The number of regular semisimple classes of special linear and unitary groups

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1 Introduction

Let G be a connected, reductive, linear algebraic group with Frobenius endomorphism F and Frobenius fixed point group G^F. A semisimple element $t \in G$ is called regular, if the connected component of its centralizer is a maximal torus. If G is simply connected, which means that G^F is a universal finite Chevalley group, then, due to a theorem of Steinberg, centralizers of semisimple elements are connected and a semisimple element t of G^F is regular if and only if its centralizer $C_{G^F}(t)$ is abelian.

In [2] the first two authors described general formulae for the number of semisimple regular classes of G^F with given centralizer, which are valid, if G is a simple, simply connected group of classical type. These formulae are somewhat technical but experiments showed that the total number of regular classes in G^F can be described in easy terms. In the present paper this will be demonstrated for groups of type A.

For $n \geq 2$ let $G_n^\epsilon(q)$ denote the general linear group $GL_n(q)$ if $\epsilon = 1$ and the unitary group $U_n(q)$ (sometimes also denoted by $U_n(q^2)$) if $\epsilon = -1$. Moreover let $S_n^\epsilon(q)$ denote the special linear or unitary groups $S_n^\epsilon(q) = \{g \in G_n^\epsilon(q) | \det(g) = 1\}$ for $\epsilon = 1$ or -1 respectively. For convenience of notation we define K^ϵ to be $\mathbf{F}_q$ if $\epsilon = 1$ and to be $\mathbf{F}_{q^2}$ if $\epsilon = -1$ and $q = p^s$ for a prime p and a positive integer s.
If G is one of these groups, then one can attach to G a finite Lie algebra $LieG$ over K^ϵ in a natural way such that G has an 'adjoint action' on $LieG$. In case $G = G_n^1(q)$ for example, $LieG$ is just the 'bracket algebra' $Mat_n(q)_{[\ ,\ ]}$, where $[x, y] := xy - yx$; the adjoint action of $G_n^1(q)$ is given by conjugation of matrices.
In G as well as in $LieG$ we have a Jordan decomposition of elements into a semisimple

and unipotent (respectively nilpotent) part. For $g \in G$ this decomposition coincides with the well known decomposition into p' - part and p - part: $g = g_{p'}g_p = g_p g_{p'}$. Here $g_{p'}$ is semisimple and has an order coprime to p whereas g_p is unipotent of order a power of p. In the Lie algebra case $x \in LieG$ can be written uniquely as $x = x_s + x_n$ such that $[x_s, x_n] = 0$, where $Ad(x_s)$ and $Ad(x_n) \in End_{K^\epsilon}(LieG)$ are semisimple and nilpotent operators, respectively. For any subset $X \subseteq G$ or $LieG$, the set of semisimple elements in X will be denoted by X_{ss}.
A conjugacy class of G or an adjoint G - orbit in $LieG$, consisting of semisimple elements will be called a **semisimple** conjugacy class or orbit. It is called **regular** if the corresponding centralizers of elements are abelian.
In this paper we will prove the following explicit formulae for the number of regular semisimple classes and orbits of $G_n^\epsilon(q)$ and $S_n^\epsilon(q)$:

Theorem 1.1 *For $G = G_n^\epsilon(q)$ or $S_n^\epsilon(q)$ and $X = G$ or $LieG$ let $\mathrm{Reg}(X)$ denote the number of regular semisimple conjugacy classes or orbits in X. Then*

$$\mathrm{Reg}(S_n^\epsilon(q)) = \begin{cases} \frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon} & \textit{if } n \textit{ is odd or } q \textit{ is even.} \\ \frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon} - \epsilon^{\frac{n}{2}+1} & \textit{if } n \textit{ is even and } q \textit{ is odd.} \end{cases}$$

$$\mathrm{Reg}(G_n^\epsilon(q)) = (q-\epsilon)\cdot\frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon}.$$

$$\mathrm{Reg}(LieS_n^\epsilon(q)) = \begin{cases} q^{n-1}-q^{n-2} & \textit{if } n>2 \textit{ or } q \textit{ is odd.} \\ 0 & \textit{if } n=2 \textit{ and } q \textit{ is even.} \end{cases}$$

$$\mathrm{Reg}(LieG_n^\epsilon(q)) = q^n - q^{n-1}.$$

The result about $\mathrm{Reg}(LieS_n^\epsilon(q))$ has already been obtained in [5]; moreover in Corollary 3.5 of that paper, Lehrer proved that the number of regular semisimple classes in Chevalley groups of universal type is always odd. This can be verified in the case of $S_n^\epsilon(q)$ by rewriting the formulae above in the following way:

$$\mathrm{Reg}(S_n^\epsilon(q)) =$$

$$\begin{cases} q^{n-1}-q^{n-2}+q^{n-3}-q^{n-4}+q^{n-5}..+(-1)^{n+1} & \text{if } \epsilon=1 \text{ and } \{n \text{ is odd or } q \text{ is even }\}. \\ q^{n-1}-q^{n-2}-q^{n-3}+q^{n-4}+q^{n-5}..+(-1)^{\left[\frac{n}{2}\right]} & \text{if } \epsilon=-1 \text{ and } \{n \text{ is odd or } q \text{ is even }\}. \\ q^{n-1}-q^{n-2}+q^{n-3}-q^{n-4}+q^{n-5}..-2 & \text{if } \epsilon=1,\ n \text{ is even and } q \text{ is odd.} \\ q^{n-1}-q^{n-2}-q^{n-3}+q^{n-4}+q^{n-5}..+2\cdot(-1)^{\frac{n}{2}} & \text{if } \epsilon=-1 \text{ and } n \text{ is even and } q \text{ is odd.} \end{cases}$$

Notice that the numbers of regular semisimple classes for exceptional universal Chevalley groups F_4, E_6, E_7 and E_8 are listed in [3].

2 Basic results

In this section we collect some basic facts and notation about semisimple conjugacy classes and orbits of finite groups of Lie type. As an excellent and up to date source for this material we recommend Humphreys' book [4]. Throughout this paper, if Ω is a set and $\sigma \in \Sigma_\Omega$ is a bijection of Ω, then Ω^σ will denote the set of σ - fixed points in Ω. The groups $G_n^\epsilon(q)$ are special series of finite groups of Lie type. The group $GL_n(q)$ is the group of fixed points $(GL_n(\overline{\mathbf{F}}_q))^{F_q}$ of the Frobenius endomorphism

$$F_q : GL_n(\overline{\mathbf{F}}_q) \longrightarrow GL_n(\overline{\mathbf{F}}_q),\ (g_{ij}) \mapsto ({g_{ij}}^q),$$

whereas the unitary group $U_n(q)$ is the group of fixed points $(GL_n(\overline{\mathbf{F}}_q))^{F_{-q}}$ of the generalized Frobenius endomorphism

$$F_{-q} : GL_n(\overline{\mathbf{F}}_q) \longrightarrow GL_n(\overline{\mathbf{F}}_q),\ g \mapsto F_q(g^{-tr}),$$

where tr denotes transposition of matrices.

As classical groups they can be described in the following way: consider the involution $\tau : \mathbf{F}_{q^2} \longrightarrow \mathbf{F}_{q^2},\ c \mapsto c^q$ and the hermitian form on the $\mathbf{F}_{q^2}$ - vector space $(\mathbf{F}_{q^2})^n$ given by

$$\langle v, w \rangle := \sum_{i=1,\dots,n} v_i \cdot {w_i}^q,$$

for all $v = (v_1, \dots, v_n),\ w = (w_1, \dots, w_n) \in (\mathbf{F}_{q^2})^n$. Then $U_n(q)$ is the subgroup

$$\{g \in GL_n(q^2) | \langle g(v), g(w) \rangle = \langle v, w \rangle \forall\ v, w \in \mathbf{F}_{q^2}^n\}.$$

The Lie algebra $LieGL_n = Lie(GL_n(\overline{\mathbf{F}}_q))$ is the "bracket-algebra" $Mat_n(\overline{\mathbf{F}}_q)_{[\ ,\ ]}$ where $[\ ,\ ]$ is the commutator of matrices and $LieSL_n = Lie(SL_n(\overline{\mathbf{F}}_q))$ is the subalgebra of matrices with trace zero.
The endomorphisms

$$F_q : LieGL_n \longrightarrow LieGL_n,\ (m_{ij}) \mapsto ({m_{ij}}^q).$$

and

$$F_{-q} : LieGL_n \longrightarrow LieGL_n,\ m \mapsto F_q(-m^{tr})$$

are compatible in an obvious sense with the adjoint action of GL_n. We denote

$$LieG_n^\epsilon(q) := (LieGL_n)^{F_{\epsilon q}}$$

and

$$LieS_n^\epsilon(q) := (LieSL_n)^{F_{\epsilon q}}.$$

For all $g \in GL_n := GL_n(\overline{\mathbf{F}}_q)$ the centralizer groups are exactly the (Zariski - open) groups of units in the the corresponding centralizer algebras, hence they are connected. It is well known that in this situation the set of conjugacy classes of $G_n^\epsilon(q)$ is in bijection with the set of $F_{\epsilon q}$ - stable conjugacy classes of GL_n. In particular each such $F_{\epsilon q}$ - stable class

contains an $F_{\epsilon q}$ - stable element. The same is true for the adjoint orbits of $G_n^\epsilon(q)$ on $LieG_n^\epsilon(q)$.
Although centralizers of arbitrary elements of $SL_n := SL_n(\overline{\mathbf{F}}_q)$ are not connected in general (e.g. consider the centralizer of a single Jordan block), centralizers of semisimple elements $s \in SL_n$ or $LieSL_n$ are still connected, due to theorems of Steinberg [6]. So any semisimple $F_{\epsilon a}$ - stable class or orbit under SL_n gives rise to a unique class or orbit under $S_n^\epsilon(q)$.
Moreover two elements in SL_n or $LieSL_n$ are conjugate by an element in GL_n if and only if they are conjugate by an element in SL_n. Combining these facts we see that the sets of semisimple classes or orbits under $G_n^\epsilon(q)$ or $S_n^\epsilon(q)$ are in bijection with the corresponding sets of $F_{\epsilon q}$ - stable classes or orbits under GL_n in GL_n, SL_n, $LieGL_n$ or $LieSL_n$, respectively.
Now let G be any connected affine algebraic group, defined over the algebraically closed field K. Then each semisimple element $s \in G$ or $\in LieG$ lies in some maximal torus T or in some maximal **toral Lie algebra**, i.e. the subalgebra $Lie\ T \subseteq Lie\ G$ of a maximal torus T of G, respectively (see [1] 11.8, pg.150). Moreover all maximal tori and toral algebras are conjugate.

From these facts one can deduce that the semisimple classes of G are in bijection with the set $T\backslash\backslash W$ and the semisimple adjoint orbits of $LieG$ are in bijection with the set of orbits $LieT\backslash\backslash W$. In both cases $W = N_G(T)/C_G(T)$ is the (finite) Weyl group of G with respect to the maximal torus T of G.
As a quotient of a torus by a finite group, the set $G_{ss}\backslash\backslash G \cong T\backslash\backslash W$ of semisimple classes of G has the structure of an affine variety, whereas $(LieG)_{ss}\backslash\backslash W$ is a vector space.

If $G = GL_n$, then T can be chosen to be the subgroup T_n of invertible, diagonal matrices with $LieT$ being just the subspace D_n of $LieG$ consisting of all diagonal matrices and W is isomorphic to the symmetric group Σ_n, embedded into GL_n as the subgroup of permutation matrices.

Definition 2.1 *Let $a \in \overline{\mathbf{F}}_q$. We define:* $\mathcal{P}(n) := \{f \in \overline{\mathbf{F}}_q[X] \mid f \text{ monic, with } deg(f) = n\}$
$\mathcal{P}^*(n) := \{f = f_0 + f_1X + ... + X^n \in \mathcal{P}(n) \mid f_0 \neq 0\}$.
$\mathcal{P}_a(n) := \{f = f_0 + ... + f_{n-1}X^{n-1} + X^n \in \mathcal{P}(n) \mid f_{n-1} = a\}$
$\mathcal{P}^a(n) := \{f = f_0 + f_1X + ... + X^n \in \mathcal{P}(n) \mid f_0 = a\}$

Now from the remarks above we see, that sending a semisimple class or orbit in GL_n or $LieGL_n$ to the characteristic polynomial of a representing element, yields the following isomorphisms of algebraic varieties:

$$(LieGL_n)_{ss}\backslash\backslash GL_n \cong D_n\backslash\backslash\Sigma_n \cong \mathcal{P}(n);$$

$$(GL_n)_{ss}\backslash\backslash GL_n \cong T_n\backslash\backslash\Sigma_n \cong \mathcal{P}^*(n);$$

$$(LieSL_n)_{ss}\backslash\backslash SL_n \cong (LieSL_n \cap D_n)\backslash\backslash\Sigma_n \cong \mathcal{P}_0(n);$$

$$(SL_n)_{ss}\backslash\backslash SL_n \cong (SL_n \cap T_n)\backslash\backslash\Sigma_n \cong \mathcal{P}^{(-1)^n}(n).$$

The Frobenius endomorphisms $F_{\epsilon q}$ on $LieGL_n$ and GL_n induce Frobenius endomorphisms on $\mathcal{P}(n)$ and $\mathcal{P}^*(n)$ with

$$F_{\epsilon q}^{+} : \mathcal{P}(n) \to \mathcal{P}(n),\ f(X) \mapsto \epsilon^n f^{(q)}(\epsilon X);$$

and

$$F_{\epsilon q}^{*} : \mathcal{P}^*(n) \to \mathcal{P}^*(n),\ f(X) \mapsto \begin{cases} f^{(q)}(X) & \text{if } \epsilon = 1; \\ f_0^{-q} X^n f^{(q)}(1/X) & \text{if } \epsilon = -1; \end{cases}$$

Here for $f = f_0 + f_1 X + f_2 X^2 + \ldots X^n$ we define $f^{(q)}(X) := f_0^q + f_1^q X + f_2^q X^2 + \ldots X^n$.

Notice that, since centralizers in GL_n or SL_n of semisimple elements t (in the groups or Lie algebras) are connected reductive subgroups of GL_n, the finite centralizers $(GL_n)_t^{F_{\epsilon q}}$ or $(SL_n)_t^{F_{\epsilon q}}$ are abelian if and only if the algebraic centralizers $(GL_n)_t$ or $(SL_n)_t$ are abelian maximal tori. This in turn is the case if and only if all eigenvalues of the elements in the class are pairwise different, or, in other words, if the corresponding characteristic polynomial is squarefree. Hence to get the number of regular semisimple classes and orbits, we have to count those Frobenius fixed points in the sets of 2.1 which are squarefree.

Definition 2.2 *Let $a \in \overline{\mathbf{F}}_q$ and C be one of the sets $\mathcal{P}(n), \mathcal{P}^*(n), \mathcal{P}_a(n)$ or $\mathcal{P}^a(n)$. Then we define*

$$\mathcal{F}C := \{f \in C \mid f \text{ is squarefree}\}.$$

We get:

Lemma 2.3

$$\mathrm{R}eg(LieG_n^{\epsilon}) = |\mathcal{F}\mathcal{P}(n)^{F_{q\epsilon}^{+}}|;\ \mathrm{R}eg(G_n^{\epsilon}) = |\mathcal{F}\mathcal{P}^*(n)^{F_{q\epsilon}^{*}}|;$$

$$\mathrm{R}eg(LieS_n^{\epsilon}) = |\mathcal{F}\mathcal{P}_0(n)^{F_{q\epsilon}^{+}}|;\ \mathrm{R}eg(S_n^{\epsilon}) = |\mathcal{F}\mathcal{P}^{(-1)^n}(n)^{F_{q\epsilon}^{*}}|.$$

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3 Squarefree polynomials over finite fields

In this section we will determine the cardinalities of the sets $\mathcal{FP}(n)^{F_{q^\epsilon}^+}$, $\mathcal{FP}^*(n)^{F_{q^\epsilon}^*}$, $\mathcal{FP}_a(n)^{F_{q^\epsilon}^+}$ and $\mathcal{FP}^a(n)^{F_{q^\epsilon}^*}$. In fact we prove the following, slightly more general result:

Proposition 3.1 $|\mathcal{FP}^a(n)^{F_{q^\epsilon}^*}| =$

$$\begin{cases} \frac{q^{n+1}-q^n+\epsilon^{\frac{n-1}{2}}\cdot(q-\epsilon)}{q^2-\epsilon} & \textit{if } n>1 \textit{ is odd and } a\neq 0 \\ \frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon}+\epsilon^{\frac{n}{2}+1} & \textit{if } n>0 \textit{ is even and } a\neq 0 \textit{ is no square.} \\ \frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon} & \textit{if } n>0 \textit{ is even, } q \textit{ is even and } a\neq 0. \\ \frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon}-\epsilon^{\frac{n}{2}+1} & \textit{if } n>0 \textit{ is even, } q \textit{ is odd and } a\neq 0 \textit{ is a square.} \\ (q-\epsilon)\cdot\frac{q^n-q^{n-1}+(-1)^n\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon} & \textit{if } n\geq 3 \textit{ and } a=0. \\ q-\epsilon & \textit{if } n=2 \textit{ and } a=0. \\ 1 & \textit{if } n=1 \textit{ or } \{n=0 \textit{ and } a=1\}. \\ 0 & \textit{if } n=0 \textit{ and } a\neq 1. \end{cases}$$

$$|\mathcal{FP}^*(n)^{F_{q^\epsilon}^*}| = \begin{cases} (q-\epsilon)\cdot\frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon} & \textit{if } n>1. \\ (q-\epsilon)^n & \textit{if } n\in\{0,1\}. \end{cases}$$

$$|\mathcal{FP}_a(n)^{F_{q^\epsilon}^+}| = \begin{cases} q^{n-1}-q^{n-2} & \textit{if } n>2. \\ q-1 & \textit{if } n=2 \textit{ and } q \textit{ is odd.} \\ q & \textit{if } n=2,\ q \textit{ is even and } a\neq 0. \\ 1 & \textit{if } n=1 \textit{ or } \{n=0 \textit{ and } a=0\}. \\ 0 & \textit{if } \{n=0 \textit{ and } a\neq 0\} \textit{ or } \{n=2,\ q \textit{ is even and } a=0\}. \end{cases}$$

$$|\mathcal{FP}(n)^{F_{q^\epsilon}^+}| = \begin{cases} q^n-q^{n-1} & \textit{if } n>1. \\ q^n & \textit{if } n\in\{0,1\}. \end{cases}$$

Proof:

Let $f(X) = f_0 + f_1 X + ... + f_{n-1} X^{n-1} + X^n \in \mathcal{P}(n)$. Then f is stable under $F^*_{\epsilon q}$ if and only if the following holds:

$f_i^q = f_i$ for $i = 1, ..., n-1$ in case $\epsilon = 1$;

$f_0^{q+1} = 1$, $f_i = f_0 \cdot f_{n-i}^q$ for $1 \leq i \leq n-1$ in case $\epsilon = -1$.

In the case $\epsilon = -1$ this implies that $f_i^{q^2} = f_i$ for $i \in \{1, ..., n-1\}$, $i \neq n-i$ and $f_{\frac{n}{2}}^q = f_{\frac{n}{2}}$ if n is even.

The polynomial f is fixed by $F^+_{\epsilon q}$ iff $f_i^q = \epsilon^{n+i} \cdot f_i$ for $i = 0, ..., n-1$.

Since a polynomial can be decomposed uniquely into the product of a squarefree polynomial and the square of a polynomial, the following equations are obviously true

$$(*) \qquad |\mathcal{P}^a(n)^{F^*_{q\epsilon}}| = \sum_{i=0}^{[\frac{n}{2}]} \sum_{\{b \in \overline{\mathbf{F}}_q | b^{q-\epsilon}=1\}} |P^b(i)^{F^*_{q\epsilon}}| \cdot |\mathcal{FP}^{a \cdot b^{-2}}(n-2i)^{F^*_{q\epsilon}}|.$$

$$(**) \qquad |\mathcal{P}_a(n)^{F^+_{q\epsilon}}| = \sum_{i=0}^{[\frac{n}{2}]} \sum_{\{b \in \overline{\mathbf{F}}_q | b=(\epsilon)\cdot b^q\}} |P_b(i)^{F^+_{q\epsilon}}| \cdot |\mathcal{FP}_{a-2b}(n-2i)^{F^+_{q\epsilon}}|.$$

Let us start with the case $|\mathcal{FP}^*(n)^{F^*_{q\epsilon}}|$ and n odd. For $n \geq 1$ the set $\mathcal{P}^a(n)^{F^*_{q\epsilon}}$ always contains $(q-\epsilon)q^{n-1}$ elements; moreover $\mathcal{P}^a(0)^{F^*_{q\epsilon}} = 1$ if $a = 1$ and zero otherwise. Then by induction and $(*)$ we have to verify the equation

$$q^{n-1} = \frac{q^{n+1} - q^n + \epsilon^{\frac{n-1}{2}} \cdot (q-\epsilon)}{q^2 - \epsilon} + \sum_{i=1}^{\frac{n-1}{2}} (q-\epsilon) \cdot q^{i-1} \cdot \frac{q^{n-2i+1} - q^{n-2i} + \epsilon^{\frac{n-2i-1}{2}} \cdot (q-\epsilon)}{q^2 - \epsilon}.$$

Multiplying by $q^2 - \epsilon$ yields

$$q^n - \epsilon \cdot q^{n-1} - \epsilon^{\frac{n-1}{2}} \cdot (q-\epsilon) = \sum_{i=1}^{\frac{n-1}{2}} q^{n-i+1} - (1+\epsilon) \cdot q^{n-i} + \epsilon \cdot q^{n-i-1} + \epsilon^{\frac{n-2i-1}{2}} \cdot (q^{i+1} - 2 \cdot \epsilon \cdot q^i + q^{i-1}).$$

For $\epsilon = 1$ we get

$$q^n - q^{n-1} - q + 1 = (q-1)^2 \cdot \sum_{i=1}^{\frac{n-1}{2}} (q^{n-i-1} + q^{i-1}) = (q-1) \cdot (q^{n-1} - 1)$$

and for $\epsilon = -1$

$$q^n + q^{n-1} - (-1)^{\frac{n-1}{2}}(q+1) = (q^2-1) \cdot (\sum_{i=1}^{\frac{n-1}{2}} q^{n-i-1}) + (q+1)^2(-1)^{\frac{n-1}{2}} \cdot (\sum_{i=1}^{\frac{n-1}{2}} (-1)^i q^{i-1})$$

$= (q+1) \cdot (q^{n-1} - (-1)^{\frac{n-1}{2}})$. These equations are obviously true and the case $|\mathcal{FP}^*(n)^{F^*_{q\epsilon}}|$ and n is odd is verified. The other cases can be done in an analogous way using the equations $(*)$ or $(**)$. By this proposition the proof of Theorem 1.1 is finished. $\circ$

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