Squarefree Polynomials over Finite Fields and Regular Semisimple Classes of Certain Classical Groups

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Abstract

Let G be $PSL_n(q), PSU_n(q), Sp_{2n}(q)$ or $PSp_{2n}(q)$, where q is a power of the prime p. Using results on the numbers of special squarefree polynomials over finite fields, we describe and count the conjugacy classes of p' - elements with abelian centralizers in G. Similar results are obtained for the semisimple orbits of G on the corresponding finite Lie algebras.

1 Introduction

Let G be a finite group and ω an element of the finite G - set Ω. We call ω or the orbit ω^G **abelian** if the centralizer G_ω of ω is abelian. If $\mathbf{G}$ is a connected, reductive, linear algebraic group with Frobenius endomorphism F and Frobenius fixed point group $G := \mathbf{G}_F$, a semisimple element $t \in G$ is called **regular**, if the connected component $(\mathbf{G}_t)^0$ of its centralizer is a maximal torus. If $\mathbf{G}$ is quasisimple and simply connected, which means that $\mathbf{G}_F$ is a universal finite Chevalley group, then, due to a theorem of Steinberg, centralizers of semisimple elements are connected and a semisimple element t of $\mathbf{G}_F$ is regular if and only if it is abelian in the sense above.

In [8] formulae have been derived for the total number of semisimple abelian conjugacy classes for the groups $GL_n(q), U_n(q), SL_n(q)$ and $SU_n(q)$ over the finite field $\mathbb{F}_q$.

In this paper these results are extended to the (generically simple) central quotient groups $PSL_n(q)$ and $PSU_n(q)$ as well as the symplectic groups groups $Sp_{2n}(q)$ and $PSp_{2n}(q)$. For example we will prove that the total number of regular semisimple conjugacy classes of $Sp_{2n}(q)$ is given by the formula

$$(-1)^n(n+1)+\sum_{i=0}^{n-1}(-1)^i(2i+1)q^{n-i}, \text{if } q \text{ is odd and by}$$

$$(\frac{q-1}{q+1})(q^n+(-1)^{n-1}), \text{if } q \text{ is even.}$$

Since our method also works for the adjoint action of $\mathbf{G}_F$ on its Lie algebra $Lie(\mathbf{G})_F$, where F is a compatible Frobenius map on $Lie(\mathbf{G})$, we get similar results about the total number of abelian semisimple orbits. The corresponding question for the orthogonal groups will be handled in a subsequent paper.

2 Basic results

In this section we give a brief summary of the general theory of semisimple conjugacy classes and orbits of finite reductive groups and their Lie algebras. We will give proofs only in those cases, where the statements we need are not easily found in the literature.

If Ω is a set on which the group G acts, then Ω/G will denote the set of G - orbits on Ω. For each $\omega \in \Omega$ we define $G_\omega := \{g \in G \mid {}^g\omega = \omega\}$. For any subset X of an algebraic group or its Lie algebra, let X_{ss} denote the subset of semisimple elements in X.

Let $\mathbf{G}$ be any connected reductive affine algebraic group, defined over the algebraically closed field $\overline{\mathbb{F}}_p$ and let F denote a Frobenius endomorphism on $\mathbf{G}$ and on $Lie(\mathbf{G})$ such that for each $X \in Lie(\mathbf{G})$ we have

$$F({}^g(X)) = F(ad(g)(X)) = ad(F(g))(F(X)) = {}^{F(g)}(F(X)).$$

Then G will denote the fixed point group $\mathbf{G}_F$ and $Lie(G) := Lie(\mathbf{G})_F$.

2.1 Centralizers of semisimple elements

It is known that each semisimple element $\delta \in \mathbf{G}$ or $Lie(\mathbf{G})$ lies in some maximal torus $\mathcal{T} := \mathbf{T}$ or $Lie(\mathbf{T})$ respectively, and that all maximal tori are conjugate (see [Bo] 11.8, pg. 150).

Let $t \in \mathbf{T}$ or $Lie(\mathbf{T})$. Then the 'connected centralizers' $\mathbf{G}_t^0$ are connected reductive algebraic subgroups of the form $\mathbf{G}_{\Phi_t} := \langle \mathbf{T}, \mathbf{X}_\alpha \mid \alpha \in \Phi_t \rangle$, where Φ_t is the set of all roots $\alpha \in \Phi$ such that $\alpha(t) = 1$ if $t \in \mathbf{T}$ or $d\alpha(t) = 0$ if $t \in Lie(\mathbf{T})$ (see [14]). Here $d\alpha \in Hom_{\overline{\mathbb{F}}_q}(Lie(\mathbf{T}), \overline{\mathbb{F}}_q)$ is the differential of α and $\mathbf{X}_\alpha$ denotes the unipotent one parameter subgroup of $\mathbf{G}$ corresponding to α. Moreover, if t is F - stable, the set of F - fixed points $(t^{\mathbf{G}})_F$ splits into a set of semisimple G - orbits.

Lemma 2.1 *Any semisimple element $s \in G$ is contained in some F - stable maximal torus $\tilde{\mathbf{T}}$ of $\mathbf{G}$. Similarly any semisimple element $\delta \in Lie(G)$ is contained in $Lie(\tilde{\mathbf{T}}) \leq Lie(\mathbf{G})$ with some $\tilde{\mathbf{T}}$ as above.*

Proof: For the group case this has been proved in [3]. In the Lie algebra case the argument is similar (see also [12] (2.6)). We include a short proof for convenience: let $\mathcal{T} := Lie(\mathbf{T})$ be a maximal toral subalgebra of $Lie(\mathbf{G})$ containing δ. Since δ is semisimple we have $Lie(\mathbf{G}_\delta) = Lie({\mathbf{G}_\delta}^0) = \mathcal{Z}_{Lie(\mathbf{G})}(\delta) := \{X \in Lie(\mathbf{G}) \mid [\delta, X] = 0\}$, hence $Lie(\mathbf{G}_\delta)$ is F - stable. Now $\mathbf{T} \leq (\mathbf{G}_\delta)^0$ so $\delta \in \mathcal{T} \subseteq Lie((\mathbf{G}_\delta)^0)$ and $F(\mathcal{T}) = {}^g\mathcal{T}$ with $g \in (\mathbf{G}_\delta)^0$. By Lang's theorem there is $h \in (\mathbf{G}_\delta)^0$ with $h^{-1}F(h) = g^{-1}$. Hence $\delta = {}^h\delta \in {}^h\mathcal{T} = F({}^h\mathcal{T})$. $\circ$

Now we fix one F - stable maximal torus $\mathbf{T}$ of $\mathbf{G}$; then any other F - stable maximal torus is of the form ${}^g\mathbf{T}$ for some $g \in \mathbf{G}$ with $\mathbf{N} := N_{\mathbf{G}}(\mathbf{T}) \ni \mathcal{L}_F(g) := g^{-1}F(g) \mapsto w \in W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Notice that we have $F \circ int(g)|_{\mathbf{T}} = int(g) \circ w \circ F|_{\mathbf{T}}$, where $int(g)$ denotes (left) conjugation with $g \in \mathbf{G}$. Hence $({}^g\mathbf{T})_F \cong (\mathbf{T})_{w \circ F}$. If for some $h \in \mathbf{G}$, ${}^h\mathbf{T}$ is G - conjugate to ${}^g\mathbf{T}$, then there is $x \in G$ with $\tilde{n} := g^{-1}xh \in \mathbf{N}$ and $\mathcal{L}_F(g) = \tilde{n}\mathcal{L}_F(h)F(\tilde{n})^{-1}$. As $\mathcal{L}_F(h) \in \mathbf{N}$, it maps to $w' \in W$ and $\tilde{n}$ to $\tilde{w}$, so $w = \tilde{w}w'F(\tilde{w})^{-1}$, hence w, w' lie in the same 'F - conjugacy class', which is an orbit under the F - twisted conjugacy action of W. It follows from a theorem of Lang that the map ${}^g\mathbf{T} \mapsto \mathcal{L}_F(g)\mathbf{T} \in W$ sets up a bijection between G classes of F - stable maximal tori in $\mathbf{G}$ or $Lie(\mathbf{G})$ and F - conjugacy classes of W (see [4] 3.23 or [2] 3.3.3). For each F - class $[w] \subseteq W$ we fix one $g \in \mathbf{G}$ with $w = \mathcal{L}_F(g)\mathbf{T}$, denote ${}^g\mathbf{T}$ by $\mathbf{T}(w)$ and $({}^g\mathbf{T})_F \cong \mathbf{T}_{wF}$ by $T(w)$. Similarly we denote $Lie(\mathbf{T}(w))_F \cong Lie(\mathbf{T})_{wF}$ by $Lie(T(w))$. We choose $g = 1$ if $w = 1$ to get $\mathbf{T}(id) = \mathbf{T}$.

Let $T(w) \leq {}^g\mathbf{T}$ with $g^{-1}F(g) \mapsto w$ be a finite maximal torus. For any $x, h \in \mathbf{G}$ we have $F({}^hx) = {}^{h\mathcal{L}_F(h)}F(x)$, hence $N_G(\mathbf{T}(w)) = N_G({}^g\mathbf{T}) =$

$$\begin{gathered} {}^g\{x \in N_{\mathbf{G}}(\mathbf{T}) \mid x = {}^{\mathcal{L}_F(g)}F(x)\} \text{ and } N_G(\mathbf{T}(w))/(\mathbf{T}(w))_F \cong \\ {}^g(\{x \in N_{\mathbf{G}}(\mathbf{T}) \mid x = {}^{\mathcal{L}_F(g)}F(x)\}/\mathbf{T}_{\mathcal{L}_F(g)F}) \cong {}^g(N_{\mathbf{G}}(\mathbf{T})_{\mathcal{L}_F(g)}/\mathbf{T}_{\mathcal{L}_F(g)}) \cong \\ (N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})_{\mathcal{L}_F(g)} = \{w' \in W \mid w'w = wF(w')\} =: C_{W,F}(w), \end{gathered}$$

where the third isomorphism follows from Lang's theorem.

Notice that $N_G(\mathbf{T}(w)) \leq N_G(T(w))$ may be a strict inclusion (e.g. if $\mathbf{T}(w)_F = 1$). But this cannot happen if $\mathbf{T}(w)$ is the centralizer of a semisimple element of G.

Lemma 2.2 *Let $t \in G$ be semisimple with $(\mathbf{G}_t)^0_F = T(w)$. Then $N_G(\mathbf{T}(w)) = N_G(T(w))$.*

Proof: Since $(\mathbf{G}_t)^0_F = T(w)$ is abelian, the connected reductive group $(\mathbf{G}_t)^0$ is an F - stable maximal torus of $\mathbf{G}$. Let $\mathbf{T}'$ be some F - stable maximal torus containing t, then $t \in \mathbf{T}' \leq (\mathbf{G}_t)^0$, hence $t \in T(w)$ and $\mathbf{T}(w) = \mathbf{T}' = (\mathbf{G}_t)^0$. Let $x \in N_G(T(w))$, then $t \in \mathbf{T}(w) \cap \mathbf{T}(w)^x$, hence $\mathbf{T}(w)^x \leq \mathbf{G}_t^0 = \mathbf{T}(w)$ and $x \in N_G(\mathbf{T}(w))$. $\circ$

If $\mathcal{T} = \mathbf{T}$ or $Lie(\mathbf{T})$ and $\delta \in \mathcal{T}(w)_F$ is regular then we have $\mathbf{G}_\delta \leq N_{\mathbf{G}}((\mathbf{G}_\delta)^0) = N_{\mathbf{G}}(\mathbf{T}(w))$. Hence

$$G_\delta/T(w) = (\mathbf{G}_\delta/\mathbf{T}(w))_F = {}^g(C_{W,F}(w)_{t'}) \cong C_{W,F}(w)_{t'},$$

where $\delta = {}^g t' \in {}^g\mathcal{T} = \mathcal{T}(w)$.

2.2 Centralizers modulo center

Now let $H := G/Z$ with $Z = Z(G)$. We want to consider the abelian p' - classes in H. We call a semisimple class or element of $tZ \in G/Z$ **regular** if t (and so any element in tZ) is regular in G. The results about regular and abelian classes of G/Z can be derived from those for G using the following easy lemma:

Lemma 2.3 *Let X be a finite group with center Z, $Y \subseteq X$ a normal subset with $ZY \subseteq Y$ and ${}^-: X \to \overline{X} := X/Z$ the canonical epimorphism. Then Z acts on the set Y/X of X - conjugacy classes in Y and there is a bijection*

$$\theta : (Y/X)/Z \to \overline{Y}/\overline{X}, x^X Z \to \bar{x}^{\overline{X}}.$$

For any $x \in Y$, $C_{\overline{X}}(\bar{x}) = M_x/Z$ with $M_x := \{g \in X \mid [x,g] \in Z\}$ and we have an exact sequence:

$$1 \hookrightarrow X_x \to M_x \to Z_{x^X} \to 1.$$

Notice that there is the following subtlety connected with the notion of 'regular elements' in the case of the central quotient groups: Although abelian p' - classes of G/Z are always regular, the converse need not be true: indeed, by 2.3, the centralizer of a regular p' - element is an extension of the abelian group G_t/Z by a subgroup of Z and thus can be nonabelian. On the other hand, even if G_t/Z is a proper subgroup of $(G/Z)_{tZ}$, this latter centralizer can still be abelian. As we will see later this only happens very rarely. In all the other cases, which can be viewed as the 'generic case', a p' - element $tZ \in G/Z$ is abelian if and only if it is regular and $(G/Z)_{tZ} = G_t/Z$. These p' - elements or classes will be called **toral**.

If $t \in T(w)$ then we have $C_H(tZ) = M_t/Z$, where $M_t := \{x \in G \mid {}^x t = tz, \exists\ z \in Z\}$ $= \mathbf{M}_{t,F}$ with $\mathbf{M}_t := \{x \in \mathbf{G} \mid {}^x t = tz, \exists\ z \in \mathbf{Z}\}$. By a Frattini type argument $\mathbf{M}_t = \mathbf{G}_t^0 C_{N_{\mathbf{G}(\mathbf{T}(w))}}(t\mathbf{Z})$ hence

$$M_t/(\mathbf{G}_t^0)_F = (\mathbf{M}_t/\mathbf{G}_t^0)_F = (\mathbf{G}_t^0 C_{N_{\mathbf{G}(\mathbf{T}(w))}}(t\mathbf{Z})/\mathbf{G}_t^0)_F \cong (W_{t'\mathbf{Z}}/W_{t'})_{wF}.$$

Here $\mathbf{T}(w) = {}^g\mathbf{T}$ and $t' = t^g$. On the other hand, the group homomorphism $W_{t'\mathbf{Z}} \to \mathbf{Z}$, $w \mapsto {}^w t' t'^{-1}$ has $W_{t'}$ as kernel, so $(W_{t'\mathbf{Z}}/W_{t'})_{wF}$ is isomorphic to a subgroup of $Z = \mathbf{Z}_{wF} = \mathbf{Z}_F$. We see that $C_{G/Z}(tZ)$ is solvable if and only if $(\mathbf{G}_t^0)_F$ is solvable. With some exceptions (over small fields with less than 5 elements) this is true if and only if t is regular in $\mathbf{G}$.

2.3 Connected centralizers

Now consider the following technical assumptions:
In the case of conjugacy classes we assume that the center of the dual group $\mathbf{G}^*$ *(in the sense of [2]) is connected.*
In the case of orbits in the Lie algebra we assume that $p = char\ \mathbb{F}_q$ *is not a torsion prime (in the sense of [14]).*

Whenever we invoke this hypothesis we will simply say that $\mathbf{G}$ is assumed to satisfy condition $(*)$.

For example SO_{2n+1} is the dual of Sp_{2n}, hence in case of conjugacy classes Sp_{2n} satisfies condition $(*)$ because $Z(SO_{2n+1}) = 1$, but SO_{2n+1} does not if the characteristic of $\overline{\mathbb{F}}_q$ is odd, since $Z(Sp_{2n}) \cong \mathbb{Z}_2$ in this case. Torsion primes are all primes dividing either $|X^*/\langle\Phi^*\rangle|$ or $|Tor(\langle\Phi^*\rangle/\langle\Phi_1^*\rangle)|$ for some closed subsystem $\Phi_1 = \langle\Phi_1\rangle \cap \Phi$ of the root system $\Phi := \Phi(\mathbf{G},\mathbf{T})$. Here $\Phi^* \subseteq X^*$ denotes the root system and character group of $\mathbf{G}^*$ and ${}^* : \Phi \to \Phi^*$ the duality bijection of root systems. In particular GL_n, SL_n as well as Sp_{2n} do not have any torsion primes. For the other quasisimple groups the torsion primes are ≤ 5 (see [14]).

Let $t \in \mathbf{T}$ or $Lie(\mathbf{T})$. Then, as a consequence of condition $(*)$, the centralizers $\mathbf{G}_t$ are connected reductive algebraic subgroups of the form $\mathbf{G}_{\Phi_t} := \langle \mathbf{T}, \mathbf{X}_\alpha \mid \alpha \in \Phi_t\rangle$, where Φ_t is the set of all roots $\alpha \in \Phi$ such that $\alpha(t) = 1$ if $t \in \mathbf{T}$ or $d\alpha(t) = 0$ if $t \in Lie(\mathbf{T})$ (see [14]). Here $d\alpha \in Hom_{\overline{\mathbb{F}}_q}(Lie(\mathbf{T}), \overline{\mathbb{F}}_q)$ is the differential of α and $\mathbf{X}_\alpha$ denotes the unipotent one parameter subgroup of $\mathbf{G}$ corresponding to α. Moreover, if t is F - stable, the set of F - fixed points $(t^{\mathbf{G}})_F$ forms a single semisimple G - orbit. Hence if $(*)$ holds then we have bijections

$$G_{ss}/G \cong (\mathbf{T}/W)_F \text{ and } Lie(G)_{ss}/G \cong (Lie(\mathbf{T})/W)_F.$$

Notice that if $(*)$ holds, $G_t = \mathbf{G}_{t,F}$ is abelian if and only if $\alpha(t) \neq 1$, or $d\alpha(t) \neq 0$ for all $\alpha \in \Phi$ respectively, i.e. if and only if t is regular.

Following Carter ([3]), we call two semisimple conjugacy classes t^G, t'^G of G or G - orbits in $Lie(G)$ of the same **genus** if the 'connected algebraic centralizers' $\mathbf{G}_t^0$ and $\mathbf{G}_{t'}^0$ are conjugate in G. The number of classes of given genus will be called a **genus number**. If (*) holds, t^G and t'^G are of the same genus if and only if the finite centralizers G_t and $G_{t'}$ are conjugate under G; if moreover $t \in G_{ss}$ or $Lie(G)_{ss}$ is regular, then G_t is a maximal torus of G. Hence the regular semisimple genera are parametrized by the set of F - conjugacy classes of W.

3 Groups of type A

Let $\tilde{\mathbf{G}}$ be the group $GL_n(\bar{\mathbb{F}}_q)$ and $\mathbf{G} = SL_n(\bar{\mathbb{F}}_q)$, viewed as connected reductive, linear algebraic groups, defined over the finite field with $q = p^s$ elements. Let $F := F_{\epsilon q}$ denote the Frobenius morphisms on $\tilde{\mathbf{G}}$ and $\mathbf{G}$, defined by $g \mapsto g^{(q)}$ if $\epsilon = 1$ and $g \mapsto (g^{(q)})^{-tr}$ if $\epsilon = -1$. Here g^{-tr} is the inverse transpose of g and $g^{(q)}$ is the matrix with every entry raised to the q - th power. Then the Frobenius fixed - point groups are the following finite groups of Lie type:

$$\begin{aligned} \tilde{\mathbf{G}}^F &= G_n^\epsilon(q) := \begin{cases} GL_n(q) & \text{if } \epsilon = 1, \\ GU_n(q) & \text{if } \epsilon = -1. \end{cases} \\ \mathbf{G}^F &= S_n^\epsilon(q) := \begin{cases} SL_n(q) & \text{if } \epsilon = 1, \\ SU_n(q) & \text{if } \epsilon = -1. \end{cases} \end{aligned}$$

For $G_n^\epsilon(q)$ all semisimple genus numbers have been determined in [5]; in [7] this has been done for $S_n^\epsilon(q)$. Moreover in [8] we gave a closed formula for the total number of regular semismple classes in each of these cases. In this section we will supplement this by proving the analoguous results for the quotient groups:

$$\begin{aligned} PG_n^\epsilon(q) := G_n^\epsilon(q)/Z(G_n^\epsilon(q)) &:= \begin{cases} PGL_n(q) & \text{if } \epsilon = 1, \\ PU_n(q) & \text{if } \epsilon = -1. \end{cases} \\ PS_n^\epsilon(q) := S_n^\epsilon(q)/Z(G_n^\epsilon(q)) &:= \begin{cases} PSL_n(q) & \text{if } \epsilon = 1, \\ PSU_n(q) & \text{if } \epsilon = -1. \end{cases} \end{aligned}$$

Let $\tilde{\mathbf{G}} = GL_n(\bar{\mathbb{F}}_q) =: GL_n$, $\mathbf{G} = SL_n(\bar{\mathbb{F}}_q) =: SL_n$. We choose $\tilde{\mathbf{T}}$ as the subgroup of invertible, diagonal matrices and $\mathbf{T} := \tilde{\mathbf{T}} \cap SL_n$.

We recall some concepts from [8].

3.1 The total numbers of abelian p' - classes in $PS_n^\epsilon(q)$

Definition 3.1 *Let K be a field and $a \in K$ an element. We define the following:*
$\mathcal{P}(\ell, K) := \{f = f_0 + f_1X + ... + X^\ell \in K[X] \mid f \text{ monic of degree } \ell\}$.
$\mathcal{P}_*(\ell, K) := \mathcal{P}_*(\ell) := \{f \in \mathcal{P}(\ell, K) \mid f_0 \neq 0\}$.
$\mathcal{P}_a(\ell, K) := \mathcal{P}_a(\ell) := \{f \in \mathcal{P}(\ell, K) \mid f_0 = a\}$.

Now the map which associates to any semisimple element $t \in \tilde{\mathbf{G}}$ its characteristic polynomial $cp(t) \in \mathcal{P}_*(n)$ yields bijections

$$(GL_n)_{ss}/GL_n \cong \tilde{\mathbf{T}}/\Sigma_n \cong \mathcal{P}_*(n),$$

$$(SL_n)_{ss}/SL_n \cong \mathbf{T}/\Sigma_n \cong \mathcal{P}_{(-1)^n}(n).$$

The Frobenius endomorphism $F_{\epsilon q}$ on GL_n induces a Frobenius endomorphism on $\mathcal{P}_*(n)$ with

$$F_{q\epsilon} : \mathcal{P}_*(n) \to \mathcal{P}_*(n),\ \ f(X) \mapsto \begin{cases} f^{(q)}(X) & \text{if } \epsilon = 1, \\ \\ f_0^{-q} X^n f^{(q)}(1/X) & \text{if } \epsilon = -1. \end{cases}$$

Here for $f = f_0 + f_1X + f_2X^2 + ...X^n$ we define $f^{(q)}(X) := f_0^q + f_1^qX + f_2^qX^2 + ...X^n$.

Now we get obvious bijections:

$$G_n^\epsilon(q)_{ss}/G_n^\epsilon(q) \cong (\mathcal{P}_*(n))_{F_{q\epsilon}},$$

$$S_n^\epsilon(q)_{ss}/S_n^\epsilon(q) \cong (\mathcal{P}_{(-1)^n}(n))_{F_{q\epsilon}}.$$

Notice that the cyclic group $\mathbb{F}_q^* \cong \mathbb{Z}_{q-1}$ acts on the set $\mathcal{P}_*(n, q)$ by the rule

$${}^c f(X) := c^n f(c^{-1}X) = \prod_{i=1}^n (X - c\alpha_i),$$

where $f = \prod_{i=1}^n (X - \alpha_i)$ is the factorization of f over $\overline{\mathbb{F}}_q$. We identify $\mathbf{Z} = Z(GL_n(\bar{\mathbb{F}}_q))$ with $\bar{\mathbb{F}}_q^*$ and $Z(G_n^\epsilon(q))$, $Z(S_n^\epsilon(q))$ with the subgroups $C(\epsilon, q) = (\bar{\mathbb{F}}_q)^{q\epsilon}$ $\cong \mathbb{Z}_{q-\epsilon}$ and $C(\epsilon, q, n) \cong \mathbb{Z}_{\gcd(q-\epsilon, n)}$.

The next lemma is a straightforward consequence of Lemma 2.3:

Lemma 3.2 *Let G be $G_n^\epsilon(q)$ or $S_n^\epsilon(q)$ and $t \in G$ a semisimple element. Then we have the following exact sequence:*

$$1 \hookrightarrow G_t \to H_t \to Z_{cp(t)} \to 1,$$

where $Z = Z(G) =: C(\epsilon, q)$ or $=: C(\epsilon, q, n)$ respectively and $H = G/Z$. Moreover there are bijections

$$(G/Z)_{p'}/(G/Z) \to \begin{cases} (\mathcal{P}_*(n))_{F_{q\epsilon}}/C(\epsilon, q) & \text{if } G = G_n^\epsilon(q), \\ (\mathcal{P}_{(-1)^n}(n))_{F_{q\epsilon}}/C(\epsilon, q, n) & \text{if } G = S_n^\epsilon(q), \end{cases}$$

given by the map $[tZ] \mapsto [cp(t)]$.

Notice that the finite stabilizers $G_n^\epsilon(q)_t$ or $S_n^\epsilon(q)_t$ are abelian if and only if the algebraic stabilizers $(GL_n)_t$ or $(SL_n)_t$ are abelian maximal tori. This in turn is the case if and only if all eigenvalues of the elements in the class are pairwise different, or, in other words, if the corresponding characteristic polynomial is squarefree. Hence to get the number of regular semisimple classes and orbits, we have to count those Frobenius fixed points in the sets of 3.1 which are squarefree. As in [8] we define for $C = \mathcal{P}_*(n)$ or $\mathcal{P}_{(-1)^n}(n)$ the sets $\mathcal{F}C := \{f \in C \mid f \textit{ is squarefree}\}$ and get for the number $\mathrm{R}eg_{ss}$ of semisimple classes or orbits:

$$\mathrm{R}eg_{ss}(G_n^\epsilon) = |(\mathcal{F}\mathcal{P}_*(n))_{F_{q\epsilon}}|, \ \mathrm{R}eg_{ss}(S_n^\epsilon) = |(\mathcal{F}\mathcal{P}_{(-1)^n}(n))_{F_{q\epsilon}}|.$$

These numbers have been determined in [8]:

Theorem 3.3 $|(\mathcal{P}_*(n))_{F_{q\epsilon}}| = (q-\epsilon)q^{n-1}$.

$|(\mathcal{P}_{(-1)^n}(n))_{F_{q\epsilon}}| = q^{n-1}$.

$|(\mathcal{F}\mathcal{P}_*(n))_{F_{q\epsilon}}| = (q-\epsilon) \cdot \frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon}$.

$$|(\mathcal{F}\mathcal{P}_{(-1)^n}(n))_{F_{q\epsilon}}| = \begin{cases} \frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon} & \text{if } n \text{ is odd or } q \text{ is even}, \\ \frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon} - \epsilon^{\frac{n}{2}+1} & \text{if } n \text{ is even and } q \text{ is odd}. \end{cases}$$

Now our results can be stated as follows:

Theorem 3.4 *Let G be $G_n^\epsilon(q)$ or $S_n^\epsilon(q)$, Z the center of G, and $H = PG_n^\epsilon(q)$ or $PS_n^\epsilon(q)$, resepectively. We define $\mathcal{C}_{p',n}^\epsilon(q)$ to be the set of all p' - classes in H, $\mathcal{C}_{p',n}^{\epsilon,reg}(q) := \{[tZ] \in \mathcal{C}_{p',n}^\epsilon(q) \mid tZ \text{ regular}\}$ and $\mathcal{C}_{p',n}^{\epsilon,tor}(q) := \{[tZ] \in \mathcal{C}_{p',n}^\epsilon(q) \mid tZ \text{ toral}\}$. Then*

$$\mathcal{C}_{p',n}^\epsilon(q) = \begin{cases} \sum_{d|(q-\epsilon,n)} \phi(d) q^{n/d-1} & \text{if } H = PG_n^\epsilon(q), \\ \frac{1}{\gcd(q-\epsilon,n)} \sum_{d|(q-\epsilon,n)} \phi(d) q^{n/d-1} & \text{if } H = PS_n^\epsilon(q). \end{cases}$$

$$\mathcal{C}^{\epsilon,reg}_{p',n}(q) = \begin{cases} \frac{1}{q-\epsilon}\sum_{d|(q-\epsilon,n)}\phi(d)f^{\epsilon}_{n/d}(q) & \textit{if } H = PG^{\epsilon}_n(q), \\ \\ \frac{1}{\gcd(q-\epsilon,n)}\sum_{d|(q-\epsilon,n)}\phi(d)g^{\epsilon}_{n/d}(q) & \textit{if } H = PS^{\epsilon}_n(q). \end{cases}$$

$$\mathcal{C}^{\epsilon,tor}_{p',n}(q) = \begin{cases} \frac{1}{q-\epsilon}\sum_{d|(q-\epsilon,n)}\mu(d)f^{\epsilon}_{n/d}(q) & \textit{if } H = PG^{\epsilon}_n(q), \\ \\ \frac{1}{\gcd(q-\epsilon,n)}\sum_{d|(q-\epsilon,n)}\mu(d)g^{\epsilon}_{n/d}(q) & \textit{if } H = PS^{\epsilon}_n(q). \end{cases}$$

Here ϕ is the Euler function, μ the arithmetical Moebius function,

$$f^{\epsilon}_n(q) = (q-\epsilon)\cdot\frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon}$$

and

$$g^{\epsilon}_n(q) := \begin{cases} \frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon} & \textit{if } n \textit{ is odd or } q \textit{ is even,} \\ \\ \frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon}-\epsilon^{\frac{n}{2}+1} & \textit{if } n \textit{ is even and } q \textit{ is odd.} \end{cases}$$

Proof: Let $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ act on the set Ω. For each divisor d of m let Z_d be the subgroup of order d in $\mathbb{Z}_m$ and define $\mathcal{O}_d := \{\omega^{\mathbb{Z}_m} \mid Z_d = \mathbb{Z}_{m\omega}\}$. Then $\biguplus_{d|d'|m}\biguplus_{o\in\mathcal{O}_{d'}} o = Fix_{\Omega}(Z_d)$ and Möbius - inversion yields:

$$|\mathcal{O}_d| = \frac{d}{m}\sum_{d|d'|m}\mu(d'/d)|Fix_{\Omega}(Z_{d'})|.$$

By the lemma of Cauchy and Frobenius ([11] 1.1.1), the total number of $\mathbb{Z}_m$ - orbits is given by

$$|\Omega/\,\mathbb{Z}_m\,| = \frac{1}{m}\sum_{d\ |\ m}\phi(d)|Fix_{\Omega}(Z_d)|.$$

Now let $f \in \mathcal{P}_*(n)$ and $z \in \bar{\mathbb{F}}_q^*$. Then ${}^z f = f = \sum_{i=0}^n f_i X^i$ if and only if $f_i = 0$ whenever $z^{n-i} \neq 1$. Notice that for $f = h(X^d)$ with $h \in \overline{\mathbb{F}}_q[X]$ and $d \in \mathbb{N}$ coprime to q, the polynomial h is squarefree if and only if f is squarefree. Hence if $C(n)$ is any one of $(\mathcal{P}_*(n))_{q\epsilon}$, $(\mathcal{P}_{(-1)^n}(n))_{q\epsilon}$, $(\mathcal{FP}_*(n))_{q\epsilon}$ or $(\mathcal{FP}_{(-1)^n}(n))_{q\epsilon}$ then for any z of order d we get $|Fix_{C(n)}(z)| = |C(n/d)|$ if d divides n and is coprime to q, and zero if d does not divide n. Notice that $|Fix_{(\mathcal{FP}_{(-1)^n}(n)_{F_{q\epsilon}})}(z)| = |\mathcal{FP}_{(-1)^n}(n/d)_{F_{q\epsilon}}|$ if d divides n. The formulae in [8] for $|\mathcal{FP}_a(n)_{F_{q\epsilon}}|$ show that $|\mathcal{FP}_{(-1)^n}(n/d)_{F_{q\epsilon}}|$ $= |\mathcal{FP}_{(-1)^{n/d}}(n/d)_{F_{q\epsilon}}|$.

Now define $\mathcal{C}^{\epsilon}_{p',n,d}(q) := \{[tZ] \in \mathcal{C}^{\epsilon}_{p',n}(q) \mid t \in G \textit{ regular},\ |H_t/(G_t/Z)| = d\}$. Then we get $\mathcal{C}^{\epsilon}_{p',n,d}(q) =$

$$\begin{cases} \frac{d}{q-\epsilon}\sum_{s,\ d|s|(q-\epsilon,n)}\mu(s/d)|(\mathcal{FP}_*(n/s))_{F_{q\epsilon}}| & \text{if } H = PG^{\epsilon}_n(q), \\ \\ \frac{d}{\gcd(q-\epsilon,n)}\sum_{s,\ d|s|(q-\epsilon,n)}\mu(s/d)|(\mathcal{FP}_{(-1)^{n/s}}(n/s))_{F_{q\epsilon}}| & \text{if } H = PS^{\epsilon}_n(q). \end{cases}$$

Now $\mathcal{C}^{\epsilon,reg}_{p',n}(q) = \sum_{d \mid \gcd(q-\epsilon,n)} \mathcal{C}^{\epsilon}_{p',n,d}$ and the second set of formulae follows from the identity $\sum_{x \mid m} x\mu(\frac{m}{x}) = \phi(m)$. Notice that $\mathcal{C}^{\epsilon,tor}_{p',n}(q) = \mathcal{C}^{\epsilon}_{p',n,1}(q)$. This gives the third part of the formulae in the theorem. The first part can be obtained easily, using the above version of Burnside's lemma. ◦

Remark 3.5 (i) *If* $\gcd(q-1,n) = 1$ *then* $GL_n(q) = PGL_n(q) \times Z_{q-1}$ *and* $SL_n(q) \cong PGL_n(q)$. *In this case our formula for the number of regular semisimple classes gives:*

$$\mathcal{C}^{\epsilon,reg}_{p',n}(q) = \frac{1}{q+1}(q^n - (-1)^n) = \frac{1}{q-1}\mathrm{Reg}_{ss}(G^1_n),$$

as it should be.

(ii) *In [13] Lehrer also derived formulae for the numbers of regular semisimple conjugacy classes of* $GL_n(q)$ *and* $PGL_n(q)$ *using a different method. In case of* $GL_n(q)$ *Lehrer's result coincides with the one of [8] which has been used here also.*

The formula in [13] for the number of regular semisimple classes of $PGL_n(q)$, *Theorem F, is given as*

$$\frac{1}{q+1}\sum_{d \mid n} d \sum_{e \mid n/d} \mu(e)(q^{n/ed} - (-1)^{n/ed}),$$

which can also be written as

$$\frac{1}{q-1}\sum_{d|n} \phi(d) f^1_{n/d}(q).$$

The difference to our formula in 3.4 is the range of summation. Notice that for $q = 2$ *the groups* $GL_n(q)$, $PGL_n(q)$ *and* $SL_n(q)$ *all coincide, as do our formulae in that case because gcd*$(q-1,n) = 1$. *Summation over all divisors of n, as in [13] Theorem F, in general gives a number of regular* $2'$ *- classes which is too big (e.g. two instead of one for* $PGL(2,2) \cong \Sigma_3$*).*

3.2 Abelian centralizers of p' - elements in $PS^{\epsilon}_n(q)$

Clearly, if $Z_{cp(t)} = 1$ for some regular semisimple $t \in T(w)$, then $(G/Z)_{tZ} \cong M_t/Z = T(w)/Z$. The next theorem shows that for all but a small number of groups $PSL_n(q)$ and $PSU_n(q)$, the converse is also true, i.e. almost all abelian centralizers of p' - elements of G/Z are finite tori. We need the following lemmas:

Lemma 3.6 *Let* R *be a principal ideal domain,* $\alpha, \beta \in R$ *with* $\gcd(\alpha,\beta) = 1$ *and* $m, n \in \mathbf{N}$ *with* $g := gcd(m,n)$. *Then for any two divisors* d, d' *of* $\beta^g - \alpha^g$ *with*

$d'' := \gcd(d, d')$ *we have*

$$\gcd(\frac{\beta^n - \alpha^n}{d}, \frac{\beta^m - \alpha^m}{d'}) = \frac{\beta^g - \alpha^g}{\mathrm{lcm}(d, d')}(\gcd(n/g, d/d'')\gcd(m/g, d'/d''))1_R.$$

Proof: The proof is straightforward. $\circ$

Lemma 3.7 *Let* $\epsilon \in \{\pm 1\}$, $d, m, q \in \mathbb{N}$ *with* $1 < q$ *and* d *a proper divisor of* m.

1) *If* $|\frac{(q\epsilon)^m - 1}{(q\epsilon)^d - 1}| \leq (q\epsilon - 1)^2$ *then*
$(\epsilon, q, m, d) \in \{(\pm 1, q, 2, 1), (-1, q, 3, 1), (-1, q, 4, 2), (-1, 2, 4, 1), (-1, 2, 6, 3)\}$.

2) $\frac{(q\epsilon)^m - 1}{(q\epsilon)^d - 1} \mid \gcd(q\epsilon - 1, m/d)\gcd(q\epsilon - 1, m) \iff$
$(\epsilon, q, m, d) \in \{(1, 3, 2, 1), (-1, q, 2, 1), (-1, 2, 3, 1) \mid q = 2, 3 \text{ or } 5\}$.

Proof: For $\epsilon = 1$, $\frac{q^m - 1}{q^d - 1} = 1 + q^d + q^{2d} + ... + q^{d(m/d-1)} \leq q^2 - 2q + 1$ implies $d = 1$ and $m = 2$. So suppose $\epsilon = -1$. Consider the case $m \equiv d \equiv 1 \bmod 2$: $q^{m-d} - q^{m-2d} + q^{m-3d} - q^{m-4d} + ... + 1 = (q^d - 1)[q^{m-2d} + q^{m-4d} + ... + q^d] + 1 \leq q^2 + 2q + 1$. Since $(q^d - 1)q^d > q(q+2)\ \forall d \geq 3$ we get $d = 1$; now $(q-1)(q^{m-3} + q^{m-5} + ... + 1) \leq q + 2$ implies $m = 3$.

Consider the case $m \equiv d \equiv 0 \bmod 2$: Then the left-hand side is $1 + q^d + ... + q^{d(m/d-1)}$ which is greater than $q^2 + 2q + 1$ for $m/d \geq 4$, hence $m/d \leq 3$. If $m/d = 3$ then $1 + q^d + q^{2d} \leq q^2 + 2q + 1$ implies the contradiction $d = 1$. Hence $m/d = 2$ and $q^d \leq q(q+2)$ implies $d = 2$.

Consider the case $m \equiv 0 \bmod 2$ and $d \equiv 1 \bmod 2$: If $m \equiv 2 \bmod 4$ then, using the previous considerations, $\frac{q^m - 1}{q^{m/2} + 1}\frac{q^{m/2} + 1}{q^d + 1} \leq (q+1)^2$ implies $m/2 = 3, d = 1$ or $d = m/2$. In the first case we get the contradiction $(q^3 - 1)(q^2 - q + 1) \leq q^2 + 2q + 1$. The second possibility leads to $m = 2, d = 1$ or to $q = 2, m = 6, d = 3$. If $m \equiv 0 \bmod 4$ then again by the previous result $\frac{q^m - 1}{q^{m/2} - 1}\frac{q^{m/2} - 1}{q^d + 1} \leq (q+1)^2$ implies $m = 4, d = 1$ and $q = 2$. Now the rest is clear. $\circ$

Corollary 3.8 *Let* $1 < q, m \in \mathbb{N}$, $\epsilon = \pm 1$ *and* $i := ord(\epsilon q) \bmod \frac{(q\epsilon)^m - 1}{q\epsilon - 1}$. *Then* $i = m$, *except* $(\epsilon, q, m, i) \in \{(-1, 2, 2, 1), (-1, 2, 3, 1), (-1, 3, 2, 1)\}$.

Proof: The exceptional cases are easily checked. In the general case i is obviously a divisor of m. Hence we get

$$|\frac{(q\epsilon)^m - 1}{q\epsilon - 1}| = |\gcd((q\epsilon)^i - 1, \frac{(q\epsilon)^m - 1}{q\epsilon - 1})| = |\frac{(q\epsilon)^i - 1}{q\epsilon - 1}| \cdot |(m/i, q\epsilon - 1)|.$$

Hence $|(q\epsilon)^m - 1| = |(q\epsilon)^i - 1| \cdot |(m/i, q\epsilon - 1)|$. From the previous lemma we get $(\epsilon, q, m, i) = (1, 3, 2, 1), (-1, 2, 3, 1)$ or $(-1, q, 2, 1)$ for $q = 2, 3$ or 5. Now the statement follows by inspection of these cases. $\circ$

Theorem 3.9 *Let $\epsilon = \pm 1$, $G = S_n^\epsilon(q)$, $Z := Z(G)$, $H = PS_n^\epsilon(q) := G/Z$. Let $t \in G$ be semisimple such that H_{tZ} is abelian. Then, up to conjugation in G, t is a regular element in some $T(w) = G_t$ and either $H_{tZ} = G_t/Z$ or $H_{tZ} = M_t/Z$, $M_t \cong T(w).\langle \overline{w} \rangle$ with $\overline{w} \in W_w$ and one of the following holds:*

1) $(\epsilon, q, n) = (1, 3, 2)$, $H \cong PSL_2(3)$ and $w = (12) = \overline{w}$; moreover $cp(t) = X^2 + 1$, $T(w) \cong \mathbb{Z}_4$, $|t| = 4$, $M_t = N_G(T(w)) \cong D_8$ and $H_{tZ} = C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

2) $(\epsilon, q, n) = (1, 5, 2)$, $H \cong PSL_2(5)$, $w = id$, and $\overline{w} = (12)$; moreover $cp(t) = X^2 + 1$, $T(id) \cong \mathbb{Z}_4$, $|t| = 4$, $M_t = N_G(T(id)) \cong D_8$, $H_{tZ} = C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

3) $(\epsilon, q, n) = (1, 4, 3)$ or $(-1, 2, 3)$, $H \cong PSL_3(4)$ or $PSU_3(2)$ respectively, $w = id$, $\overline{w} = (132)$; moreover $cp(t) = X^3 - 1$, $T(id) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ $|t| = 3$, $M_t \cong \mathcal{E}(3)^{1+2}$ extraspecial of exponent 3 and $H_{tZ} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

4) $(\epsilon, q, n) = (-1, 2, 6)$, $H \cong PSU_6(2)$, $w = (123)(456)$ and $\overline{w} = (132)(456)$; moreover $cp(t) = X^6 + X^3 + 1$, $T(w) \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ $|t| = 9$, and $H_{tZ} \cong \mathbb{Z}_9 \times \mathbb{Z}_3$.

Proof: Since H_{tZ} is abelian we can assume that $t \in T(w) = G_t$ for some $w \in W \cong \Sigma_n$. Furthermore, by 2.2 we can assume that $G_t < M_t \leq N_G(T(w))_{tZ}$. Since $T(w) = {}^g(\mathbf{T}_{wF})$ with $g \in \mathbf{G}$ and $g^{-1}F(g) \mapsto w$, we can identify $T(w)$ with the set of functions

$$T(w) = \{f : \underline{n} \to \bar{\mathbb{F}}_q^* \mid f(i) = f(w^{-1}(i))^{q\epsilon}, \prod_i f(i) = 1\}$$

and $N_G(T(w)_{tZ}/T(w)$ with $(\Sigma_n)_{w,tZ}$; in particular $W_t = id$. We can assume that w is a product of cycles of the form $(i\ i+1\ i+2\ ...\ i+k-1)$. For any $x \in \Sigma_n$ let *supp* x denote the subset of non - fixed points of x in $\underline{n}$. Since $G_t < M_t$ there is $1 \neq \bar{w} \in (\Sigma_n)_{w,tZ}$; then $\bar{w}$ permutes the w - orbits of the same length. Moreover *supp* $\bar{w} = \underline{n}$: indeed suppose $\bar{w}(i) = i$, then ${}^{\bar{w}}t = t(z, z, ..., z)$ implies ${}^{\bar{w}}t(i) = t(\bar{w}^{-1}(i)) = t(i) = t(i)z$, hence $z = 1$ and we get the contradiction $\bar{w} \in W_t = 1$. (We write $(z, z, ..., z)$ as a shorthand for $\mathrm{diag}(z, z, ..., z)$).

First suppose $w = id$: if $\epsilon = 1$ then the existence of a regular element $t \in T(w)$ implies $q > 3$. So there is $h := \mathrm{diag}(\omega, \omega^{-1}, 1, 1, ..., 1) \in T(id)$ with $\omega \neq \omega^{-1}$. If

$n \geq 4$ then ${}^{\bar{w}}h = h(z, ..., z)$ implies $z = 1$ and $\bar{w}_{|\{1,2\}} = id$, a contradiction. So $n \leq 3$. If $n = 3$ then $t' := \mathrm{diag}(\alpha, \alpha^{-1}, 1) \in T(id)$ with $|\alpha| = q - \epsilon$ is regular. Moreover ${}^{\bar{w}}t' = t'(z, z, z)$ implies $\alpha^3 = 1$, $\bar{w} = (132)$ and we get $(\epsilon, q, n) = (1, 4, 3)$ or $(-1, 2, 3)$. In both cases $T(id) = \langle t, {}^{\bar{w}} t\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, $M_t = T(id) : \langle \bar{w}\rangle$ is extraspecial of order 27 and $H_{tZ} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If $n = 2$ then $\epsilon = 1$ and $\bar{w} = (12)$. Notice that $t = (c, c^{-1})$ with $c \neq c^{-1}$ and $c \in \mathbb{F}_q^*$, so $q > 3$. For any $t' = (\alpha, \alpha^{-1})$ with $\alpha \in \mathbb{F}_q$ and $|\alpha| = q - 1$ we get ${}^{\bar{w}}t' = t'(z, z)$, $\alpha = z\alpha^{-1} = z^2\alpha$ and $\alpha^4 = 1$. Hence $q = 5$ which leads to the case of $PSL_2(5)$.

For each $1 < k < n$ let $\alpha_k \in \bar{\mathbb{F}}_q^*$ be an element of order $\frac{(q\epsilon)^k - 1}{q\epsilon - 1}$. If (ϵ, q, k) is different from $(-1, 2, 2), (-1, 2, 3)$ and $(-1, 3, 2)$ then by 3.8 the entries in the vector $\bar{\alpha}_k := (\alpha_k, \alpha_k^{q\epsilon}, ..., \alpha_k^{(q\epsilon)^{k-1}})$ are pairwise different and their product is 1.

Now we assume that $q > 3$ whenever $\epsilon = -1$. Let i be the minimal length of a w orbit. Suppose there is a w - orbit $\mathcal{O}$ of length $k > i$. Then we define $f : \underline{n} \to \bar{\mathbb{F}}_q^*$ to be identically 1 on $\underline{n}/\mathcal{O}$ and $\bar{\alpha}_k$ on $\mathcal{O}$. Now $f \in T(w)$ and ${}^{\bar{w}}f = f \cdot (z, z, ..., z)$ implies ${}^{\bar{w}}f = f$ and we conclude that $\bar{w}$ is the identity on $\mathcal{O}$, contradicting *supp* $\bar{w} = \underline{n}$. We conclude that w is homogeneous, i.e. $1 \neq w$ consists of cycles of length $i > 1$. The element $\bar{w}$ acts transitively on the set of w - orbits: indeed if the w - cycles $\mathcal{O}_1$ and $\mathcal{O}_2$ lie in different $\bar{w}$ - orbits, then we put $h_{\mathcal{O}_1} := \bar{\alpha}_i$ and 1 elsewhere and get ${}^{\bar{w}}h = h$, yielding the contradiction $\bar{w}_{\mathcal{O}_1} = id$. Moreover if $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ are w - cycles, then again ${}^{\bar{w}}h = h$ with h as above, yielding $\bar{w} = 1$, hence there are at most 2 cycles in w and ${}^{\bar{w}}h = {}^{\bar{w}}(\bar{\alpha}_i, 1, ..., 1) = (1, ..., 1, \bar{\alpha}_i^{(q\epsilon)^d}) = (\bar{\alpha}_i, 1, ..., 1)(z, z, ..., z)$ implies $\alpha_i = z^{-1} = z^{-q\epsilon}$ and $\alpha_i = \alpha_i^{(q\epsilon)^d} = z$, hence $z^2 = 1$ and $|\alpha_i| = |\frac{(q\epsilon)^i - 1}{q\epsilon - 1}| \mid 2$, which gives the contradiction $\epsilon = -1, q \leq 3$, $i = 2$ and $n = 4$.

Now we can assume that $w = (123...n)$ and $\bar{w} = w^d$ for some divisor d of n. For $h := \bar{\alpha}_n$ we get ${}^{\bar{w}}h = h(z, z, ..., z)$, which implies $|\alpha^{|(q\epsilon)^d - 1|}| = |z| \mid |q\epsilon - 1|$. Using 3.6 we get $|\alpha^{|(q\epsilon)^d - 1|}| = \frac{|(q\epsilon)^n - 1|}{|(q\epsilon)^d - 1| \, |\gcd(q\epsilon - 1, n/d)|} \mid |\gcd(q\epsilon - 1, n)|$. From 3.7 we conclude

$$(\epsilon, q, n, d) \in \{(1, 3, 2, 1), (-1, q, 2, 1), (-1, 2, 3, 1) \mid q = 2, 3 \text{ or } 5\}.$$

From our running assumptions $q > 3$ and $n > 2$ if $\epsilon = -1$; hence $(\epsilon, q, n, d) = (1, 3, 2, 1)$, giving the case of $PSL_2(3)$.

Now we can assume $\epsilon = -1$, $n > 2$, $q \leq 3$ and $w \neq 1$. We begin with the case $q = 2$: Regularity of t implies that $|\underline{n}/supp\ w| \leq 3$. Moreover, if w contains a cycle of length 2, then $T(w)$ cannot contain a regular element: indeed, $\alpha^4 = \alpha$ implies $\alpha^{-2} = \alpha$. Let again i be the minimal length of a w orbit. If $i \geq 3$ then the same argument as above shows, that w is homogeneous and if $i \geq 4$ there can only be two w - orbits, giving the contradiction $\epsilon = -1, q \leq 3$, $i = 2$ and $n = 4$. So we suppose first that $i = 3$. Then the condition $\gamma^{-2^3} = \gamma$ implies that a regular element t can only exist in $T(w)$ if $n = 6$ and $t = \mathrm{diag}(\alpha, \alpha^{-2}, \alpha^4, \alpha^2, \alpha^{-4}, \alpha^8)$ with $|\alpha| = 9$. We get $w = (123)(456)$ and

$$T(w) = \{\text{diag}(\gamma, \gamma^{-2}, \gamma^4, \beta, \beta^{-2}, \beta^4) \mid \gamma^3 = \beta^{-3}, |\gamma| \mid 9\} = \langle t, y\rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_3,$$

with $y = \text{diag}(1, 1, 1, \alpha^3\alpha^3, \alpha^3)$. Now $W_w = \langle (123), (456), (14)(25)(36)\langle$ of order $3^2 2$. Since ${}^{\bar{w}}t = t(z, z, z, z, z, z)$ implies $|\bar{w}| \mid 3^2$. Since ${}^{(123)}tt^{-1} \ni Z$ and ${}^{(132)(456)}tt^{-1} \in Z$ we see that $\bar{w} \in \langle (132)(456)\rangle$. We have $M_t = \langle t, y\rangle : \langle \bar{w}\rangle$ with ${}^{\bar{w}}t = t^{-1}y$; hence $Z = \langle t^3y^{-1}\rangle < Z(M_t) = \langle t^3, y\rangle$, so $H = PSU_6(2)$ and $H_{tZ} = M_t/Z \cong \mathbb{Z}_9 \times \mathbb{Z}_3$.

Now we suppose that the minimal length of w - orbits is $i = 1$. The existence of regular elements in $T(w)$ requires that w has at most 3 fixed points. If w has only one fixed point, then this is also a fixed point of $\bar{w}$, a contradiction. Suppose that w has exactly two fixed points. Then the regular element t is of the form $(1, \alpha, \tilde{t})$ with $1 \neq \alpha$ and $\det(\tilde{t}) = \alpha^{-1}$, or $t = (\alpha, \alpha^{-1}, \tilde{t})$ with $\det(\tilde{t}) = 1$. In the first case ${}^{\bar{w}}t = t(z, ..., z)$ implies $z = \alpha, \alpha z = 1$, hence the contradiction $\alpha = 1$. The second case yields $\alpha z = \alpha^{-1}$, $z = \alpha^2 = \alpha^{-2}$, hence the contradiction $\alpha = 1$. So w has exactly 3 fixed points, permuted by $\bar{w}$, $t = (1, \alpha, \alpha^{-1}, \tilde{t})$ with $\alpha^3 = 1$ and $det(\tilde{t}) = 1$. Now consider $t' = (1, \alpha, \alpha^{-1}, 1, \ldots, 1)$; then ${}^{\bar{w}}t' = t'(z, \ldots z)$ implies that $z = 1$ and $\bar{w}$ has the fixed points $1, 2, 3$ which is a contradiction.

Now we can assume $\epsilon = -1$, $n > 2$, $q = 3$ and $w \neq 1$. Suppose that the minimal length i of a w - orbit is ≥ 2. Again we conclude that w is homogeneous. If moreover $i \geq 3$ then we arrive at the contradiction $\epsilon = -1, q = 3$, $i = 2$ and $n = 4$. So we assume $i = 2$ and $w = (12)(34)...(n-1\ n)$ and the elements in $T(w)$ look like $\text{diag}(\alpha, \alpha^{-3}, \beta, \beta^{-3}, ...)$ with $\alpha^8 = \beta^8 = 1$. For $|\alpha| = 8$ we have $1\alpha\alpha^2...\alpha^7 = -1$, so we conclude $n \leq 6$. If $n = 6$ then $\{h_1 := (-1, -1, 1, 1, 1, 1), h_2 := (1, 1, -1, -1, 1, 1), h_3 := (1, 1, 1, 1, -1, -1)\} \subseteq T(w)$, and ${}^{\bar{w}}h_j = h_j(z, ..., z)$ implies $z = 1$, so we see that $\bar{w} = w$. But for $\tilde{t} = \text{diag}(\alpha, \alpha^{-3}, \alpha, \alpha^{-3}, \alpha^2, \alpha^{-6}) \in T(w)$ with $|\alpha| = 8$ we get ${}^{\bar{w}}\tilde{t}\tilde{t}^{-1} = (-1, -1, -1, -1, 1, 1) \notin Z(G)$. Hence $n = 4$, $w = (12)(34)$, $T(w) = \langle t, s\rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$ with $t = (\alpha, \alpha^{-3}, -\alpha, -\alpha^{-3})$. Since t is regular, $|\alpha| = 8$, but then we $\alpha + -\alpha^{-1}$, a contradiction.

Finally we assume that $\epsilon = -1$, $q = 3$, $n > 2$ and w has fixed points. Since $\bar{w}$ does not have fixed, w has at least two of them. Notice that $q + 1 = 4$, hence there is $\beta^3 = \beta^{-1}$ with $\beta \neq \beta^{-1}$. If $n \geq 4$, then we have for $h = (\beta, \beta^{-1}, 1, ..., 1) \in T(w)$, ${}^{\bar{w}}h = h$ and $\bar{w}$ has fixed points $1, 2$, a contradiction. So $n = 3$ but then we get the contradiction $1 \neq |Z| = \gcd(q - \epsilon, n) = \gcd(4, 3) = 1$. $\circ$

4 Groups of type C

In this section we consider groups and Lie algebras of type C_n, $n > 1$. Here $\mathbf{G} = Sp_{2n}$ is simply connected and has no torsion primes (see [14]), so condition $(*)$ holds for semisimple conjugacy classes as well as for semisimple orbits in $Lie(\mathbf{G})$. We start with the description of $\mathbf{G}$ as a classical group:

Let $K = \overline{\mathbb{F}}_q$, $V := K^{2n}$ with basis $\{e_i \mid i = 1, ..., 2n\}$, where $e_i := (0, ...0, 1, 0..., 0)^{tr}$ is the i - th standard basis column vector. Let $\check{E} := \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \in GL_{2n}(K)$ with the unit matrix $E \in GL_n(K)$. We consider the bilinear form $\langle v_1, v_2 \rangle := v_1^{tr} \circ \check{E} \circ v_2$. Then

$$\langle e_i, e_j \rangle = \begin{cases} -1 & n < i = n + j \leq 2n, \\ 1 & 1 \leq i = j - n \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle e_i, e_j \rangle = -\langle e_j, e_i \rangle$ and $\langle e_i, e_i \rangle = 0$, hence $\langle , \rangle$ is a nondegenerate symplectic form on V and the group

$$\mathbf{G} := Sp_{2n} := \{g \in GL_{2n}(K) \mid \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle \ \forall \ v_i \in V\}$$

is called the **symplectic group** on V with its Lie algebra $Lie(\mathbf{G}) = Lie(Sp_{2n}) :=$

$$\{\mathcal{X} \in Mat_{2n}(K) \mid \langle \mathcal{X}v_1, v_2 \rangle + \langle v_1, \mathcal{X}v_2 \rangle = 0 \ \forall \ v_i \in V\}.$$

The subgroup

$$\mathbf{T} := \{t(d) := \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \in GL_{2n}(K) \mid d := diag(d_1, ..., d_n), \ d_i \in K^*\}$$

is a maximal torus with Lie algebra

$$Lie(\mathbf{T}) = \{\mathcal{X} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in Mat_{2n}(K) \mid a = diag(a_1, \ldots, a_n)\}.$$

Let $X := Hom(\mathbf{T}, K^*)$ written additively and $\epsilon_i \in X$ be defined by $\epsilon_i(t(d)) := d_i$. Then the root system $\Phi(\mathbf{G}, \mathbf{T})$ is given as

$$\Phi_{C_n} := \{\pm\epsilon_i \pm \epsilon_j, \ \pm 2\epsilon_i \mid i \neq j = 1, 2, ..., n\} \leq X.$$

For $\mathcal{G} = \mathbf{G}$ or $Lie(\mathbf{G})$ consider the standard Frobenius endomorphisms

$$F : \mathcal{G} \to \mathcal{G}, (a_{ij}) \mapsto (a_{ij}^q),$$

then $G = G_F = Sp_{2n}(q)$ and $Lie(G) = Lie(\mathbf{G})_F$. Notice that the adjoint action of $\mathbf{G}$ on $Lie(\mathbf{G})$ satisfies

$$ad(F(g))(F(\mathcal{X})) = F(ad(g)(\mathcal{X})),$$

for $g \in \mathbf{G}$ and $\mathcal{X} \in Lie(\mathbf{G})$ and induces an adjoint action of G on $Lie(G)$.

4.1 Regular genus numbers of $Sp_{2n}(q)$

The G - conjugacy classes of F - stable maximal tori of $\mathbf{G}$ coincide with the regular semisimple genera and are parametrized by the set of conjugacy classes of $W \cong \mathbb{Z}_2^n : \Sigma_n$ (F acts trivially on W). This latter set itself is parametrized by 'double partitions' $(n_i^+, n_i^-)_{i=1,\ldots,n}$ of the number n: Let $w \in W$ be a representative of the class w^W then w contains n_i^+ positive and n_i^- negative pairwise disjoint i - cycles with $\sum_{i=1}^n i \cdot (n_i^+ + n_i^-) = n$. Here a negative i - cycle $(j_1, j_2, ..., j_i)^-$ maps $j_1 \mapsto j_2 \mapsto j_3 ... \mapsto j_i \mapsto -j_1 \mapsto -j_2 ...$ and has order $2i$. If $t = (t_1, \ldots, t_n) := diag(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \in \mathbf{T}$ or $t = diag(t_1, \ldots, t_n, -t_1, \ldots, -t_n) \in Lie(\mathbf{T})$ then $t^{(1,2,\ldots,i)^\pm} = (t_i^{\pm 1}, t_1, t_2, \ldots, t_{i-1}, t_{i+1} \ldots)$ or $(\pm t_i, t_1, t_2, \ldots, t_{i-1}, t_{i+1} \ldots)$ respectively. If w^W is parametrized by $(n_i^+, n_i^-)_{i=1,\ldots,n}$, we can choose w such that $t \in \mathbf{T}_{wF}$ if and only if

$$t = (\ldots, t_j, t_j^q, \ldots, t_j^{q^{i-1}}, \ldots, s_k, s_k^q, \ldots, s_k^{q^{i-1}}, \ldots),$$

with $j = 1, \ldots, n_i^+, k = 1, \ldots, n_i^-, i = 1, \ldots, n$, $t_j^{q^i} = t_j$ and $s_k^{q^i} = s_k^{-1}$. Hence

$$T(w) \cong \mathbf{T}_{w \cdot F} \cong \prod_{i=1}^n (\mathbb{Z}_{q^i-1})^{n_i^+} \times (\mathbb{Z}_{q^i+1})^{n_i^-}.$$

Definition 4.1 *Let* $\mathcal{D}\Lambda(n) := \{(n_i^+, n_i^-)_{i=1,\ldots,n} \mid \sum_i (n_i^+ + n_i^-) i = n\}$ *be the set of double partitions of the integer* n *and* $\phi : W/W \to \mathcal{D}\Lambda(n)$ *the parametrization described above. For each* $\lambda = (n_i^+, n_i^-)_i \in \mathcal{D}\Lambda(n)$ *let* $\mathcal{C}_\lambda$ *denote the number of regular semisimple classes or orbits with genus* λ*, i.e. the corresponding genus number.*

Now we define certain sets of polynomials which will be needed later:

Definition 4.2 *Let* K *be an arbitrary field,* $f = f_0 + f_1 X + \ldots + f_{\ell-1} X^{\ell-1} + X^\ell \in K[X]$ *a monic polynomial of degree* ℓ *and* $f = \prod_{i=1}^\ell (X - \alpha_i)$ *be the decomposition of* f *in* $\overline{K}[X]$*. Suppose that* $f(0) \neq 0$*; then the monic polynomial* $f^* := \prod_{i=1}^\ell (X - \alpha_i^{-1}) = f_0^{-1} X^\ell f(\frac{1}{X})$ *is called the* **reciprocal** *of* f*. The monic polynomial* f^- *is defined as* $f^- := \prod_{i=1}^\ell (X + \alpha_i) = (-1)^\ell f(-X)$*. If* $f^* = f$ *then* f *is called* **selfreciprocal***, if* $f^- = f$ *it is called* **symmetric***. Notice that if* f *is selfreciprocal, then* $f_0 = (-1)^\ell \prod_{i=1}^\ell \alpha_i = (-1)^{\ell+c}$*, where* c *denotes the multiplicitiy of* -1 *as root of* f*. We define the following:*
$Irr_{(*)}(\ell, q) := \{f \in \mathcal{P}_{(*)}(\ell, \mathbb{F}_q) \mid f \text{ irreducible}\}$.
$\mathcal{S}r(\ell, K) := \{f \in \mathcal{P}_*(\ell, K) \mid f \ selfreciprocal\}$.
$\mathcal{S}r_1(\ell, K) := \{f \in \mathcal{S}r(\ell, K) \mid f_0 = 1\}$.
$\mathcal{S}r_\pm(\ell, K) := \{f \in \mathcal{S}r(\ell, K) \mid X \pm 1 \not| f\}$.
$\mathcal{S}ym(\ell, K) := \{f \in \mathcal{P}(\ell, K) \mid f \ symmetric\}$.
$\mathcal{S}ym_*(\ell, K) := \{f \in \mathcal{S}ym(\ell, K) \mid f_0 \neq 0\}$.

$\mathcal{FS}r_{(\pm)}(\ell,K) := \{f \in \mathcal{S}r_{(\pm)}(\ell,K) \mid f \text{ } squarefree\}$.
$\mathcal{FS}ym_{(*)}(\ell,K) := \{f \in \mathcal{S}ym_{(*)}(\ell,K) \mid f \text{ } squarefree\}$.

Let $sr_{\ell,(1),(\pm)} := sr_{\ell,(1),(\pm)}(q) := |\mathcal{S}r_{(1),(\pm)}(\ell,\mathbb{F}_q)|$, $fsr_{\ell,(\pm)} := fsr_{\ell,(\pm)}(q) := |\mathcal{FS}r_{(\pm)}(\ell,\mathbb{F}_q)|$ *and* $fsy_{\ell,(*)} := fsy_{\ell,(*)}(q) := |\mathcal{FS}ym_{(*)}(\ell,\mathbb{F}_q)|$.

Lemma 4.3 *Let K be a field of odd characteristic. Then we have the following bijection:*

$$\theta : \{f \in K[X] \mid f \text{ monic } f(-1) \neq 0 \neq f(0)\} \to \{f \in K[X] \mid f \text{ monic } f(\pm 1) \neq 0\},$$

$$f \mapsto (-1)^{deg(f)} f(-1)^{-1} f\left(\frac{X+1}{1-X}\right)(1-X)^{deg(f)},$$

where the inverse is given by

$$\chi : g \mapsto g(1)^{-1} g\left(\frac{X-1}{X+1}\right)(X+1)^{deg(g)}.$$

Moreover

$$f^{*\theta} = (-1)^{deg(f)} f^{\theta-}.$$

In particular there is a bijection

$$\Psi : \mathcal{FS}r(2n,K) \to \mathcal{FS}ym_*(2n,K),\ f \mapsto \begin{cases} f^\theta & \text{if } X+1 \not| f, \\ (x^2-1)(\frac{f}{x^2-1})^\theta & \text{otherwise.} \end{cases}$$

Proof: Consider the bijection

$$\sigma : \overline{K}/\{-1,0\} \to \overline{K}/\{-1,1\},\ \alpha \mapsto \frac{\alpha-1}{\alpha+1},$$

with inverse

$$\tau : \beta \mapsto \frac{\beta+1}{1-\beta}.$$

Then for $f = \prod_{i=1}^n (X-\alpha_i) \in \overline{K}[X]$ we have $f^\theta = \prod_{i=1}^n (X - \sigma(\alpha_i))$ if $\alpha_i \notin \{-1,0\}$. Similarly we see that χ is induced by τ. If $f \in \mathcal{FS}r(2n,K)$, then $X+1$ (or $X-1$) divides f if and only if $f = (X^2-1)g$ with $g \in \mathcal{FS}r_\pm(2n-2,K)$; thus Ψ is a well defined bijection. $\circ$

For each $t = diag(t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1}) \in \mathbf{T}$ or $\delta = diag(d_1,\ldots,d_n,-d_1,\ldots,-d_n)$ in $Lie(\mathbf{T})$ let $cp(t) := char.pol(t) = \prod_{i=1}^n (X-t_i)(X-t_i^{-1})$ and $cp(d) := char.pol(d) = \prod_{i=1}^n (X-d_i)(X+d_i)$. Then $f := cp(t)$ is selfreciprocal with $f_0 = 1$ and $g := cp(d)$ is symmetric (i.e. $g(X) = g(-X)$). Of course two $\mathbf{G}$ - conjugate elements t,t' or d,d' have the same characteristic polynomial. On the other hand it is easy to see that each $f \in \mathcal{S}r_1(2n,\mathbb{F}_q)$ or $\mathcal{S}ym(2n,\mathbb{F}_q)$ gives rise to a single W - orbit in $\mathbf{T}$ or

$Lie(\mathbf{T})$ respectively and hence represents a unique semisimple $\mathbf{G}$ orbit. Moreover an element $t \in \mathbf{T}$ is regular if and only if $t_i t_j^{\pm 1} \neq 1 \neq t_i^2$ for all $i \neq j \in \{1, \ldots, n\}$. Similarly an element $\delta \in Lie(\mathbf{T})$ is regular if and only if $d_i \pm d_j \neq 0 \neq 2d_i$ for all $i \neq j \in \{1, \ldots, n\}$. Now the general theory yields the following bijections

$$G_{ss}/G \cong (\mathcal{S}r_1(2n, \overline{\mathbb{F}}_q))_F = \mathcal{S}r_1(2n, \mathbb{F}_q), \ G_{ss,reg}/G \cong \mathcal{FS}r_{\pm}(2n, \mathbb{F}_q),$$

$$Lie(G)_{ss}/G \cong \mathcal{S}ym(2n, \mathbb{F}_q), \ Lie(G)_{ss,reg}/G \cong \mathcal{FS}ym_*(2n, \mathbb{F}_q),$$

with q odd in the Lie algebra case. In particular we get

$$|G_{ss,reg}/G| = fsr_{2n,\pm}(q), \ |Lie(G)_{ss,reg}/G| = \begin{cases} fsy_{2n,*}(q) & \text{if } q \text{ is odd,} \\ 0 & \text{if } q \text{ is even.} \end{cases}$$

Let $()^q \in Aut(\overline{\mathbb{F}}_q)$ be the Frobenius endomorphism and assume that q is odd in the Lie algebra case. Then t^G represents a regular semisimple class or orbit of genus $(n_i^+, n_i^-)_{i=1,\ldots,n}$ if and only if $cp(t) \in \mathcal{S}r_1(2n, \mathbb{F}_q)$ (respectively $\mathcal{S}ym_*(2n, \mathbb{F}_q)$) is multiplicity free. This means that its roots build exactly n_i^+ pairs $(\mathcal{O}, \mathcal{O}^{-1})$ of distinct $()^q$ - orbits of length i, with $\alpha \in \mathcal{O} \iff \alpha^{-1} \in \mathcal{O}^{-1}$ $(\alpha \notin \{0, \pm 1\})$ in the group case and $\alpha \in \mathcal{O} \iff -\alpha \in \mathcal{O}^{-1}$ $(\alpha \neq 0)$ in the Lie algebra case, and n_i^- $()^q$ - orbits $\mathcal{O}$ of length $2i$, with $\alpha \in \mathcal{O} \iff \alpha^{-1} \in \mathcal{O}$ in the group case and $\alpha \in \mathcal{O} \iff -\alpha \in \mathcal{O}$ in the Lie algebra case. In the group case this condition is equivalent to the fact that for each i, the factorization of $cp(t) \in \mathbb{F}_q[X]$ is given as

$$cp(t) = \prod_{i=1}^{n_i^+} (g_{j_i} g_{j_i}^*) \cdot \prod_{i=1}^{n_i^-} h_{k_i},$$

with $g_{j_i} \in Irr_*(i,q)/\mathcal{S}r(i,q)$ and $h_{k_i} \in Irr_*(2i,q) \cap \mathcal{S}r(2i,q)$. In the case of $Lie(G)$ and q odd it is equivalent to

$$cp(t) = \prod_{i=1}^{n_i^+} (g_{j_i} g_{j_i}^-) \cdot \prod_{i=1}^{n_i^-} h_{k_i},$$

with $g_{j_i} \in Irr_*(i,q)/\mathcal{S}ym(i,q)$ and $h_{k_i} \in Irr_*(2i,q) \cap \mathcal{S}ym(2i,q)$. Thus for each $\lambda = (n_i^+, n_i^-)_i \in \mathcal{D}\Lambda(n)$ we get

$$\mathcal{C}_\lambda = \prod_{i=1}^{n} \binom{\frac{1}{2}X^+(i,q)}{n_i^+} \cdot \binom{X^-(2i,q)}{n_i^-},$$

where $X^+(i,q) = |Irr_*(i,q)/\mathcal{S}r(i,q)|$ and $X^-(2i,q) = |Irr_*(2i,q) \cap \mathcal{S}r(2i,q)|$ in the group case and where $X^+(i,q) = |Irr_*(i,q)/\mathcal{S}ym(i,q)|$ and $X^-(2i,q) = |Irr_*(2i,q) \cap \mathcal{S}ym(2i,q)|$ in the Lie algebra case.

The cardinalities are given as follows:

Lemma+Definition 4.4 *For $\ell \in \mathbb{N}$ and q we define*

$$\chi(\ell,q) := |Irr_*(\ell,q)| = \frac{1}{\ell}\sum_{d|\ell}\mu(d)(q^{\ell/d}-1).$$

$\chi^+(\ell,q) = |Irr_*(\ell,q)/\mathcal{S}r(\ell,q)|$, $\tilde{\chi}^+(\ell,q) := |Irr_*(\ell,q)/\mathcal{S}ym(\ell,q)|$.
$\chi^-(\ell,q) := |Irr_*(\ell,q)\cap\mathcal{S}r(\ell,q)|$, $\tilde{\chi}^-(\ell,q) := |Irr_*(\ell,q)\cap\mathcal{S}ym(\ell,q)|$.

Then we have

$$\chi^+(\ell,q) = \begin{cases} \frac{1}{\ell}(q^{\frac{\ell}{2}}-1)^2 & \textit{if } 1<\ell=2^m \textit{ and } q \textit{ is odd,}\\ \chi(\ell,q) & \textit{if } 1<\ell \textit{ is odd,}\\ q-3 & \textit{if } \ell=1 \textit{ and } q \textit{ is odd,}\\ q-2 & \textit{if } \ell=1 \textit{ and } q \textit{ is even,}\\ \chi^-(\ell,q^2)-2\chi^-(\ell,q) & \textit{otherwise.}\end{cases}$$

$$\chi^-(\ell,q) = \begin{cases} \frac{1}{\ell}(q^{\frac{\ell}{2}}-1) & \textit{if } 1<\ell=2^m \textit{ and } q \textit{ is odd,}\\ 0 & \textit{if } 1<\ell \textit{ is odd,}\\ 2 & \textit{if } \ell=1 \textit{ and } q \textit{ is odd,}\\ 1 & \textit{if } \ell=1 \textit{ and } q \textit{ is even,}\\ \frac{1}{\ell}\sum_{d|\frac{\ell}{2},d \text{ odd}}\mu(d)q^{\frac{\ell}{2d}} & \textit{otherwise.}\end{cases}$$

Let q be odd, then $\tilde{\chi}^+(\ell,q) = \begin{cases} \chi^+(\ell,q) & \textit{if } \ell>1,\\ q-1 & \textit{if } \ell=1.\end{cases}$

$\tilde{\chi}^-(\ell,q) = \begin{cases} \chi^-(\ell,q) & \textit{if } \ell>1,\\ 0 & \textit{otherwise.}\end{cases}$

Proof: The result for $\chi(\ell,q)$, as well as for the cardinality $\chi^-(\ell,q) = |\mathcal{S}r(\ell,q)\cap Irr_*(\ell,q)|$ is easy to derive via Moebius inversion (e.g. see [10] 2.7.4. pg. 77). Now the rest can be verified directly. In particular for odd q, 4.3 yields $|Irr_*(2n,q)\cap\mathcal{S}ym(2n,q)| = \chi^-(2n,q)$ and $|Irr_*(2i+1,q)\cap\mathcal{S}ym(2i+1,q)| = 0 = \chi^-(2i+1,q)$ for $i>0$. $\circ$

4.2 Cardinalities of squarefree selfreciprocal polynomials

Now we are going to determine the total numbers of regular semisimple classes and orbits. Notice that, by the general theory of semisimple conjugacy classes, we must have

$$|G_{ss}/G| = q^r,$$

where r is the semisimple rank of $\mathbf{G}$, i.e. the dimension of a maximal torus in $\mathbf{G}$ (see [2]). This is confirmed in (i) of the following theorem:

Theorem 4.5 *Suppose that q is a prime power. Then we have*

(i) $sr_\ell(q) = \begin{cases} 1 & \text{if } \ell = 0, \\ (q+1)q^{n-1} & \text{if } \ell = 2n > 0 \text{ and } q \text{ is odd,} \\ 2q^n & \text{if } \ell = 2n+1 \text{ and } q \text{ is odd,} \\ q^n & \text{if } 0 < \ell \in \{2n, 2n+1\} \text{ and } q \text{ is even.} \end{cases}$

$sr_{\ell,1}(q) = q^n$, *where* $\ell \in \{2n, 2n+1\}$.

(ii) $sr_{\ell,\pm}(q) = \begin{cases} 1 & \text{if } \ell = 0, \\ q-2 & \text{if } \ell = 2 \text{ and } q \text{ is odd,} \\ (q-1)^2 q^{n-2} & \text{if } \ell = 2n > 2 \text{ and } q \text{ is odd,} \\ (q-1)q^{n-1} & \text{if } \ell = 2n > 0 \text{ and } q \text{ is even,} \\ 0 & \text{if } \ell = 2n+1. \end{cases}$

(iii) $fsr_\ell(q) = \begin{cases} 1 & \text{if } \ell = 0, \\ (\frac{q-1}{q+1})(q^n + (-1)^{n-1}) & \text{if } \ell = 2n \text{ and } q \text{ is odd,} \\ 2fsr_{2n,\pm}(q) & \text{if } \ell = 2n+1 \text{ and } q \text{ is odd,} \\ fsr_{2n,\pm}(q) & \text{if } \ell \in \{2n, 2n+1\} \text{ and } q \text{ is even.} \end{cases}$

If q is odd, then $fsy_{2n,*}(q) = fsr_{2n}(q)$.

(vi) $fsr_{\ell,\pm}(q) = \begin{cases} (-1)^n(n+1) + \sum_{i=0}^{n-1}(-1)^i(2i+1)q^{n-i} & \text{if } \ell = 2n \text{ and } q \text{ is odd,} \\ (\frac{q-1}{q+1})(q^n + (-1)^{n-1}) & \text{if } \ell = 2n \text{ and } q \text{ is even,} \\ 0 & \text{if } \ell \text{ is odd.} \end{cases}$

Remark 4.6 *Notice that* $(\frac{q-1}{q+1})(q^n + (-1)^{n-1})$ *can also be expressed in the form*

$$q^n - 2q^{n-1} + 2q^{n-2} - \ldots + (-1)^{n-1}2q + (-1)^n.$$

Proof: First assume that q is odd.

(i): Notice that $f = f_0 + f_1X + \ldots + f_{\ell-1}X^{\ell-1} + X^\ell \in \mathcal{S}r(\ell, q)$ if and only if $f_0^{-1}X^\ell f(\frac{1}{X}) = f$ or, in other words, $f_{\ell-i} = f_i/f_0$ for $0 \le i \le \ell$. In particular we see that $f_0 = \pm 1$. If $\ell = 2n$ and $f_0 = -1$ then $f_n = 0$ and we have q^{n-1} possible choices for f. If $\ell = 2n$ and $f_0 = 1$ we have q^n choices and if $\ell = 2n+1$ we have $2q^n$ possibilities (q^n for $f_0 = 1$ and $f_0 = -1$).

(ii): Each element f of $\mathcal{S}r(\ell, q)$ is a product of $(X-1)^i(X+1)^j$ and an element of $\mathcal{S}r_\pm(\ell - k, q)$ with $k = i + j$. In particular we get

$$sr_\pm(\ell, q) = sr(\ell, q) - \sum_{k \ge 1} sr_\pm(\ell - k, q)(k+1).$$

From this recursion formula the result can easily be seen.

(iii): Each element $f \in \mathcal{S}r_{(\pm)}(\ell, q)$ can be decomposed uniquely into a product of a square u^2 and a squarefree polynomial g. The roots of g in $\overline{\mathbb{F}}_q$ are exactly those that appear in f with an odd multiplicity, so α is a root of g if and only α^{-1} is also a root of g (both with multiplicity one). Hence $g \in \mathcal{FS}r_{(\pm)}(\ell - 2i, q)$ and $u \in \mathcal{S}r_{(\pm)}(i, q)$ and we get the following identity:

$$sr_{\ell,(\pm)} = \sum_{i=0}^{[\frac{\ell}{2}]} sr_{i,(\pm)} \cdot fsr_{\ell-2i,(\pm)}. \tag{$*$}$$

Since the values $\mathcal{S}r_{(\pm)}(i, q)$ are known, the statements of (iii) and (iv) can be proved by inserting the hypothesis for the $fsr_{\ell-2i,(\pm)}$ and verifying the identity (*).

First we consider the case $\ell = 2n$. After multiplication with $q + 1$ we have to verify

$$(q+1)^2 q^{n-1} = (q-1) \sum_{0 \le i < n} sr_i \cdot (q^{n-i} + (-1)^{n-i+1}) + sr_n(q+1).$$

If n is even then (*) is equivalent to

$$(q+1)^2 q^{n/2-1} \left(\frac{q^{n/2}-1}{q-1}\right) = \sum_{0 \le i < n} sr_i \cdot (q^{n-i} + (-1)^{n-i+1}).$$

The right hand side can be written as

$$q^n - 1 + (q+1)[q^{n-2} - 1 + q(q^{n-4} - 1) + q^2(q^{n-6} - 1) + \ldots + q^{n/2-2}(q^2 - 1)] +$$
$$[2q^{n-1} + 2 + 2q^{n-2} + 2q + 2q^{n-3} + 2q^2 + \ldots + 2q^{n/2} + 2q^{n/2-1}] =$$
$$q^n - 1 + (q+1)\left[\frac{1}{q-1}(-q^{n/2-1} + 1 + q^{n/2}(q^{n/2-1} - 1))\right] + 2\left(\frac{q^n - 1}{q-1}\right).$$

Hence (*) is equivalent to the equation

$$(q^n-1)(q-1) + (q+1)(q^{n-1} - q^{n/2} - q^{n/2-1} + 1) + 2q^n - 2 = (q+1)^2 q^{n/2-1}(q^{n/2} - 1),$$

which can be verified by direct calculation.

Now suppose that n is odd. Then (*) is equivalent to the equation

$$\left(\frac{q+1}{q-1}\right) q^{\frac{n-1}{2}} (q^{\frac{n+1}{2}} + q^{\frac{n-1}{2}} - 2) = \sum_{0 \le i < n} sr_i \cdot (q^{n-i} + (-1)^{n-i+1}).$$

Here the right hand side can be written as

$$q^n + 1 + (q+1)[q^{n-2} + 1 + q(q^{n-4} + 1) + q^2(q^{n-6} + 1) + \ldots + q^{(n-1)/2-1}(q+1)] +$$

$$[2q^{n-1}-2+2q^{n-2}-2q+2q^{n-3}-2q^2+\ldots+2q^{(n+1)/2}-2q^{(n-1)/2-1}]=$$

$$q^n+1+(q+1)[\frac{q^{n-1}-1}{q-1}]+\frac{2}{q-1}[-(q^{(n-1)/2}-1)+q^{(n+1)/2}(q^{(n-1)/2}-1)].$$

So (*) is equivalent to

$$(q+1)q^{(n-1)/2}(q^{(n+1)/2}+q^{(n-1)/2}-2)=$$

$$(q-1)(q^n+1)+(q+1)(q^{n-1}-1)+2[q^n-q^{(n+1)/2}-q^{(n-1)/2}+1],$$

which is obviously true. Now let $\ell=2n+1$; then $\mathcal{FS}r_{\pm}(\ell,q)=\emptyset$, since ± 1 are excluded as roots. Hence each element of $\mathcal{FS}r(\ell,q)$ is divisible with multiplicity one by either $X-1$ or $X+1$ and we get $fsr_{2n+1}=2fsr_{2n,\pm}$.

(iv) We have only to prove the statement about $fsr_{2n,\pm}$. By the arguments above it is clear that we have $fsr_{2n,\pm}=fsr_{2n}-fsr_{2n-2,\pm}$ which, by induction, is equal to $q^n-2q^{n-1}+2q^{n-2}-\ldots+(-1)^{n-1}2q+(-1)^n$ $-(q^{n-1}-3q^{n-2}+5q^{n-3}-\ldots+(-1)^{n-2}(2n-3)q+(-1)^{n-1}n)=$

$$q^n-3q^{n-1}+5q^{n-2}-7q^{n-3}+\ldots+(-1)^{n-1}(2n-1)q+(-1)^n(n+1).$$

Now we assume that q is even. In this case each element of $\mathcal{S}r^(\ell,q)$ is of the form $(X-1)^k g$ with $g\in\mathcal{S}r_{\pm}(\ell-k,q)$ and we get the recursion

$$sr_{\ell,\pm}=sr_\ell-\sum_{k=0}^{\ell-1}sr_{k,\pm}.$$

Again for obvious reasons $sr_{\ell,\pm}=0=fsr_{\ell,\pm}$ is zero whenever ℓ is odd. Hence we have $fsr_{2n+1}=fsr_{2n,\pm}=fsr_{2n}$. The proof of the statement about fsr_{2n} is similar to the case where q is odd and is therefore not repeated here. The fact that $fsy_{2n,*}(q)=fsr_{2n}(q)$ if q is odd follows immediately from 4.3. ∘

Now we can summarize the results on regular semisimple genus numbers as follows:

Theorem 4.7 *Let $G=Sp_{2n}(q)$ with prime power $q=p^s$ and Weyl group $W\cong\mathbb{Z}_n\wr\Sigma_n$. Then for each conjugacy class of W paramtrized by $\lambda=(n_i^+,n_i^-)_{i=1,\ldots,n}\in\mathcal{D}\Lambda(n)$ the corresponding genus number is given by*

$$\mathcal{C}_\lambda=\prod_{i=1}^n\binom{\frac{1}{2}X^+(i,q)}{n_i^+}\cdot\binom{X^-(2i,q)}{n_i^-},$$

where $X^{\pm}(\ell,q)=\begin{cases}\chi^{\pm}(\ell,q) & \text{in the case of } G \text{ - classes,}\\ \tilde{\chi}^{\pm}(\ell,q) & \text{in the case of } G \text{ - orbits on } Lie(G) \text{ and } q \text{ odd,}\\ 0 & \text{in the case of } G \text{ - orbits on } Lie(G) \text{ and } q \text{ even.}\end{cases}$

EXAMPLE: $G = Sp_4(q)$

$\lambda \in \mathcal{D}\Lambda(2)$	$C_\lambda(G), q$ odd	$C_\lambda(G), q$ even	$C_\lambda(Lie(G)), q$ odd
$(1^2, -)$	$\frac{1}{8}(q^2-8q+15)$	$\frac{1}{8}(q^2-6q+8)$	$\frac{1}{8}(q^2-4q+3)$
$(2, -)$	$\frac{1}{4}(q^2-2q+1)$	$\frac{1}{4}(q^2-2q)$	$\frac{1}{4}(q^2-2q+1)$
$(1, 1)$	$\frac{1}{4}(q^2-4q+3)$	$\frac{1}{4}(q^2-2q)$	$\frac{1}{4}(q^2-2q+1)$
$(-, 1^2)$	$\frac{1}{8}(q^2-4q+3)$	$\frac{1}{8}(q^2-2q)$	$\frac{1}{8}(q^2-4q+3)$
$(-, 2)$	$\frac{1}{4}(q^2-1)$	$\frac{1}{4}q^2$	$\frac{1}{4}(q^2-1)$
total	q^2-3q+3	q^2-2q+1	q^2-2q+1

Moreover the total numbers $\mathcal{C}_{ss,reg}$ of regular semisimple classes in G or orbits in $Lie(G)$ satisfy the following polynomial identities:

$|\mathcal{C}_{ss,reg}| = \sum_{(n_i^+, n_i^-) \in \mathcal{D}\Lambda(n)} \mathcal{C}_\lambda =$

$$\begin{cases} (-1)^n(n+1) + \sum_{i=0}^{n-1}(-1)^i(2i+1)q^{n-i} & \text{in the case of } G \text{ - classes and } q \text{ odd,} \\ (\frac{q-1}{q+1})(q^n + (-1)^{n-1}) & \text{in the cases of } G \text{ - classes and } q \text{ even,} \\ & \text{and } G \text{ - orbits on } Lie(G),\ q \text{ odd,} \\ 0 & \text{for } G \text{ - orbits on } Lie(G) \text{ and } q \text{ even.} \end{cases}$$

Remark 4.8 (i)*: The formula in 4.7 for the Lie - algebra case has also been obtained by Lehrer with different methods ([12] (4.11) ff.). In section* 5 *of this paper a map ω from the set of semisimples conjugacy classes of $\mathbf{G}_F$ to the set of conjugacy classes of W is defined. If F is split, then the preimages $\omega^{-1}([w])$ consist of those classes whose algebraic centralizers have a maximal split torus which, when viewed as an F - stable maximal torus of $\mathbf{G}$, is parametrized by $[w]$. Hence the formulae in [12] for the cardinalities of these preimages are sums of genus numbers in our sense.*

(ii)*: Notice that the polynomial giving $\mathcal{C}_{ss,reg}$ in the group case for even q coincides with that giving $\mathcal{C}_{ss,reg}$ in the Lie algebra case and odd q. Although, as the above example shows, in both cases the individual genus numbers are different.*

4.3 The simple groups $PSp_{2n}(q)$

Let $G = Sp_{2n}(q)$ and $Z := Z(G)$. In this section we use the previous results to determine the number of abelian p' - classes for the central quotient group $H := G/Z(G)$. Notice that these groups are simple whenever $(n, q) \notin \{(1,2), (1,3), (2,2)\}$. Moreover, $|Z| = \gcd(q-1, 2)$ so $H \cong G$, if q is even. Hence in this section we will assume that q is odd.

The action of $\mathbb{F}_q^*$ on the set $\mathcal{P}_*(n,q)$ given by ${}^c f(X) := c^n f(c^{-1}X)$ (see 3) induces an action of the subgroup $\mathbb{Z}_2 \cong \langle -1 \rangle \leq \mathbb{F}_q^*$ on the sets $\mathcal{S}r(n,q)$, $\mathcal{S}r_\pm(n,q)$ and $\mathcal{FS}r_{(\pm)}(n,q)$. Since $fsr_{\ell,\pm} = 0$ if ℓ is odd, we can assume that ℓ is even in the following lemma:

Lemma 4.9 *Let q be an odd prime power, $\ell \in 2\,\mathbb{N}$, $\mathcal{FS}r_{\pm,-1}(\ell,q) := \mathcal{FS}r_\pm(\ell,q)_{\langle -1\rangle}$ be the set of -1 - fixed points and $fsr_{\ell,\pm,-1}$ its cardinality. Then*

$$fsr_{\ell,\pm,-1} = \begin{cases} fsr_{\frac{\ell}{2},\pm} & \text{if } q \text{ is odd and } \ell \equiv 0 \bmod 4, \\ fsr_{\frac{\ell}{2}-1,\pm} & \text{if } q \text{ is odd and } \ell \equiv 2 \bmod 4. \end{cases}$$

Proof: We have ${}^{(-1)}f(X) = f(-X)$ for each $f \in \mathcal{FS}r_\pm(\ell,q)$. Then $f(-X) = f(X) \in \mathcal{FS}r_\pm(\ell,q)$ if and only if $f = h(X^2)$ with $h \in \mathcal{FS}r(\ell/2,q)$ such that $X-1$ does not divide h. Let $\tilde{f}sr_\ell$ denote the number of elements in $\mathcal{FS}r(\ell,q)$ which are not divisible by $X-1$ and let g be such a polynomial. If ℓ is even, then $X+1$ cannot divide g either, so $\tilde{f}sr_\ell = fsr_{\ell,\pm}$. If ℓ is odd, then $X+1$ must divide g and $\tilde{f}sr_\ell = fsr_{\ell-1,\pm}$. From this the lemma is immediate. $\circ$

From the results in 2 it is clear that for $t \in T(w)$ we have $M_t = G_t$ if and only if $cp(t) \notin \mathcal{FS}r_{\pm,-1}(2n,q)$. The following lemma shows that, apart from one exception, the classes tZ^H with $cp(t) \in \mathcal{FS}r_\pm(2n,q)/\mathcal{FS}r_{\pm,-1}(2n,q)$ are exactly the abelian p' - classes in the finite simple group $PSp_{2n}(q)$.

Theorem 4.10 *Let $G = Sp_{2n}(q)$, $n > 1$, $Z := Z(G)$ and $H = G/Z = PSp_{2n}(q)$. Let $t \in G$ be semisimple such that H_{tZ} is abelian. Then, up to conjugation in G, t is a regular element in some $T(w) = G_t$ and either $H_{tZ} = G_t/Z$ or $G = Sp_4(3)$, $H = PSp_4(3) \cong U_4(2)$, $w = (12), |t| = 8$, $cp(t) = X^4+1$, $T(w) \cong \mathbb{Z}_8$, $H_{tZ} = T(w)/Z.\langle (12)(-id)\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.*

Proof: Let $\pi : W \to \Sigma_n$ be the natural epimorphism and $w = (\sigma(w), \pi(w)) \in \{\pm 1\} : \Sigma_{\underline{n}} \cong W$. We identify $T(w)$ with the subgroup $\mathbf{T}_{wF} = \{t \in \overline{\mathbb{F}}_q^{\underline{n}} \mid {}^{wF}t(j) = t(w^{-1}(j))^{\sigma(w)(j)q} = t(j)\ \forall j \in \underline{n}\}$.

Suppose that $\pi(w')(j) = j$ for some $1 \leq j \leq n$. By the assumption there is a regular element $\tilde{t} \in T(w)$ satisfying ${}^{w'}\tilde{t}Z = \tilde{t}Z$ but ${}^{w'}\tilde{t} \neq \tilde{t}$. Hence ${}^{w'}\tilde{t} = -\tilde{t} \in G$ and we conclude that ${}^{w'}\tilde{t}(j) = \tilde{t}(j)^{\pm 1} = -\tilde{t}(j) \neq 0$. This implies that w' contains the negative 1 - cycle $(j)^-$ and $\tilde{t}(j)^{-1} = -\tilde{t}(j)$, so $\tilde{t}(j) = \epsilon$ is a primitive 4^{th} root of unity. Since $\epsilon^q = \epsilon^{\pm 1}$ and $\tilde{t}$ is regular, w also must contain $(j)^-$. But $n > 1$, so the element $t'' = (1,\ldots,1,\epsilon,1,\ldots,1)$ with ϵ in position j is contained in $T(w)$ and satisfies ${}^{w'}t'' = t''^{-1} \neq t''Z$, a contradiction. If w contains $(j)^\pm$, then $d := (1,\ldots,1,c,1,\ldots,1) \in T(w)$ for some c of order $q \mp 1$. Now ${}^{w'}dZ = dZ$ implies that

${}^{w'}d = -d = (-1, \ldots, -1, -c, -1, \ldots, -1)$, hence $c = -1$, $n = 2$, $q = 3$, $d = (1, -1)$, $w' = (12)$ and $w = id$. But in this case $T(id) = \{(\pm 1, \pm 1)\}$ does not contain a regular element, which is a contradiction. So we have shown that neither $\pi(w)$ nor $\pi(w')$ have fixed points in $\underline{n}$, i.e. $n_1^{\pm 1}(w') = 0 = n_1^{\pm 1}(w)$.

For each $1 < i \le n$ let $\underline{t}_i^{\pm} := (t_\pm, t_\pm^q, \ldots, t_\pm^{q^{i-1}}, 1 \ldots, 1) \in \mathbf{T}$ with $t_\pm \in \overline{\mathbb{F}}_q^*$ of order $q^i \mp 1$. Suppose now that $n_i^{\pm}(w) > 0$ for some $1 < i < n$. Then there is some suitable $t = (1, \ldots, 1, \underline{t}_i^{\pm}, 1, \ldots, 1) \in T(w)$. Since $t_\pm \notin \{\pm 1\}$, ${}^{w'}tZ = tZ$ implies that ${}^{w'}t = t$. Let $\mathcal{O}_i$ denote the support of t; since $\underline{t}_\pm^{q^\ell} \ne \underline{t}_\pm^{\pm q^k}$ for any $1 \le k \le \ell < i$ we see that $\pi(w')_{|\mathcal{O}_i} = id_{|\mathcal{O}_i}$, in contradiction to the fact that $n_1(w') = 0$. Hence we see that $w = (12 \ldots n)^{\pm}$ and $w' = w^k \cdot (-id)^\ell$ for some k and ℓ.

Suppose that $w = (12 \ldots n)^-$; then $w' = w^k$ for some $k \le n$. Since $\underline{t}_n^- \in T(w)$, ${}^{w'}\underline{t}_n^- Z = \underline{t}_n^- Z$ implies ${}^{w'}\underline{t}_n^- = (\underline{t}_n^-)^{q^{-k}} = (-id)\underline{t}_n^-$; hence $q^n + 1$ divides $2(q^k - 1)$. The only possibility is $n = k = 1$ (and $q = 3$), which is a contradiction. Hence we have $w = (12 \ldots n)$ and $w' = w^k \cdot (-id)^\ell$ for some $k, \ell \le n$. If ℓ is even, then we conclude in the same way as above, that $q^n - 1$ divides $2(q^k - 1)$, hence $n = k$ and $w' = id$, a contradiction. So ℓ is odd and we conclude that $q^n - 1$ divides $2(q^k + 1)$. This implies that $q^k(q^{n-k} - 2) \le 3$. If $n = k$ then $(q^n - 1)x = 2(q^n + 1)$ for some $x < 3$ (because $q > 2$ and $1 < n$), which is impossible. Hence $n > k$ and we conclude that $q = 3$, $k = 1$ and $n = 2$. $\circ$

Now a similar argument as in the proof of 3.4 yields:

Corollary 4.11 *Let $H = PSp_{2n}(q)$ and $q = p^s$ odd. Then the number $\mathcal{C}_{ab,p'}(2n, q)$ of abelian p' - classes in H is given by*

$$\mathcal{C}_{ab,p'}(2n, q) = \begin{cases} 2 & \textit{if } n = 2 \textit{ and } q = 3, \\ \frac{1}{2}(fsr_{2n,\pm}(q) - fsr_{n,\pm}(q)) & n \textit{ is even and } (n, q) \ne (2, 3), \\ \frac{1}{2}(fsr_{2n,\pm}(q) - fsr_{n-1,\pm}(q)) & \textit{if } n \textit{ is odd.} \end{cases}$$

Remark 4.12 *Notice that in case of $PSp_4(q)$ the 'generic formula' for $\mathcal{C}_{ab,p'}(4, q)$ is $\frac{1}{2}(q^2 - 4q + 5)$ which specializes to 1 for $q = 3$, but $\mathcal{C}_{ab,p'}(4, 3) = 2$ because of the extra class appearing in 4.10.*

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