is constant); it is defined by the pair of diagnostic equations (which lack time-derivatives).

(a) *Geometric features when f is constant*

The semigeostrophic equations with constant f have several geometric features that are revealed by using geostrophic coordinates,

$$\xi^1 = x^1 + f^{-1}u^2, \qquad \xi^2 = x^2 - f^{-1}u^1. \tag{5.1}$$

Hoskins (1975) showed that, from the Eulerian viewpoint,

$$\frac{\partial \Phi}{\partial \xi^\alpha} = \frac{\partial \phi}{\partial x^\alpha}, \tag{5.2}$$

where $\phi = g\eta$ is the geopotential and

$$\Phi = \phi + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}. \tag{5.3}$$

Note that (5.2) follows immediately from the definition of the geostrophic wind by the diagnostic equations

$$u^1 = -f^{-1}\frac{\partial \phi}{\partial x^2}, \qquad u^2 = f^{-1}\frac{\partial \phi}{\partial x^1}. \tag{5.4}$$

The consequence of (5.2) and (5.4) is that, for each t, the transformation

$$\left(x^1, x^2, \phi, \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}\right) \mapsto \left(\xi^1, \xi^2, \Phi, \frac{\partial \Phi}{\partial \xi^1}, \frac{\partial \Phi}{\partial \xi^2}\right),$$

† See Delahaies (2009) for further details.

is a strict contact transformation, because

$$\mathrm{d}\Phi - \frac{\partial \Phi}{\partial \xi^\alpha}\mathrm{d}\xi^\alpha = \mathrm{d}\phi - \frac{\partial \phi}{\partial x^\alpha}\mathrm{d}x^\alpha.$$

From the Lagrangian viewpoint, semigeostrophic dynamics is governed by

$$\xi^1_{,t} = u^1 = -f^{-1}\frac{\partial \Phi}{\partial \xi^2}, \qquad \xi^2_{,t} = u^2 = f^{-1}\frac{\partial \Phi}{\partial \xi^1}. \tag{5.5}$$

At first sight, this appears to be a canonical Hamiltonian system whose Hamiltonian is $f^{-1}\Phi$. However, each ξ^α is a length, which contravenes the usual idea in mechanics that the Hamiltonian is energy and that the dependent variables are position and momentum. For semigeostrophic flow, Φ is an energy density (energy per unit mass), so it seems sensible to use this as the Hamiltonian and to have as dependent variables a position and its canonically conjugate momentum density. Bearing in mind that we later intend to allow f to depend on the latitude variable x^2, we use ξ^1 as the position. Then, with

$$A = -f\xi^2 = u^1 - fx^2, \qquad B = \xi^1 = x^1 + f^{-1}u^2, \tag{5.6}$$

equation (5.5) amounts to the canonical system

$$A_{,t} = -\frac{\partial \Phi}{\partial B}, \qquad B_{,t} = \frac{\partial \Phi}{\partial A}. \tag{5.7}$$

Note that, by construction, the Hamiltonian Φ is independent of the Coriolis parameter.

The following observations are also helpful. First, using (A,B) as variables simplifies the time-derivative part of the multisymplectic Lagrangian (3.26), so that

$$\tilde{L}_{\mathbf{sg}}[\mathbf{x}, \mathbf{u}] = AB_{,t} + w^\alpha_\beta(x^\beta_{,\alpha} - x^\beta_\alpha) - \left[\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau)\right]. \tag{5.8}$$

Second, using (3.2), (3.6) with (3.21) the time-component of the multisymplectic two-form reduces to

$$\omega = \mathrm{d}A \wedge \mathrm{d}B,$$

so that the potential vorticity is

$$\mathcal{Q} = \frac{\partial(A, B)}{\partial(m^1, m^2)}.$$

Third, as f is a constant, we have not altered the contact transformation property by using (A, B) instead of (ξ^1, ξ^2).

(*b*) *An extension that preserves the Hamiltonian and contact structures*

A major distinction between the multisymplectic shallow-water and semigeostrophic equations is the rank of W, which is 4 and 2 repectively. Consequently, for the shallow water equations there are four prognostic (dynamic) equations, and two pairs of dependent variables are needed. For semigeostrophic flow, however, one can find a single pair of variables (A, B) that are dependent variables in the prognostic equations; moreover, the diagnostic part of the semigeostrophic equations, as expressed by the contact transformation, requires only the same single pair of variables. These facts lie behind the useful geometric structures that Hoskins' coordinates first revealed.

It makes sense, therefore, to see which other systems of equations have the same features in common. We shall leave the Hamiltonian Φ unchanged, but we allow A and B to be arbitrary independent functions of $\mathbf{x}$ and $\mathbf{u}$. Just as for the semigeostrophic model with constant f, the multisymplectic Lagrangian is

$$\tilde{L} = AB_{,t} + w_i^\alpha x^i_{,\alpha} - \left[\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + e(\tau) + w_i^\alpha x_\alpha^i\right], \tag{5.9}$$

because we want Φ to remain unchanged. Then the Euler-Lagrange equations corresponding to variations of δx^β, δu^β, δw^α_β and δx^β_α are

$$\begin{aligned}
\frac{\partial(A,B)}{\partial(x^1,x^2)}x^2_{,t} + \frac{\partial(A,B)}{\partial(x^1,u^\alpha)}u^\alpha_{,t} - w^\alpha_{1,\alpha} &= 0, & (5.10)\\
\frac{\partial(A,B)}{\partial(x^2,x^1)}x^1_{,t} + \frac{\partial(A,B)}{\partial(x^2,u^\alpha)}u^\alpha_{,t} - w^\alpha_{2,\alpha} &= 0, & (5.11)\\
\frac{\partial(A,B)}{\partial(u^1,x^\alpha)}x^\alpha_{,t} + \frac{\partial(A,B)}{\partial(u^1,u^2)}u^2_{,t} &= u^1, & (5.12)\\
\frac{\partial(A,B)}{\partial(u^2,x^\alpha)}x^\alpha_{,t} + \frac{\partial(A,B)}{\partial(u^2,u^1)}u^1_{,t} &= u^2, & (5.13)\\
x^\beta_{,\alpha} &= x^\beta_\alpha, & (5.14)\\
w^\alpha_\beta &= -\frac{\partial e(\tau)}{\partial x^\beta_\alpha}. & (5.15)
\end{aligned}$$

This system can be written as (3.20), except that the non-zero block W is now

$$W = \begin{pmatrix}
0 & \frac{\partial(A,B)}{\partial(x^1,x^2)} & \frac{\partial(A,B)}{\partial(x^1,u^1)} & \frac{\partial(A,B)}{\partial(x^1,u^2)}\\
\frac{\partial(A,B)}{\partial(x^2,x^1)} & 0 & \frac{\partial(A,B)}{\partial(x^2,u^1)} & \frac{\partial(A,B)}{\partial(x^2,u^2)}\\
\frac{\partial(A,B)}{\partial(u^1,x^1)} & \frac{\partial(A,B)}{\partial(u^1,x^2)} & 0 & \frac{\partial(A,B)}{\partial(u^1,u^2)}\\
\frac{\partial(A,B)}{\partial(u^2,x^1)} & \frac{\partial(A,B)}{\partial(u^2,x^2)} & \frac{\partial(A,B)}{\partial(u^2,u^1)} & 0
\end{pmatrix}.$$

Equation (5.15) amounts to

$$w^\alpha_{1,\alpha} = \frac{\partial\phi}{\partial x^1}, \qquad w^\alpha_{2,\alpha} = \frac{\partial\phi}{\partial x^2}, \tag{5.16}$$

and therefore the system (5.10)–(5.13) is equivalent to

$$\begin{aligned}
B_{,t}\frac{\partial A}{\partial x^\beta} - A_{,t}\frac{\partial B}{\partial x^\beta} &= g\frac{\partial\eta}{\partial x^\beta}, & (5.17)\\
B_{,t}\frac{\partial A}{\partial u^\beta} - A_{,t}\frac{\partial B}{\partial u^\beta} &= u^\beta, & (5.18)
\end{aligned}$$

which can be expressed as

$$B_{,t}\mathrm{d}A - A_{,t}\mathrm{d}B = \mathrm{d}\Phi. \tag{5.19}$$

As before, W is of rank 2, so we can treat Φ as a function of (A, B) for each t. Consequently, the system is again Hamiltonian:

$$A_{,t} = -\frac{\partial \Phi}{\partial B}, \qquad B_{,t} = \frac{\partial \Phi}{\partial A}. \tag{5.20}$$

Substituting (5.20) into (5.17) and (5.18), we find that once again

$$\mathrm{d}\Phi - \frac{\partial \Phi}{\partial A}\mathrm{d}A - \frac{\partial \Phi}{\partial B}\mathrm{d}B = \mathrm{d}\phi - \frac{\partial \phi}{\partial x^\alpha}\mathrm{d}x^\alpha,$$

and so the transformation

$$\left(x^1, x^2, \phi, \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}\right) \mapsto \left(A, B, \Phi, \frac{\partial \Phi}{\partial A}, \frac{\partial \Phi}{\partial B}\right),$$

is a strict contact transformation.

So whenever the multisymplectic Lagrangian is of the same form as for constant-f semigeostrophic flow, the contact and Hamiltonian structures are preserved. Furthermore, the time-component of the multisymplectic two-form is again

$$\omega = \mathrm{d}A \wedge \mathrm{d}B,$$

and thus there is an analogue of potential vorticity that is preserved by the flow, namely

$$\mathcal{Q} = \frac{\partial(A, B)}{\partial(m^1, m^2)}.$$

(*c*) *Variable-f equations*

The discussion above is entirely mathematical, without reference to any physical constraints. We now consider how to perturb the constant-f semigeostrophic variables (A, B) in a way that allows the Coriolis parameter to vary with x^2. As we have seen, the main problem is that there is a great deal of freedom. Here is one rationale that leads to a solution; we cannot claim that this is the only or best solution, but it has the advantage of introducing errors that are no larger than the errors in the constant-f case.

For constant f,

$$A_{,t} = -\frac{\partial \phi}{\partial x^1}, \qquad B_{,t} = u^1.$$

As neither of these depend explicitly on x^2, we seek to preserve these equations. A straightforward calculation shows that this requires

$$A = u^1 + F(x^2, u^2), \qquad B = x^1 + G(x^2, u^2),$$

for some functions F and G. Suppose that, as for constant f, $A - u^1$ is independent of u^2 and $B - x^1$ is a multiple of u^2. Then

$$A = u^1 + \tilde{F}(x^2), \qquad B = x^1 + \tilde{G}(x^2)u^2.$$

On dimensional grounds, the functions $\tilde{F}$ and $\tilde{G}$ must have the same dimensions as fx^2 and $1/f$ respectively. Indeed, for constant f,

$$\tilde{F} = -fx^2, \qquad \tilde{G} = 1/f.$$

Even with all of these constraints, there remains a gauge freedom: we can add a constant to $\tilde{F}$ without changing the explicit form of the equations. We may exploit this freedom by choosing

$$\tilde{F} = -\int^{x^2} f(y)\mathrm{d}y, \qquad \tilde{G} = 1/f(x^2).$$

The integral is indefinite, so A and B are local functions, as required.

So with

$$A = u^1 - \int^{x^2} f(y)\,\mathrm{d}y, \qquad B = x^1 + \frac{u^2}{f(x^2)}, \tag{5.21}$$

the Eulerian form of (5.17), (5.18) is

$$\begin{aligned} f(x^2)x^2_{,t} - u^1_{,t} &= g\frac{\partial\eta}{\partial x^1}, & (5.22)\\ -f(x^2)x^1_{,t} + \frac{f'(x^2)}{[f(x^2)]^2}u^2u^1_{,t} - u^2_{,t} &= g\frac{\partial\eta}{\partial x^2}, & (5.23)\\ x^1_{,t} - \frac{f'(x^2)}{[f(x^2)]^2}u^2x^2_{,t} + \frac{u^2_{,t}}{f(x^2)} &= u^1, & (5.24)\\ x^2_{,t} - \frac{u^1_{,t}}{f(x^2)} &= u^2. & (5.25)\end{aligned}$$

Therefore the constraint velocity is

$$\begin{aligned} u^1 &= -\frac{g}{f(x^2)}\frac{\partial\eta}{\partial x^2} - \frac{f'(x^2)}{[f(x^2)]^2}\left(u^2\right)^2,\\ u^2 &= \frac{g}{f(x^2)}\frac{\partial\eta}{\partial x^1},\end{aligned}$$

and the potential vorticity is

$$\mathcal{Q} = \frac{\partial(A,B)}{\partial(m^1,m^2)} = \frac{1}{\eta}\left(f(x^2) + \frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2} + \frac{\partial(u^1, u^2/f(x^2))}{\partial(x^1,x^2)}\right).$$

This is a new variable-f approximation, which preserves that contact and Hamiltonian structure of the constant-f system. Furthermore, a routine calculation shows that the error introduced by this approximation is of precisely the same order of magnitude as the error due to the neglect of planetary curvature. Consequently, this model is no worse than the constant-f semigeostrophic model.

If $f(x^2) = f_0 + \beta x^2$, the system (5.22)-(5.25) provides a β-plane semigeostrophic model that differs slightly from the β-plane version of the planetary geostrophic equations derived in Roulstone & Sewell (1997), principally because it contains a term that ensures that the contact structure is preserved.

6. Summary and concluding remarks

We have adapted Salmon's approach of incorporating a balance condition into a variational principle to the multisymplectic framework and thus obtained a local formulation of near-local balanced models. Starting with Salmon's L_1 dynamics, we then considered the more general class of near-local constraints used in McIntyre & Roulstone (2002), including the $\sqrt{3}$-model. These descriptions are not unique: instead of inserting the balance condition into the Lagrangian through the use of Lagrange multipliers, we could have replaced the velocity field by the suitably expressed balance condition directly into the Lagrangian. As shown for L_1 dynamics and the generalization of semigeostrophic theory, the local formulation paves the way for constructing conservation laws from the structural conservation law.

In this paper we have only considered near-local constraints, in which the constraint is expressed in terms of the local value of the geopotential and a finite number of its derivatives. Higher accuracy typically requires nonlocal constraints; examples include the nonlocal second-order balanced models derived in Allen & Holm (1996) and Vanneste & Bokhove (2002).

We have also proposed a multisymplectic system which generalizes the system presented in Bridges *et al.* (2005) for semigeostrophic theory. This generalization enabled us to present a local extension of the f-plane semigeostrophic theory to variable Coriolis parameter, and we have proved that this generalization carries a contact structure. The generalization of semigeostrophic theory presented in this paper was influenced by our prior knowledge of the formulation of semigeostrophic theory in terms of canonical coordinates, namely Hoskins' geostrophic coordinates. McIntyre & Roulstone (2002) found that complex canonical coordinates exist for the class of near-local constraints considered here; they lead to interesting complex geometries. It would be useful to extend our approach to this more general class of complex canonical coordinates'.

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