Finite p'-Semiregular Groups

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0 Introduction

About 60 years ago H. Zassenhaus [17] classified finite near - fields. In doing so he also gave a full description of those finite groups which have a representation D (over $\mathbb{C}$) such that $D(g)$ has no eigenvalue 1 for any nontrivial element $g \in G$. In our terminology such a group is called semiregular. One of the results in [17] characterizes $SL_2(5)$ as the only perfect semiregular group.

Clearly, if the underlying field $\mathbb{F}$ has characteristic p dividing the group order, then p - elements of G always act with eigenvalue 1 on any $\mathbb{F}G$ - module. So in this situation the notion of 'semiregularity' naturally modifies to a local version which in this paper is called 'p' - semiregularity' (cf. 1.1). Of course this localized semiregularity can also be studied in the complex case. In particular a finite group is semiregular if and only if it is p' - semiregular for all primes p.

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So in this paper we generalize H. Zassenhaus' result and classify the finite p' - semiregular groups. Moreover we obtain group theoretical criteria for p' - semiregularity which generalize those given in [17].

It turns out that in particular for non - solvable groups, there are many more locally semiregular groups than semiregular ones, as can be seen in 4.1 and 5.6.

There is an interesting interrelation between finite locally semiregular groups and a certain class of finite primitive permutation groups which has been investigated by H. Bender [2] and which plays an essential role in the course of the classification of finite simple groups.

Consider a faithful primitive permutation group G on a finite set Ω. Then the socle $\mathrm{soc}(G)$ of G is either elementary abelian or a direct product L^m of some non-abelian finite simple group L. Moreover, if $\mathrm{soc}(G) = V = \mathbb{F}_\ell^n$ is elementary abelian (ℓ a prime), then G is the semi-direct product of V and any point stabilizer G_a, $a \in \Omega$. In this case, one can identify Ω with V, $a \in \Omega$ with the zero vector of V, then embed G_0 in $GL(V)$ such that the action of $G_a = G_0$ on Ω and the linear action of G_0 on V are compatible.

Let $\mathcal{F}(p)$ denote the set of finite, primitive permutation groups with every two - point stabilizer $G_{a.b}$ being a p - group. Then the connection mentioned above reads as follows:

Theorem 0.1 *Assume* $\mathrm{soc}(G) = V$ *is an elementary abelian* ℓ*-group. Then* $G \in \mathcal{F}(p)$ *if and only if the pair* (G_0, V) *is* p'*-semiregular and* V *is an irreducible* G_0 *- module.*

The results of the present paper include a full classification of the groups contained in $\mathcal{F}(p)$ having abelian socle. Observe, that $O_p(G_0) = 1$ whenever $p = \ell$, but in case $p \neq \ell$ the group $O_p(G_0)$ may be nontrivial; this situation actually arises and requires additional attention.
A forthcoming paper will deal with those groups in $\mathcal{F}(p)$ the socle of which is nonabelian.

1 Preliminary results

G will always denote a finite group and p a prime. The notation used is standard and can be found in [12] or [7]. In particular $\pi(G)$ denotes the set of prime divisors of $|G|$; for any set π of primes, π' is the complement of π in the set of all primes; $O_\pi(G)$ denotes the largest normal π - subgroup of G and $O(G) := O_{\{2\}'}(G)$; $Z(G)$ is the centre of G and $F(G)$ is the Fitting - subgroup of G (i.e. the largest nilpotent normal subgroup of G). Furthermore, $\mathrm{Aut}_G(H) := N_G(H)/C_G(H)$ denotes the **automizer** of a subgroup H in G.

For the sake of convenience the term "quaternion group" is understood to include the generalized quaternion groups of order greater than 8.

Definition 1.1 *Let* $V := V_{\mathbb{F}}$ *be a finite - dimensional* $\mathbb{F}G$ *- module over some field* $\mathbb{F}$*. Then the pair* (G, V) *is called* p' - **semiregular** *if every nontrivial* p' *- element of* G *acts*

without any fixed points on $V\backslash\{0\}$.
G is called p' - semiregular over $\mathbb{F}$, if there is a p' - semiregular pair (G,V) as above and it is called p' - semiregular if there is a p' - semiregular pair over some field K.
A pair (G,V) or group G is called **semiregular** *if it is p' - semiregular for any prime p, i.e. if no nontrivial element of G has a fixed point on V.*

Notice that (G,V) is p' - semiregular if and only if no p' - element $g \neq 1$ of G has eigenvalue 1 on V. Putting $r := \ char\ \mathbb{F}$ we note that if r divides $|G|$ then (G,V) can only be p' - semiregular if $r = p$. Whenever r does not divide $|G|$ the pair (G,V) is called **separably** p' - semiregular, because then the group algebra $\mathbb{F}G$ is separable; if $r = p$ divides $|G|$ we call (G,V) **inseparable**. If p does not divide $|G|$ then clearly p' - semiregularity and semiregularity coincide.
Clearly, for any p' - semiregular pair (G,V) any pair (H,W) with $H \leq G$ and W an H-submodule of V is also p' - semiregular. If moreover H is a p' - group then (H,W) is semiregular. Notice that the kernel of G on V is always a normal p - subgroup of G.

We start with some auxiliary statements.

Lemma 1.2 *The following are equivalent.*

(i) *The pair (G,V) is p' - semiregular.*

(ii) *The pair $(G,\bar{V})$ is p' - semiregular, where $\bar{V} = V \otimes_{\mathbb{F}} \bar{\mathbb{F}}$.*

(iii) *The spectrum* $\mathrm{Spec}(g,V)$ *of any p'-element g of prime order in G acting on V does not contain 1.*

(iv) $\mathrm{Spec}(g,V) \not\ni 1$ *whenever $g \in G$ is not a p-element.*

Proof: (i) $\Leftrightarrow$ (ii): By definition.
(i) $\Rightarrow$ (iii): By definition.
(iii) $\Rightarrow$ (i): Let g be a p'-element, $g \neq 1$. Then some power g^k of g is a p'-element of prime order. Due to (iii), $\mathrm{Spec}(g^k,V) \not\ni 1$, hence $\mathrm{Spec}(g,V) \not\ni 1$.
(i) $\Rightarrow$ (iv): Let $g \in G$ be not a p-element. Then some power g^k of g is a non-identity p'-element. Due to (i), $\mathrm{Spec}(g^k,V) \not\ni 1$, and so $\mathrm{Spec}(g,V) \not\ni 1$.
(iv) $\Rightarrow$ (i): Obvious. $\circ$

In view of this lemma in what follows we can freely suppose that the field $\mathbb{F}$ is algebraically closed.

Lemma 1.3 (i) *G is p' - semiregular if and only if there is a p' - semiregular pair $(G,V_{\mathbb{F}})$ such that $char\ \mathbb{F} = p$ and $V_{\mathbb{F}}$ is an absolutely irreducible $\mathbb{F}G$ - module.*

(ii) *Let $\mathbb{F}$ be a field of characteristic p. Then G is p' - semiregular over $\mathbb{F}$ if and only if $G/O_p(G)$ is p' - semiregular over $\mathbb{F}$.*

(iii) *Let $\mathbb{F}$ be an arbitrary field and let G be p - solvable. Then G is p' - semiregular over $\mathbb{F}$ if and only if $G/O_p(G)$ is p' - semiregular over $\mathbb{F}$.*

(iv) *An arbitrary group G is p' - semiregular if and only if $G/O_p(G)$ is p' - semiregular.*

Proof: (i): "If" is clear. So we assume that (G, V_K) is p' - semiregular with *char* $K \neq p$. Then *char* K does not divide $|G|$; so we can assume that K is the complex field and G is irreducible on V_K. Now reduction $\bmod p$ gives a p' - semiregular pair over L with *char* $L = p$. Taking a suitable extension field $\mathbb{F}$ of L we find an absolutely irreducible submodule $W \leq V \otimes_L \mathbb{F}$ which is also p' - semiregular.
(ii): Since $O_p(G)$ is contained in the kernel of every irreducible module over a field of characteristic p, the statement follows.
(iii): Now suppose G is a p - solvable group and (G, V) is a p' - semiregular pair over an arbitrary field $\mathbb{F}$ with V irreducible. If *char* $\mathbb{F} = p$, then $O_p(G)$ is in the kernel and we can assume $O_p(G) = 1$. If *char* $\mathbb{F}$ does not divide $|G|$, then we can assume $\mathbb{F} = \mathbb{C}$. In this case, by the theorem of Fong - Swan - Rukolaine ([4] 22.1 pg.513), each simple constituent W of a reduction $\bmod p$ of V is liftable to a p' - semiregular pair $(G/O_p(G), \tilde{W})$ over $\mathbb{C}$. Hence G is p' - semiregular over $\mathbb{F}$ if and only if $G/O_p(G)$ is p' - semiregular over $\mathbb{F}$.
(iv): This follows immediately from (i). $\circ$

As an immediate consequence we obtain:

Corollary 1.4 *Assume V is a faithful G-module over a field of characteristic p. Then the following are equivalent.*

(i) *The pair (G, V) is p' - semiregular.*

(ii) *V has a chain $0 = V_0 \subset V_1 \subset \ldots \subset V_m = V$ of G-submodules such that for each $i = 0, 1, ..., m-1$ the group $O_p(G)$ acts trivially and the group $G/O_p(G)$ acts faithfully, irreducibly and p' - semiregularly on V_{i+1}/V_i.*

$\circ$

Remarks: 1) The last statement in 1.3 does not specify a field $\mathbb{F}$. In fact specifiying the complex field without any assumption on G invalidates the assertion. For example the simple group $A_5 \cong SL_2(4)$ is not $2'$ - semiregular over $\mathbb{C}$ but its covering group $SL_2(5)$ is even a semiregular group.
2) There exist inseparably p' - semiregular pairs (G, V) with $G = G'$ but $O_p(G) \neq 1$. For example, consider the 4-dimensional symplectic group $Sp_4(q)$ ($q = p^a$, $p > 2$), acting on the natural symplectic module $V = \mathbb{F}_q^4$. Consider a maximal totally isotropic subspace U and put $G = \{g \in Sp_4(q) \mid g(U) = U, \det(g|_U) = 1\}$. Then $O_p(G)$ is an elementary abelian group of order p^{3a}, and $G/O_p(G) \simeq SL_2(q)$. One verifies directly that $G = G'$ and G acts faithfully and p' - semiregularly on V.

The following result is well known (cf. [7] Theorems 3.3.3 and 5.4.10) :

Lemma 1.5 *Let (G,V) be p' - semiregular and let $r \in \pi(G)\backslash\{p\}$. Then the r - rank of G is one, hence Sylow r - subgroups of G are cyclic or quaternion groups.*

$\circ$

Next we see that it suffices to study indecomposable groups:

Lemma 1.6 *Let $\mathbb{F}$ be an arbitrary field and let $G = G_1 \times G_2 \times ... \times G_r$ be a direct product. Then (G,V) is p' - semiregular over $\mathbb{F}$, if and only if all G_i are p' - semiregular over $\mathbb{F}$ and $\gcd(|G_i|_{p'}, |G_j|_{p'}) = 1$ for all $i \neq j$.*

Proof: If (G,V) is p' - semiregular over $\mathbb{F}$, then the p' - parts of $|G_i|$ are pairwise coprime, because elementary abelian p' - subgroups of G must be cyclic.
Assume now that the G_i are p' - semiregular over $\mathbb{F}$ and the p' - part of their orders are pairwise coprime. Without loss of generality we choose p' - semiregular pairs (G_i, V_i) over $\mathbb{F}$ and an arbitrary p' - element $g := (g_1, ..., g_r) \in G$. Then all g_i are p' - elements and the eigenvalues of g on $V := V_1 \otimes ... \otimes V_r$ are of the form $\prod_i \lambda_i$ where λ_i is an eigenvalue of g_i. Since the orders of the g_i are pairwise coprime, this product is 1 if and only if each $\lambda_i = 1$, whence each g_i is the identity in G_i. $\circ$

As we shall encounter p - groups of symplectic type, we record the following result.

Lemma 1.7 *Let p be a prime and let $P = E * Z$ be a p - group of symplectic type with cyclic centre Z and E extraspecial of order $p^{1+2\ell}$ such that $Z_0 := E \cap Z = \Omega_1(Z)$; moreover, let A be a nontrivial p' - subgroup of $C_{\mathrm{Aut}(P)}(Z_0)$. Then the following hold:*

(i) *$P = Q\ C$ where $Q := [P,A]$ is extraspecial of order p^{1+2m} with $1 \leq m \leq \ell$ and where $C := C_P(A) \geq Z$ with $Z(Q) = Q \cap C = Z_0$ and $C_{Q/Z_0}(A) = 1$.*

(ii) *Let H denote the semidirect product $P : A$ and suppose furthermore that $A = \langle a \rangle$ is cyclic and $C_A(x) = 1$ for each $x \in Q\backslash Z_0$. Then for any faithful $\psi \in \mathrm{Irr}_{\mathbb{C}}\, H$ the restriction ψ_A contains the trivial character, except possibly when $|A| = p^m + 1$ with $|Q| = p^{1+2m}$ in which case we have*

$$\psi(a) = -\epsilon p^{(1/2)\dim_{\mathbb{F}_p} \ker((a-1)_{|P/Z})}$$

with ϵ a p - th root of unity.

Proof: Notice that the canonical homomorphism $\mathrm{Aut}(Z)_{p'} \to \mathrm{Aut}(Z_0)$ is injective, so A acts trivially on Z. An easy commutator calculation shows that $[Q,C] = 1$ and therefore $Z(Q) = Z(P) \cap Q = Z_0$; consequently Q is extraspecial.
Since H is an epimorphic image of the exterior direct product of $Q : A$ and C, any $\phi \in \mathrm{Irr}_{\mathbb{C}}(H)$ is of the form $\phi_0 \otimes \phi_1$ with $\phi_0 \in \mathrm{Irr}_{\mathbb{C}}(C)$ and $\phi_1 \in \mathrm{Irr}_{\mathbb{C}}(Q : A)$. Now we can apply [12] 17.13 pg. 574, which describes $\mathrm{Irr}_{\mathbb{C}}(Q : A)$.

It turns out that the restriction of ϕ_1 to A always contains the trivial character except possibly when $|A| = p^m + 1$, in which case $\phi_1(a) = -\epsilon$ with ϵ a suitable p - th root of unity. Observe that $C = ZX$ where $X = Z_0$ or X is extraspecial. In any case $|X| = p^{1+2k}$ where $2k = \dim_{\mathbb{F}_p} \ker((a-1)_{|P/Z})$. Now the claim follows. $\circ$

According to 1.2 (iii) p' - semiregularity of (G, V) is controlled by the action of the minimal p' - subgroups of G on V. Hence we define the following characteristic subgroup $\mathcal{M}_{p'}(G)$ of G which plays a crucial role in our investigation:

Definition 1.8 $\mathcal{M}_{p'} := \mathcal{M}_{p'}(G)$ *denotes the characteristic subgroup*

$$\langle g \in G \mid |g| \in \pi(G) \backslash \{p\} \rangle.$$

It is clear that for any pair (G, V) over any field, (G, V) is p' - regular if and only if $(\mathcal{M}_{p'}, V_{\mathcal{M}_{p'}})$ is. Moreover we have the following:

Proposition 1.9 *Let* $\mathbb{F}$ *be an arbitrary field and* $P := O_p(\mathcal{M}_{p'}(G))$.

(i) *G is p' - semiregular over $\mathbb{F}$ if and only if $\mathcal{M}_{p'}$ is p' - semiregular over $\mathbb{F}$. In particular if $(\mathcal{M}_{p'}, W)$ is a p' - semiregular pair over $\mathbb{F}$, then $(G, \mathrm{ind}^G_{\mathcal{M}_{p'}}(W))$ is p' - semiregular.*

(ii) *Let $(\mathcal{M}_{p'}, W)$ be p' - semiregular over $\mathbb{F}$ with W irreducible and faithful. Then for any $N \lhd \mathcal{M}_{p'}$ the restriction W_N is homogeneous.*

If in addition $\mathbb{F}$ is algebraically closed then every abelian $N \lhd \mathcal{M}_{p'}$ is a cyclic subgroup of $Z(\mathcal{M}_{p'})$; in particular $Z(\mathcal{M}_{p'})$ is cyclic.

If moreover $(\mathcal{M}_{p'}, W)$ is separable then one of the following is true:

a) $Z(\mathcal{M}_{p'})$ contains P;

b) $P = E * Z$ with $Z := Z(P) \leq Z(\mathcal{M}_{p'})$, E an extraspecial p - group; and $\mathcal{M}_{p'} = PC_{\mathcal{M}_{p'}}(P)$ or $p = 2$ and $C_{\mathcal{M}_{p'}}(P)$ contains all elements of prime order r for which r is not a Fermat - prime.

Proof: Suppose $(\mathcal{M}_{p'}(G), W)$ is a p' - semiregular pair, then for $V := \mathrm{ind}^G_{\mathcal{M}_{p'}}(W)$ the restriction $V|_{\mathcal{M}_{p'}}$ is a direct sum of G - conjugates of W, which are all p' - semiregular. Hence V is p' - semiregular. This proves (i).

Now suppose (ii) and set $M := \mathcal{M}_{p'}$.
Let $N \lhd M$ and $g \in M \backslash N$ be an element of prime order $r \neq p$. Then g is contained in the inertia group $I(S)$ of every irreducible constituent S of $W|_N$, else W would contain the induced module $\mathrm{ind}_N^{\langle N, g \rangle}(S)$ with its obvious g - fixed points. Therefore $M \leq I(S)$.

If $\mathbb{F}$ is algebraically closed and if N is abelian then N acts faithfully on the one dimensional space S affording a monomorphism $\lambda : N \hookrightarrow \mathbb{F}^*$. Hence N is cyclic and $\lambda(n^g) = \lambda(n)$ for all elements g of prime order $r \neq p$; this implies $N \leq Z(M)$.
If in addition (M, W) is separable we can assume $\mathbb{F} = \mathbb{C}$ and W affords the irreducible character χ. Now χ_P is homogeneous and all abelian, characteristic subgroups of P are cyclic and central in M.
By [12] 13.10, pg.357, P is a central product $P = E * R$ with $|Z(E)\Omega_1(Z(R))| = p$ where
- E is extraspecial or $E = 1$
- R is cyclic or $p = 2$ and R is quaternion, dihedral, quasidihedral of order ≥ 16.
If R is not cyclic, then $p = 2$ and $N := Z(C_P(\Phi(P)))$ *char* P is the unique maximal cyclic subgroup of index 2 in R; as $N \not\leq Z(M)$, we derive a contradiction. Hence $P = E * Z$ where $Z := Z(P)$ is cyclic.

Suppose $E \neq 1$ and let $g \in M$ be of prime order $r \neq p$, inducing a nontrivial automorphism of P. As $Z \leq Z(M)$, g centralizes $Z_0 := \Omega_1(Z)$. Now we apply 1.7 with $A = \langle g \rangle$ and conclude that each irreducible constituent μ of $\chi|_{\langle g \rangle}$, contains the trivial character $1_{\langle g \rangle}$ unless $|g| = r = p^m + 1$ for a suitable $m \in \mathbb{N}$.

Hence all nontrivial r - elements of M must have an order of the form $p^* + 1$. This can only happen if $p = 2$ and all these orders are Fermat - primes. $\circ$

As an immediate consequence we get:

Corollary 1.10 *Suppose that G is a finite p' - semiregular group for some prime p. Then each p' - subgroup whose order is a product of two primes is cyclic.*

$\circ$

We shall see that the converse holds for solvable groups, but not in general. In case of semiregular solvable groups this has been shown in [17] (loc. cit. Satz 11). This also motivates the following

Definition 1.11 *We say that the group G has property $\mathcal{Z}_{p'}$ if any subgroup of order qr is cyclic whenever $q, r \in \pi(G)\backslash\{p\}$.*

It will be convenient to have the automorphism groups of the Sylow r - subgroups of p' - semiregular groups at our disposal. So we recall the following well known results without proof.

Proposition 1.12 (i) *Let R be a cyclic group of prime power order $r^m > 1$. Then* $\mathrm{Aut}(R) \cong (\mathbb{Z}_{r^m})^*$ *is abelian of order $(r-1)r^{m-1}$.*

(ii) *Let*

$$Q := Q_{2^n \cdot 2} := \langle\ x, y \mid x^{2^n} = y^4 = 1,\ y^2 = x^{2^{n-1}},\ x^y = x^{-1}\ \rangle$$

be a quaternion group. Then $\mathrm{Aut}(Q) \cong \mathbb{Z}_{2^n} \cdot \mathrm{Aut}(\mathbb{Z}_{2^n})$ *for* $n > 2$ *and* $\mathrm{Aut}(Q_8) \cong \Sigma_4$*, where an additional element* σ *of order* 3 *permutes the subgroups* $\langle x\rangle, \langle y\rangle$ *and* $\langle xy\rangle$*. In this case, the semidirect product* $Q_8 : \langle\sigma\rangle$ *is isomorphic to* $SL_2(3)$*.*

$\circ$

We close this section with a technical lemma on groups having property $\mathcal{Z}_{p'}$:

Lemma 1.13 *Let* G *be a finite group and define*

$$O_p^p := O_p^p(G) = O^p(G/O_p(G)) \cong O^p(G)/O_p(O^p(G)) \cong O^p(G)O_p(G)/O_p(G).$$

Then the following hold:

(i) *G has property $\mathcal{Z}_{p'}$ if and only if O_p^p has property $\mathcal{Z}_{p'}$.*

(ii) *Suppose H is a normal subgroup of O_p^p of prime index r (necessarily $\neq p$). Then G has property $\mathcal{Z}_{p'}$ if and only if H has property $\mathcal{Z}_{p'}$ and one of the following holds:*

a) r divides $|H|$ and H contains every minimal p' -subgroup of O_p^p.

b) $H = O_{r'}(O_p^p)$ and $O_p^p = H \times Z$ with $|Z| = r$.

c) $r = 3$, $H = O_{3'}(O_p^p) = O_{\{2,3\}'}(O_p^p) \times O_2(O_p^p)$ and $O_p^p = O_{\{2,3\}'}(O_p^p) \times S$ with $S \cong SL_2(3)$.

Proof: The first claim is obvious because any p'-section of G corresponds to an isomorphic p'-section of O_p^p.
Now we can assume $G = O_p^p(G)$. It is easy to see that $SL_2(3) \cong Q_8 : \mathbb{Z}_3$ has property $\mathcal{Z}_{p'}$ for any prime p. Suppose H has property $\mathcal{Z}_{p'}$, then the groups described in (b) and (c) also have this property. In case (a) H contains each subgroup of order qr for distinct primes $q, r \in \pi(G)\backslash\{p\}$ and so G has property $\mathcal{Z}_{p'}$.

Now suppose that G has property $\mathcal{Z}_{p'}$ and let $R \in Syl_r(G)$. If $r \in \pi(H)$ then $R \cap H \neq 1$ contains every element of order r in R, because R is cyclic or quaternion; so case (a) holds.
Henceforth suppose that $\gcd(|H|, r) = 1$. For any prime q dividing $|H|$ there is an R-invariant Sylow q-subgroup P_q of H. Now property $\mathcal{Z}_{p'}$ implies that R centralizes $\Omega_1(P_q)$ for $q \neq p$ and so $P_q \leq C_H(R)$ for all $q \notin \{2, p\}$.
Assume now that $P_2 \leq C_H(R)$. Then $|H : C_H(R)|$ is a power of p and so $H = C_H(R)P_p$; hence $[R, H] = [R, P_p] \leq P_p$. Since $[R, H] \trianglelefteq < R, H >= G$ and $O_p(G) = 1$, we get $[R, H] = 1$ and so case (b) holds.
Finally assume that $P_2 \not\leq C_H(R)$. Using the structure of $Aut(P_2)$ we see that $P_2 \cong Q_8, r = 3$ and $P_2R \cong SL_2(3)$. As $r = 3$ does not divide $|H|$, Frobenius' Normal Complement Theorem implies $H = O(H)P_2$. The previous argument applied to $O(H)R$ yields $R \leq C_G(O(H))$. Since $C_G(H) \trianglelefteq G$, we finally get $G = O(H) \times P_2R$; this is case (c). $\circ$

2 The groups of r - rank one ($r \neq p$)

In this section we consider the groups of r - rank one (i.e. the groups whose Sylow r - subgroups are cyclic or quaternion) for all $r \in \pi(G)\backslash\{p\}$. In view of the results in 1.3 and 1.4 we may assume that $O_p(G) = 1$. Recall that if a Sylow 2 - group is cyclic then G is 2 - nilpotent and hence solvable by the Odd Order Theorem.

2.1 The solvable groups

The main result in this section is

Theorem 2.1 *Let G be solvable, $p \in \pi(G)$ with $O_p(G) = 1$ and let $P \in \mathrm{Syl}_p(G)$. If all Sylow q - groups are cyclic or quaternion for all $q \neq p$, then $G'' \leq O_2(G)$, $G'''' = 1$ and exactly one of the following is true:*

(i) *$G = O_{p'}(G) : P$ with $P' = 1$. Moreover if $p = 2$ or all Sylow 2 - subgroups of G are cyclic, then $G'' = 1$.*

(ii) *$p = 3$, G is a nonsplit extension of $SL_2(3) \times O(G)$ by $\mathbb{Z}_2$ with Sylow 2 - group Q_{16}. Here $O(G)$ is of type* (i) *and $G'' = O_2(G) \cong Q_8$.*

Proof: If $G'' = 1$ then G' is an abelian p'-subgroup and hence $G = O_{p'}(G)P$ and $P' = 1$. Henceforth we may assume that $G'' \neq 1$.
Let $q \in \pi(G)\backslash\{2, p\}$ and set $G_0 = G/O_{q'}(G)$. Then $F(G_0)$ is a cyclic q-group and so, as $C_{G_0}(F(G_0)) \leq F(G_0)$, we get $G_0'' = 1$ and hence $G'' \leq O_{q'}(G)$. From this we deduce $G'' \leq O_{\{2,p\}}(G)$; in particular $p \neq 2, O(G'') = 1$ and $O_2(G'') \neq 1$.
Put $G_1 = O_{2',2}(G), R_0 = O(G), R_2 = O_2(G)$ and let $T \in Syl_2(G)$; so $G_1 = R_0 : T_1$ with $T_1 = T \cap G_1$. As T and T_1 are cyclic or quaternion, $G_1'' \leq R_0 \cap G'' = 1$ and thus $G \neq G_1$. Since G is 2-constrained, G/G_1 is isomorphic to a nontrivial subgroup of $Out(T_1)$ that has no nontrivial normal 2-subgroup; this forces $T_1 \cong Q_8, G/R_0 \cong SL_2(3)$ or Σ_4 and T is quaternion of order 8 or 16, respectively. Now we have $G'' \leq G_1 \cap O_{\{2,p\}}(G) = R_2$ and so $\{G'', R_2\} \subseteq \{T_1', T_1\}$.

If $G' \leq O_{p'}(G)$ then obviously G is of type (i). Henceforth we can assume that $G' \not\leq O_{p'}(G)$; in particular, $O_p(G'/G'') \neq 1$ and $S := O_{2,p}(G) \neq R_2$.
As $O_p(G') = 1$, p divides $|Aut_{G'}(G'')|$ and thus $G'' = R_2 = T_1$ as well as $p = 3$ and $S \cong SL_2(3)$. Moreover, since $SL_2(3)'' \cong \mathbb{Z}_2$, we have $G/R_0 \cong \Sigma_4, T \cong Q_{16}, G'''' = 1$ and $R_0G' = R_0 \times S$ with index 2 in G. So G is of type (ii). $\circ$

Now we can assume that the r - rank of G is one for all $r \in \pi(G)$. These groups (including the nonsolvable perfect ones) have already been classified by Zassenhaus in [17] where he gives generators and relations for each type.

For example if all Sylow subgroups of G are cyclic, then $F(G)$ is cyclic and so $F(G)$ is the unique maximal abelian normal subgroup of G. Hence $C_G(F(G)) = F(G)$ and $G/F(G)$ is again cyclic, whence G is metacyclic and 2 - nilpotent. In [17] one can find the following description of these metacyclic groups:

$$\mathcal{M}^{(n,m,r)} := \langle\ a, b \mid a^n = b^m = 1, a^b = a^r\ \rangle$$

with $r^m \equiv 1\ mod(n)$ and $g.c.d.(n, m(r-1)) = 1$. In this case $G' = \langle a \rangle$.

Not all of the groups occuring in Zassenhaus' list are metacyclic. In the following we give some information on the structure of these groups. For the sake of convenience we introduce the following notation:

$$F_{ev} := O_2(G) \text{ and } F_{odd} := O(F(G));$$

thus

$$F(G) = F_{ev} \times F_{odd}.$$

Proposition 2.2 *Suppose that all Sylow subgroups of the finite group G are cyclic or quaternion. Then the following are equivalent:*

α) *G does not involve a section isomorphic to $SL_2(3)$;*
β) *G is 2 - nilpotent;*
γ) *G is supersolvable;*

Let $P_2 \in \mathrm{Syl}_2(G)$ then the following statements hold whenever one of α), β) *or* γ) *is true:*

(i) *G is metacyclic if and only if one of the following holds:*

a) $P_2 \in \mathrm{Syl}_2(G)$ *is cyclic;*

b) $P_2 \in \mathrm{Syl}_2(G)$ *is quaternion and either $F_{ev} = P_2$ or F_{ev} is maximal cyclic of index 2 in P_2.*

(ii) *G is not metacyclic if and only if $P_2 \in \mathrm{Syl}_2(G)$ is quaternion and either $F_{ev} = (P_2)'$ or F_{ev} is a quaternion subgroup of index 2 in P_2.*

(iii) *G contains a self - centralizing cyclic normal subgroup.*

(iv) *$Z(P_2) \leq Z(G)$ if P_2 is a quaternion group.*

Proof: Easy exercise left to the reader. $\circ$

2.2 The perfect groups

In this paragraph we assume that $G = G'$ with $O_p(G) = 1$ and that G has r - rank one for all $r \in \pi(G)\backslash\{p\}$.

Lemma 2.3 *The largest normal solvable subgroup* $\mathrm{sol}(G)$ *coincides with the centre* $Z(G)$ *of* G.

Proof: Put $R := \mathrm{sol}(G)$ and $Z := Z(G)$. Assume $R \neq 1$ and consider its derived series

$$R = R_0 \triangleright R_1 \triangleright \ldots \triangleright R_{m-1} \triangleright R_m = 1.$$

Here, $R_{i+1} = [R_i, R_i] \triangleleft G$.

Clearly, $S := R_{m-1}$ is abelian with $O_p(S) = O_p(G) = 1$; hence S is cyclic. As $\mathrm{Aut}_G(S)$ is abelian and $G = G'$ we get $S \leq Z \leq R$. If $m = 1$, then $R = S$ and so $R = Z$. Suppose $m \geq 2$ and put $T := R_{m-2}$; so $T' = S$ centralizes T, and thus T is nilpotent. In particular $O_p(T) = 1$ and so T must be cyclic contrary to $m \geq 2$. $\circ$

Lemma 2.4 *Assume* $p \neq 2$. *Then* $G/Z(G)$ *is a non-abelian finite simple group with dihedral Sylow 2-subgroups. Furthermore,* $Z(G) \geq \mathbb{Z}_2$.

Proof: Put $Z = Z(G)$ and consider a normal series

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = Z$$

for G, with $H_i = G_i/G_{i+1}$ simple. Consider a Sylow 2-subgroup P of G, and set $P_i = P \cap G_i$. First we remark that $P_{i+1} \in \mathrm{Syl}_2(G_{i+1})$ and $Q_i = P_i/P_{i+1} \in \mathrm{Syl}_2(H_i)$. Next we observe that H_0 is non-abelian. (Indeed, if H_0 is abelian, then $G' \leq G_1$ and so G is not perfect.) In particular, Q_0 is not cyclic. Assume P_1 is not cyclic. Then P_1 contains an element $z = x^i y$ for some i (cf. 1.12 (ii)). As $P_1 \triangleleft P$, P_1 contains $xzx^{-1}z^{-1} = x^2$. Now it is easy to see that $(P : P_1)$ is at most 2, and so Q_0 is cyclic, a contradiction. Hence, P_1 is cyclic. In this case, G_1 has a normal 2-complement, which implies that $G_1 \leq \mathrm{sol}(G) = Z$, i.e. $n = 1$. We have also seen that $P_1 \leq \langle x \rangle$. If $P_1 = 1$, then Q_0 is quaternion, therefore, according to the Brauer-Suzuki Theorem, H_0 is not simple, a contradiction. Hence, $P_1 \ni x^{2^{n-1}} = y^2$, and $Z \geq \mathbb{Z}_2$. This implies Q_0 is a dihedral 2-group (of order at least 4), and $H_0 = G/Z$ is a non-abelian simple group with dihedral Sylow 2-subgroups. $\circ$

Proposition 2.5 *Assume* $p \neq 2$. *Then one of the following holds:*

(i) $G \cong SL_2(p^a)$;

(ii) $G \cong SL_2(r)$ *for some odd prime* $r \neq p$;

(iii) $G \cong 2A_7$ *and* $p = 3$.

Proof: Put $Z := Z(G)$ and note that by 2.4 and the Gorenstein-Walter Theorem, G/Z is either $L_2(r^a)$ for some odd prime r, or A_7. Recall that $Z \geq \mathbb{Z}_2$, and the Schur multiplier of $L_2(r^a)$ is $\mathbb{Z}_2$ if $r^a \neq 9$ and $\mathbb{Z}_6$ if $r^a = 9$. Also, the Schur multiplier of A_7 is $\mathbb{Z}_6$. If $G/Z = L_2(r^a)$ and $r \neq p$, then from the fact that Sylow r-subgroups of G are cyclic it follows that $a = 1$, and so (ii) holds. If $r = p$, then, using the condition $O_p(G) = 1$, we get (i). Finally, if $G/Z = A_7$, then Sylow 3-subgroups of G are not cyclic, therefore $p = 3$ and $G = 2A_7$. $\circ$

Similarly, we have

Proposition 2.6 *Assume $p = 2$. Then one of the following holds.*

(i) $G \cong SL_2(2^a)$ *for some* $a \geq 2$.

(ii) $G \cong L_2(r)$ *for some prime* $r \geq 7$.

(iii) $G \cong {}^2B_2(2^{2a+1})$ *for some* $a \geq 1$.

(iv) $G \cong J_1$.

(v) $G \cong {}^2B_2(2^{2a+1}) \times M$ *for some* $a, b \geq 1$ *and* $M \in \mathcal{S}(a)$. *Here,* $\mathcal{S}(a)$ *consists of the groups* $SL_2(2^b)$, $b \geq 2$, $L_2(r)$, $r \geq 7$ *a prime, whose order is coprime to* $|\,{}^2B_2(2^{2a+1})\,|$.

Proof: First we remark that each non-abelian composition factor L of G belongs to the list

$$\mathcal{C} = \{SL_2(2^a), a \geq 2; L_2(r), r \text{ a prime}, r \geq 7; {}^2B_2(2^{2a+1}), a \geq 1; J_1\}.$$

Indeed, for every odd prime r, Sylow r-subgroups of L are cyclic. Now a fundamental theorem of M. Aschbacher [1] says that $L \in \mathcal{C}$.

Put $Z = Z(G)$ and consider $T = \mathrm{soc}(G/Z)$; write $T = L_1^{n_1} \times \ldots \times L_m^{n_m}$ for minimal normal subgroups $L_i^{n_i}$ of T. Here, every L_i is a non-abelian simple group, since $Z = \mathrm{sol}(G)$. If a certain n_i is greater than 1, then a Sylow r-subgroup of G is not cyclic for some odd prime divisor r of $|L_i|$, a contradiction. Therefore, $n_1 = \ldots = n_m = 1$. Similarly, the odd parts of the orders of $L_1, \ldots, L_m$ are coprime. But $|SL_2(2^a)|$ and $|L_2(r)|$ ($r \geq 7$ a prime) are divisible by 3, $|\,{}^2B_2(2^{2a+1})\,|$ is divisible by 5, $|J_1|$ is divisible by $3 \cdot 5$. This implies that $m \leq 2$. Moreover, if $m = 2$, then $T = {}^2B_2(2^{2a+1}) \times M$ with $M \in \mathcal{S}(a)$.

Now let L be a minimal normal subgroup of $\bar{G} = G/Z$. We already know that $L \in \mathcal{C}$. In particular, $\mathrm{Out}(L) = \mathrm{Aut}(L)/L$ is solvable. The group $\bar{G}$ acts on L via conjugation with kernel K. As $\bar{G}$ is perfect, we must have $\bar{G}/K = L$, i.e. $\bar{G} = K \times L$. Also, $K' = K$. If $K = 1$, then $\bar{G} = L$. Suppose $K \neq 1$. The condition $\mathrm{sol}(G) = Z$ implies that $\mathrm{soc}(K)$ is the direct product of non-abelian minimal normal subgroups of K. The Sylow subgroups of odd order of K are cyclic. Furthermore, we have shown that $m \leq 2$. Alltogether, these remarks imply that $\mathrm{soc}(K) = M \in \mathcal{C}$ is non-abelian simple and $T = L \times M$. Since $L \not\cong M$, $\mathrm{Out}(L \times M) = \mathrm{Out}(L) \times \mathrm{Out}(M)$ is solvable. Again, $\bar{G}$ acts on $L \times M$, and $\bar{G}$ is perfect. Hence, $\bar{G} = L \times M$.

Now we assume that $m = 1$. Then $\bar{G} = L \in \mathcal{C}$. In particular, the Schur multiplier of L is a 2-group. But $O_2(G) = 1$, therefore $Z = 1$, $G = L$, and we arrive at the possibilities (i) – (iv).

Finally, suppose that $m = 2$. Then $G/Z = L \times M$ with $L = {}^2B_2(2^{2a+1})$, $M \in \mathcal{S}(a)$. We observe that the Schur multiplier of G/Z is a 2 - group because the Sylow q - subgroups of G/Z are cyclic for odd q. As G is a perfect and $O_2(G) = 1$, we get $Z = 1$ and $G = L \times M$. $\circ$

3 Solvable p' - semiregular Groups

So far, we have used only the necessary conditions on the rank of Sylow subgroups of G. Now we test such groups G for p' - semiregularity. We start with the solvable groups and assume $O_p(G) = 1$.

By 2.1 a solvable p' - semiregular group is of the form $X \times Y$ or $(X \times Y) : 2$ with $X \in \{1, SL_2(3)\}$ and $Y = O_{p'}(Y)P$ with $P \in \mathrm{Syl}_p(Y)$. In this case $O_{p'}(Y)$ must be a semiregular solvable group. Moreover the 2 - extension can occur only if $p = 3$ and $X \cong SL_2(3)$; in this case the Sylow 2 - group is isomorphic to Q_{16}. Clearly, if $X \cong SL_2(3)$ then $p = 3$ and $Y = O_{3',3}(G)$.

Notice that in all cases the group $\mathcal{M}_{p'}(G)$ is contained in $X \times O_{p'}(Y)$. By 1.3 it suffices to consider inseparably p' - semiregular pairs (G, V); it is easy to see that X is regular, hence p' - semiregular. So from 1.9 we get:

Lemma 3.1 *Let $G = X \times Y$ or $G = (X \times Y) : 2$ be a solvable group as described above (or in 2.1) and $\mathbb{F}$ an arbitrary field. Then there exists a p' - semiregular pair (G, V) over $\mathbb{F}$ if and only if there is a semiregular pair $(O_{p'}(Y), W)$.*

$\circ$

Now we prove the generalization of [17] (loc. cit. Satz 11) for solvable p' - semiregular groups. The case $p \notin \pi(G)$ includes Zassenhaus' original result:

Theorem 3.2 *Let G be a solvable group. Then G is p' - semiregular if and only if it has property $\mathcal{Z}_{p'}$.*

Proof: 'Only if' is clear. Now let G be a minimal counterexample for the converse, i.e. G has property $\mathcal{Z}_{p'}$ but is not p' - semiregular. We can assume $O_p(G) = 1$ and $G = O^p(G)$; moreover, since G is solvable, there is a maximal normal subgroup H of prime index r which is p' - semiregular. Now 1.13 provides a contradiction:

Indeed in case (i) $\mathcal{M}_{p'}(G) \leq H$ implies that G is p' - semiregular and in the other cases G is the direct product of p' - semiregular groups. $\circ$

Remark 3.3 *It may be worth noting that the assumption in 3.2 on the solvability of G is not redundant. Indeed all groups $SL_2(r)$ with r a Fermat - prime have the property $\mathcal{Z}_{2'}$ but for example $SL_2(17)$ is not $2'$ - semiregular (see 4.4).*

Suppose G is a finite group such that for each $r \in \pi(G)\backslash\{p\}$ there is a unique subgroup of order r. Then clearly G is p' - semiregular and moreover a pair (G, V) is p' - semiregular, if and only if V is a faithful G - module.

This characterizes a special class of solvable p' - semiregular groups:

Proposition 3.4 *Let G be a finite group and put $\bar{G} := G/O_p(G)$. Then the following are equivalent:*

(i) *Every minimal p' - subgroup is unique in $\bar{G}$.*
(ii) *G is p' - semiregular, solvable and $\bar{G}$ does not contain $SL_2(3)$ as a subgroup unless $p = 3$.*

Proof: We can assume that $G = O_p^p(G)$. Suppose (i) holds. Then the results in the previous section imply $G^\infty = 1$, hence G is solvable. Now (ii) follows.
Suppose (ii) and let G be an indecomposable minimal counterexample. Since G is solvable, we can assume that there is a maximal normal subgroup $H \leq G$ of prime index. Now 1.13 provides a contradiction. $\circ$

Now, by 3.1 we have reduced the classification of p' - semiregular solvable groups to the case of solvable semiregular groups, which have been classified in [17]. For completeness we offer some information on the structure of these groups. It is clear from chapter 1 that a finite semiregular group G is 2 - nilpotent if and only if it is supersolvable.

Lemma 3.5 (i) *Let $G = O(G) : P_2$ be a finite 2 - nilpotent group with $P_2 \in \mathrm{Syl}_2(G)$. Then G is semiregular if and only if $O(G)$ is semiregular and either P_2 is cyclic with $\Omega_1(P_2) \leq Z(G)$ or P_2 is a quaternion group. Moreover if G is semiregular, all its minimal subgroups are unique and $O(G)$ is metacyclic.*

(ii) *Let G be a finite solvable non - 2 - nilpotent group. Then G is semiregular if and only if $O(G)$ is semiregular, $O_2(G) \cong Q_8$ and either $G/O(G) \cong SL_2(3)$ or $G/O(G) \cong 2\Sigma_4$ with quaternion Sylow 2 - subgroups.*

Proof: (i) "Only if" follows immediately from 1.13 Conversely assume $O(G)$ and P_2 are semiregular. By 2.1 we can assume that $\Omega_1(P_2) \leq Z(G)$. Since $O(G)$ is of odd order,

each group of prime order $p \neq 2$ lies in $O(G)$ and is uniquely determined therein by 3.4 As all Sylow 2 - groups are conjugate, G contains only one involution by hypothesis. So 3.4 applies again.
(ii) Now let G be a finite solvable semiregular group which is not 2 - nilpotent. An application of propositions 1.4 and 1.5 shows, that in this case $F_{ev} \cong Q_8$ and $\mathrm{Aut}_G(F_{ev})$ is isomorphic to A_4 or Σ_4. Let $P_2 \in \mathrm{Syl}_2(G)$ and note that P_2 is quaternion containing F_{ev}; in particular $C_{P_2}(F_{ev}) = Z(P_2)$ is a Sylow 2 - group of $C_G(F_{ev})$. Order arguments now show that $|P_2/F_{ev}| \leq 2$. Moreover, $C_G(F_{ev}) = Z(P_2) \times O(G)$ and consequently $N := F_{ev}C_G(F_{ev}) = F_{ev} \times O(G)$ is normal in G with $G/N \cong \mathbb{Z}_3$ or Σ_3. Using 1.13 for the existence the claims follow. $\circ$

4 Perfect p' - semiregular Groups: The inseparable case

Throughout this section we suppose that G is a finite group and the pair (G,V) is *inseparably p' - semiregular* over a field $\mathbb{F}$ of characteristic p. We also assume that V is a faithful G - module.

We define $\mathcal{R}$ to be the set of all primes r satisfying the following conditions:

(a) $r = 2^a \cdot 3^b + 1$ for $a \geq 2$, $b \geq 0$;

(b) $(r+1)/2$ is a prime.

Here are several elements of $\mathcal{R}$: 5, 13, 37, 73, 193, 1153. We do not know whether $\mathcal{R}$ is infinite or not.

The aim of this section is to prove the following result which, together with 1.4, gives us a full description of perfect p' - semiregular groups in characteristic p.

Theorem 4.1 *Let G be a nontrivial perfect finite group with $O_p(G) = 1$. Let (G,V) be an inseparably p' - semiregular pair with G - module V. Then one of the following holds:*

(i) *$G \cong SL_2(p^a)$ for some $a \geq 1$ (and $p^a > 3$).*

(ii) *$G \cong {}^2B_2(2^{2a+1})$ for some $a \geq 1$, and $p = 2$.*

(iii) *$G \cong {}^2B_2(2^{2a+1}) \times SL_2(2^{2b+1})$ for some $a, b \geq 1$ and $\gcd(2a+1, 2b+1) = 1$, and $p = 2$.*

(iv) *$G \cong SL_2(r)$ with $r \in \mathcal{R} \cup \{7, 17\}$, and $p = 3$.*

Conversely, if (G,p) satisfies any of the conditions (i) – (iv), *then there exists a faithful absolutely irreducible G-module V in characteristic p such that (G,V) is p' - semiregular.*

Remark 4.2 *Shortly after submitting this paper we learned about the work of R. Guralnick and R. Wiegand in [8]. Motivated by questions on the multiplicative structure of*

Galois field extensions they also classified p'-semiregular groups G such that the underlying field of the corresponding G-modules has characteristic $p \geq 0$; however, in their list (loc.cit. Prop. 4.1) the groups $SL_2(r)$ with $r \in \mathcal{R}$ are missing.

For the proof of 4.1 we shall suppose that the pair (G, V) is p' - semiregular, G acts absolutely irreducibly on V, $G = G'$, and $O_p(G) = 1$. Now we have to test each G occuring in 2.5 and 2.6 for p' - semiregularity. In particular by 1.10 G must have property $\mathcal{Z}_{p'}$.

Lemma 4.3 (i) *If $p = 3$, then $G \not\cong 2A_7$.*

(ii) *If $p = 2$, then $G \not\cong J_1$.*

Proof: (i) Assume $p = 3$ and $G = 2A_7$. By 1.4 there exists a faithful Brauer character $\varphi \in \mathrm{IBr}_3(G)$ which gives us a $3'$ - semiregular representation of G. Due to [14], $\mathrm{IBr}_3(G)$ contains exactly 5 characters. If $\deg\varphi = 4$, then $\mathrm{Spec}(g, \varphi) \ni 1$ for $|g| = 7$, a contradiction. (Here and below, $\mathrm{Spec}(g, \varphi)$ denotes the spectrum of g (counting multiplicities) in a representation with character φ.) If $\deg\varphi = 6$ or 36, then $\mathrm{Spec}(g, \varphi) \ni 1$ for $|g| = 5$, again a contradiction.

(ii) Notice that J_1 contains a Frobenius group of order 21 and so does not have property $\mathcal{Z}_{2'}$. ∘

Lemma 4.4 *Assume $p = 2$ and $G \cong L_2(r)$ for some odd prime r. Then $r = 5$ and so $G \cong SL_2(4)$ as listed in 4.1.*

Proof: By 1.10 and inspection of the Borel subgroup of G the prime r must be Fermat, i.e. $r = 2^\alpha + 1$ with α a power of 2.
The character table of $2G = SL_2(r)$ is given in [6]. We shall keep the notation for the characters used there. The 2-decomposition matrix for G has been determined in [3]. ¿From that we get the following description of the 2-blocks for G:

– $(r-1)/4$ blocks of defect 0. They contain the ordinary characters θ_{2j} with $1 \leq j \leq (r-1)/4$, $\deg\theta_{2j} = r - 1$;

– the principal block B with three Brauer characters 1, β_1, β_2. Furthermore, B contains the ordinary characters 1, $\psi = St$, ξ_1, ξ_2 and some of the χ_{2i}'s. Also, $\beta_t = \xi_t \bmod 2 - 1$, $\deg\beta_t = (r-1)/2$, $t = 1, 2$.

Assume that $\varphi \in \mathrm{IBr}_2(G)$ is $2'$ - semiregular. First, $\varphi \neq \gamma_s$, as $\mathrm{Spec}(c, \chi_{2i})$ contains 1 with multiplicity 2, where $|c| = r$. Next suppose $\varphi = \theta_{2j} \bmod 2$ or $\varphi = \beta_t$. Observe that G has a real element g of order 3. Then the $2'$ - semiregularity of φ implies that $\varphi(g) = -\deg\varphi/2$. Looking at the character table of G, we see that $|\varphi(g)| \leq 2$, which implies $\deg\varphi \leq 4$. But $\deg\varphi = r - 1$ or $(r-1)/2$; therefore only $r = 5$ is possible. ∘

Proposition 4.5 *Assume that $p \neq 2$ and $G \cong SL_2(r)$ for some prime $r \neq p$ with $r \equiv -1(\mathrm{mod} 4)$. Then $p = 3$ and $r = 7$. Conversely, $G \cong SL_2(7)$ has just two irreducible $3'$ - semiregular representations over $\overline{\mathbb{F}_3}$, and both of them have dimension* 6.

Proof: Recall that by 1.10 and the structure of the Borel subgroup of G we have $r - 1 = 2p^{\alpha}$ with $\alpha > 0$. In particular $r \geq 7$. We shall keep the notation for complex characters of G used in [6]. Let φ be any faithful irreducible p-Brauer character of G.

1) First we show that φ is the reduction $\mathrm{mod}\, p$ of a certain of the ordinary characters χ_i, θ_j (i, j odd), or ξ_t, $t = 1, 2$. For, let B denote the p-block containing φ. Then B contains only faithful irreducible characters of G, that is, some of χ_i, θ_j with i, j odd, and $\xi_{1,2}$. For $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ we consider the *central character*

$$\omega_{\chi}(g) = \frac{\chi(g)|g^G|}{\chi(1)},\ g \in G$$

of G. Recall that two irreducible characters χ, χ' belong to the same block if and only if

$$\omega_{\chi}(g) \equiv \omega_{\chi'}(g)(\mathrm{mod} \pi)$$

for all p'-elements g of G. (In this case we shall write $\chi \approx \chi'$.) Here we are considering a splitting p-modular system $(\mathbb{K}, R, k)$ for G, and the maximal ideal of R is generated by π. G has exactly $r + 4$ conjugacy classes with representatives 1, z, c, d, zc, zd, a^l, b^m, $1 \leq l \leq (r-3)/2$, $1 \leq m \leq (r-1)/2$; $|z| = 2$, $|c| = |d| = r$, $|a| = r - 1$, $|b| = r + 1$ (see [6]).

(a) Since

$$\omega_{\chi_i}(c) - \omega_{\theta_j}(c) = r \not\equiv 0(\mathrm{mod} \pi),$$

$\chi_i \not\approx \theta_j$.

(b) Suppose $\theta_j \approx \xi_t$. In this case, b and b^2 are p'-elements. So

$$X = \omega_{\xi_t}(b) - \omega_{\theta_j}(b) = r(\sigma^j + \sigma^{-j}) \text{ and } Y = \omega_{\xi_t}(b^2) - \omega_{\theta_j}(b^2) = r(\sigma^{2j} + \sigma^{-2j})$$

are divisible by π. Here, σ is a primitive $(r + 1)$th root of 1. In particular, π divides $X^2 - rY = 2r^2$, again a contradiction. Therefore we have $\theta_j \not\approx \xi_t$.

(c) Now suppose that $\chi_i \approx \xi_t$. Since $r - 1 = 2p^{\alpha}$, it is not difficult to see that the reduction $\mathrm{mod}\, p$ of χ_i and $\xi_1 + \xi_2$ coincide. We conclude that either $\chi_i \not\approx \xi_t$ or $\chi_i \equiv (\xi_1 + \xi_2) \mathrm{mod}\,(p)$.

(d) Return to the p-block B containing φ. If some θ_j belongs to B, then the ordinary characters of B are only among the $\theta_{j'}$'s. Inspecting the Brauer tree of B, we see that $\varphi = \theta_j \bmod p$. In the case, where B contains some of the χ_i's and the ξ_t's, a look at the Brauer tree of B shows that either $\varphi = \chi_i \bmod p$, or $\varphi = \xi_t \bmod p$.

2) Now we assume in addition that φ is p' - semiregular. Remark at once that both, $\mathrm{Spec}(c, \chi_i)$ and $\mathrm{Spec}(c, \xi_t)$ contain 1; therefore $\varphi = \theta_j \bmod p$ for some odd j.

Suppose that $r+1$ is divisible by an odd prime q. It is clear that $q \neq p$. Then consider the p'-element $y = b^m$ with $m = (r+1)/q$ and the cyclic subgroup D generated by y. Here we have

$$(\theta_j|_D, 1_D)_D = \frac{(r-1) - \sum_{s=1}^{q-1}(\sigma^{jms} + \sigma^{-jms})}{q} \geq \frac{r+1-2q}{q} > 0,$$

a contradiction. (Here we have used the fact that $r+1$ is divisible by 4, and so $r+1 \geq 4q$.) This means: $r+1 = 2^\beta$ for some $\beta \geq 2$.

As $r \geq 7$ and r is a prime, β is an odd prime. In this case, $p^\alpha = 2^{\beta-1} - 1$ is divisible by 3, hence $p = 3$. Observe that $3^\alpha + 1$ is congruent to 2 or 4 modulo 8 according to as α is even or odd. This implies that $2^{\beta-1} \leq 4$, i.e. $\beta = 3$, $r = 7$, $G = SL_2(7)$.

3) Conversely, let $p = 3$ and $G = SL_2(7)$. Then G has four faithful irreducible 3-Brauer characters; among them just two (namely those of degree 6) are $3'$ - semiregular (see [14]). ◦

Proposition 4.6 *Assume that $p \neq 2$ and $G \cong SL_2(r)$ for some prime $r \neq p$ with $r \equiv 1(\mathrm{mod} 4)$. Then $p = 3$, and $r = 17$ or $r \in \mathcal{R}$. Conversely, assume that $G \cong SL_2(r)$ and $r \in \mathcal{R} \cup \{17\}$. Then G has just two irreducible $3'$ - semiregular representations over $\overline{\mathbb{F}_3}$, and both of them have dimension $(r-1)/2$.*

Proof: As in the proof of the previous result we see that $r - 1 = 2^\alpha p^\beta$ with $\alpha > 1$ and $\beta \geq 0$. Again we use the character table of G given in [6]. Consider a splitting p-modular system $(\mathbb{K}, R, k)$ for G, and let π generate the maximal ideal of R. G has exactly $r+4$ conjugacy classes with representatives 1, z, c, d, zc, zd, a^l, b^m, $1 \leq l \leq (r-3)/2$, $1 \leq m \leq (r-1)/2$; $|z| = 2$, $|c| = |d| = r$, $|a| = r-1$, $|b| = r+1$.

1) First we show that φ is the reduction mod p of a certain of the ordinary characters χ_i, θ_j (i, j odd), or η_s, $s = 1, 2$. For, let B denote the p-block containing φ. Then B contains only faithful irreducible characters of G, that is, some of χ_i, θ_j with i, j odd, and $\eta_{1,2}$.

(a) Since

$$\omega_{\chi_i}(c) - \omega_{\theta_j}(c) = r \not\equiv 0(\mathrm{mod}\, \pi),$$

$\chi_i \not\approx \theta_j$.

(b) Since p does not divide $r+1$, b is a p'-element, and

$$\omega_{\chi_i}(b) - \omega_{\eta_s}(b) = -2r \not\equiv 0(\mathrm{mod}\, \pi),$$

hence $\chi_i \not\approx \eta_s$.

(c) Now suppose that $\theta_j \approx \eta_s$. As

$$\omega_{\theta_j}(c) - \omega_{\eta_s}(c) = -(r+1)\sqrt{r}/2 \equiv 0(\mathrm{mod}\, \pi),$$

p divides $r+1$. We claim: if q is any odd prime divisor of $r+1$, say, $r+1=2Aq^\alpha$ with $\alpha \geq 1$ and $(A, 2q)=1$, and $q \neq p$, then j is divisible by q^α. (Assume the contrary: $j=Bq^\beta$ with $\beta \leq \alpha-1$ and $(B,q)=1$. Then consider the p'-element $x=b^m$ with $m=(r+1)/q^{\beta+1}$. We have

$$\omega_{\eta_s}(x)-\omega_{\theta_j}(x)=r\sigma^{-jm}(\sigma^{jm}-1)^2 \equiv 0(\mathrm{mod}\pi),$$

where σ is a primitive $(r+1)$th root of 1. By our choice, σ^{jm} is a primitive qth root of 1, and so $\sigma^{jm}-1$ divides q. We have seen that rq^2 is divisible by p, a contradiction.) From our claim it follows that $C|j$, if $r+1=2Cp^\gamma$ with $\gamma \geq 1$ and $(C,2p)=1$. Using this observation it is not difficult to see that the reductions $\mathrm{mod}\, p$ of θ_j and $\eta_1+\eta_2$ coincide. We conclude that either $\theta_j \not\approx \eta_s$, or $\theta_j \equiv (\eta_1+\eta_2) \bmod p$.

(d) Return to the p-block B containing φ. If some χ_i belongs to B, then the ordinary characters of B are only among the $\chi_{i'}$'s. Inspection of the Brauer tree of B yields that $\varphi=\chi_i \bmod p$. In the case, where B contains some of the θ_j's and the η_s's, a look at the Brauer tree of B shows that either $\varphi=\theta_j \bmod p$, or $\varphi=\eta_s \bmod p$.

2) Now we assume in addition that φ is p' - semiregular. Remark at once that $\mathrm{Spec}(c,\chi_i)$ contains 1, therefore $\varphi=\theta_j \bmod p$ for some odd j, or $\varphi=\eta_s \bmod p$ for $s=1,2$. Here we consider the former possibility.

Now suppose that $r+1$ is divisible by an odd prime $q \neq p$. Consider the p'-element $y=b^m$ with $m=(r+1)/q$ and the cyclic subgroup D generated by y. If $q<(r+1)/2$, then

$$(\theta_j|_D, 1_D)_D = \frac{(r-1)-\sum_{s=1}^{q-1}(\sigma^{jms}+\sigma^{-jms})}{q} \geq \frac{r+1-2q}{q} > 0,$$

a contradiction. If $r+1=2q$, then $m=2$, and so $\sigma^{jm} \neq 1$ as $2 \leq jm = 2j \leq r-1$. In this case we also have

$$(\theta_j|_D, 1_D)_D = \frac{(r-1)-\sum_{s=1}^{q-1}(\sigma^{jms}+\sigma^{-jms})}{q} = \frac{r+1}{q} > 0$$

as well, a contradiction. Therefore $r+1=2p^\gamma$ for some $\gamma \geq 1$ and $r=2^\alpha+1$ is a Fermat prime ≥ 5. Therefore $\alpha=2^\delta$ with $\delta \geq 1$. In this case, $p^\gamma = 2^{2^\delta-1}+1$ is divisible by 3, hence $p=3$. Suppose $\delta \geq 3$. Then $3^\gamma-1$ is divisible by 16, which implies that $4|\gamma$. But, in that case, $3^\gamma-1$ is divisible by 5 and so it is not a power of two, again a contradiction. Therefore, $\delta=1,2$ and $r=5,17$. Now a direct verification using [14] shows that $SL_2(r)$ (with $r=5,17$) has no faithful irreducible 3-Brauer characters of degree $r-1=\deg\theta_j$. This last contradiction shows that $\varphi \neq \theta_j \bmod p$.

3) At this point we assume that $\varphi=\eta_s \bmod p$ is p' - semiregular. Remark at once that $\mathrm{Spec}(x,\eta_s) \not\ni 1$ for $x=c,d,zc,zd$, and also for 2-elements $x \in G\backslash\{1\}$. Observe that, if $x=a^{l'} \neq 1$ is a p'-element, then x is a 2-element, and so $\mathrm{Spec}(x,\varphi) \not\ni 1$.

Suppose that $r+1$ is divisible by an odd prime q and $q \neq p$. Then consider the p'-element

$y = b^m$ with $m = (r+1)/q$ and the cyclic subgroup D generated by y. Here we have

$$(\theta_j|_D, 1_D)_D = \frac{(r-1)/2 - (q-1)}{q} = \frac{r+1-2q}{2q}.$$

In particular, if $q < (r+1)/2$, then $\mathrm{Spec}(y, \varphi) \ni 1$, again a contradiction. Therefore $r+1 = 2q$. Our arguments show that: either $r+1 = 2p^\gamma$ for some $\gamma \geq 1$, or $(r+1)/2$ is a prime. Observe in this case that, if $x = b^{m'} \neq 1$ is a p'-element, then $\mathrm{Spec}(x, \varphi) \not\ni 1$.

It remains to show that $p = 3$ and $r \in \mathcal{R} \cup \{17\}$. First we suppose that $r+1 = 2p^\gamma$ for some $\gamma \geq 1$. The same arguments as in part 2) show that $r = 5, 17$. (Recall that $5 \in \mathcal{R}$.) Next, we assume that $r+1 = 2q$ for some (odd) prime $q \neq p$. In this case, remark that either $r-1$ or $r+1$ is divisible by 3, as $r \neq 3$ is a prime. Therefore, either $q = 3$ or $p = 3$. If $q = 3$, then $r = 5$. If $p = 3$, then both $r = 2^\alpha 3^\beta + 1$, $q = 2^{\alpha-1}3^\beta + 1$ are prime. In other words, $r \in \mathcal{R}$.

4) Conversely, let $p = 3$ and $G = SL_2(r)$ with $r \in \mathcal{R} \cup \{17\}$. Then G has just two $3'$ - semiregular faithful irreducible 3-Brauer characters; these characters are $\eta_{1,2}$ mod 3 and have degree $(r-1)/2$. $\circ$

Proposition 4.7 *Let $G \cong SL_2(p^a)$, $p^a > 2$ and $\mathbb{F} = \overline{\mathbb{F}_p}$. Then G is p' - semiregular over $\mathbb{F}$. Moreover, if $p \neq 3$, then an absolutely irreducible G-module is p' - semiregular over $\mathbb{F}$ if and only if this module has dimension 2. In general, any p' - semiregular irreducible Brauer character of G is not stable under nontrivial field automorphisms of G.*

Proof: 1) It is clear that $G = SL_2(p^a)$ acts faithfully and p' - semiregularly on the natural module $V = \mathbb{F}_{p^a}^2$. Also, all G-modules obtained from V by means of Frobenius twist are p' - semiregular. Now let U be a p' - semiregular irreducible $\mathbb{F}G$-module. By Steinberg's Tensor Product Theorem,

$$U \simeq V_{i_0} \otimes V_{i_1}^{(p)} \otimes \ldots \otimes V_{i_{a-1}}^{(p^{a-1})},$$

where $0 \leq i_t \leq p-1$. Here, V_0 denotes the trivial module, and V_i, $1 \leq i \leq p-1$, denotes the ith symmetric power of $V_1 = V$.

Assume $p \neq 3$. Then consider an element $g \in G$ of order 3, and let ω denote a cubic root of unity. Observe that $\mathrm{Spec}(g, V) = \{\omega, \omega^2\}$, $\mathrm{Spec}(g, V_i) = \{\omega^{-i+2j} \mid 0 \leq j \leq i\}$. In particular, $\mathrm{Spec}(g, V_i)$ contains all cubic roots of unity if $i \geq 2$. Therefore, $\mathrm{Spec}(g, U) \ni 1$ if $i_t \geq 2$ for some t. This means that $i_t \in \{0, 1\}$. As $\dim U \geq 2$, one may suppose that $i_1 = 1$. If $i_2 = i_3 = \ldots = 0$, then $U = V$, and we are done. In the contrary case, one may suppose that $i_t = 1$ for $1 \leq t \leq b$, and $i_t = 0$ for $b+1 \leq t \leq a$, where $2 \leq b \leq a$. Then $\mathrm{Spec}(g, U) \ni \mu = \omega^{m_1 + \ldots + m_b}$ for any $m_i = \pm 1$. If b is even, then choose $m_i = 1$ for i odd, $m_i = -1$ for i even. If b is odd, then $b \geq 3$, and we can choose $m_1 = m_2 = m_3 = m_5 = m_7 = \ldots = 1$, $m_4 = m_6 = m_8 = \ldots = -1$. In any case, we get $\mu = 1$, a contradiction.

2) Now we suppose that a p' - semiregular irreducible module $U = V_{i_0} \otimes V_{i_1}^{(p)} \otimes \ldots \otimes V_{i_{a-1}}^{(p^{a-1})}$ is stable under a field automorphism $\Phi : \lambda \to \lambda^{p^b}$, b a divisor of $a = bc$. Then

$$V_{i_0} \otimes V_{i_1}^{(p)} \otimes \ldots \otimes V_{i_{a-1}}^{(p^{a-1})} = U = U^{(p^b)} = V_{i_0}^{(p^b)} \otimes V_{i_1}^{(p^{b+1})} \otimes \ldots \otimes V_{i_{a-1}}^{(p^{a-1+b})} \simeq$$

$$\simeq V_{i_{a-b}} \otimes V_{i_{a-b+1}}^{(p)} \otimes \ldots \otimes V_{i_{a-1}}^{(p^{b-1})} \otimes V_{i_0}^{(p^b)} \otimes V_{i_1}^{(p^{b+1})} \otimes \ldots \otimes V_{i_{a-b-1}}^{(p^{a-1})}.$$

Hence, $V_{i_{a-b}} \simeq V_{i_0} \simeq V_{i_b} \simeq V_{i_{2b}} \simeq \ldots$, and $V_{i_{b+t}} \simeq V_{i_t}$ in general.

Before moving further, we observe that our lemma has been proved completely for the case where $p \neq 3$ or $a = 1$. If $a > 1$ and $p = 3$ then $p^a - 1$ has some prime divisor r, which does not divide $p^d - 1$ for any d, $1 \leq d < a$, see [18]. Choose an element $g \in G$ of order r, and let ε denote a primitive rth root of unity. It is clear that, for any t, $0 \leq t \leq b-1$, $\mathrm{Spec}(g, V_{i_t})$ contains some power ε^{m_t}. In this case, $\mathrm{Spec}(g, U)$ contains

$$\mu = \left(\varepsilon^{m_0+m_1+\ldots+m_{b-1}}\right)^{1+p^b+p^{2b}+\ldots+p^{(c-1)b}} = \varepsilon^n,$$

where $n = \sum_i m_i \cdot \frac{p^a-1}{p^b-1}$. If $b < a$, then $r|n$ by the choice of r, and so $\mu = 1$, a contradiction. Hence, $b = a$, and $\Phi = 1$. $\circ$

Remark 4.8 *If $p = 3$, then p' - semiregular irreducible Brauer characters need not have degree 2. For example, the 6-dimensional module $V_2 \otimes V_1^{(3)}$ of $SL_2(27)$ is $3'$ - semiregular. The 6-dimensional module $V_2 \otimes V_1^{(9)}$ of $SL_2(81)$ also is $3'$ - semiregular.*

Proposition 4.9 *${}^2B_2(2^{2a+1})$ is $2'$ - semiregular in characteristic 2 for any $a \geq 1$. Moreover, an absolutely irreducible ${}^2B_2(2^{2a+1})$-module in characteristic 2 is $2'$ - semiregular if and only if this module has dimension 4. In particular, any $2'$ - semiregular irreducible Brauer character of ${}^2B_2(2^{2a+1})$ is not stable under nontrivial field automorphisms of this group.*

Proof: Set $G = {}^2B_2(q)$ and $q = 2^{2a+1}$.

1) Consider the standard embedding of G in $Sp_4(q) = Sp(V)$, $V = \mathbb{F}_q^4$. Assume (G, V) is not $2'$ - semiregular. Then by 1.2 $\mathrm{Spec}(g, V) \ni 1$ for some element $g \in G$ of odd prime order r. In particular, g fixes some nonzero vector $v \in V$. The stabilizer of v in $Sp_4(q)$ has order $q^4(q^2-1)$. This implies that r divides $(q^2-1, (q^2+1)(q-1)) = q-1$. Therefore, without loss of generality, one may suppose that g is contained in the cyclic subgroup

$$C = \left\{\mathrm{diag}(x^{1+\vartheta^{-1}}, x^{\vartheta^{-1}}, x^{-\vartheta^{-1}}, x^{-1-\vartheta^{-1}}) \mid c \in \mathbb{F}_q^*\right\}$$

of order $q-1$ of G. Here, $\vartheta(x) = x^{2^{a+1}}$ for $x \in \mathbb{F}_q$. Direct computation then shows that $\mathrm{Spec}(g, V)$ can contain 1 only in the case where $g = 1$, a contradiction.

It is clear that all G-modules V_i, $1 \leq i \leq 2a+1$, obtained from V by means of Frobenius twist also are $2'$ - semiregular.

2) Now let U be a $2'$ - semiregular irreducible $\overline{\mathbb{F}_2}G$-module. Then $\dim U > 1$. If $\dim U = 4$, then $U \simeq V_i$ for some i. Assume $\dim U > 4$. Then it is well known that $U \simeq V_{i_1} \otimes \ldots \otimes V_{i_s}$ for some $s \geq 2$ and $1 \leq i_1 < \ldots < i_s \leq 2a+1$. Consider an element $g \in G$ of order 5, and let ε denote a primitive 5th root of unity. Observe that $\mathrm{Spec}(g, V_i) = \{\varepsilon^j \mid 1 \leq j \leq 4\}$. Therefore, $\mathrm{Spec}(g, U) \ni \mu = \varepsilon^{a_1 + \ldots + a_s}$ for any a_i, $1 \leq a_i \leq 4$. Choosing $a_3 = a_5 = \ldots = 1$, $a_4 = a_6 = \ldots = 4$, $a_1 = 1$ and $a_2 = 4$ if s is even, and $a_1 = a_2 = 2$ if s is odd, we see that $\mu = 1$, a contradiction. $\circ$

Completion of the proof of Theorem 4.1

Assume (G, V) is p' - semiregular in characteristic p with $G = G'$ and $O_p(G) = 1$. Then by Propositions 2.5, 2.6, 4.5, 4.6 and Lemmas 4.3, 4.4, the pair (G, p) is as listed in 4.1. Conversely, suppose that (G, p) satisfies any of the conditions (i) – (iv) of 4.1. If (G, p) satisfies (i) or (ii), then G is p' - semiregular by Propositions 4.7, 4.9. If (G, p) satisfies (iii), then G is the direct product of two subgroups $A = {}^2B_2(2^{2a+1})$, $B = SL_2(2^{2b+1})$. We already know that A, B are $2'$ - semiregular. The condition $(2a+1, 2b+1) = 1$ implies that the odd parts of $|A|$, $|B|$ are coprime. Therefore, G is $2'$ - semiregular. (Namely, if (A, U) and (B, V) are $2'$ - semiregular, then $(G, U \otimes V)$ is $2'$ - semiregular.) Finally, if (G, p) satisfies (iv), then G is p' - semiregular by Propositions 4.5, 4.6. $\circ$

5 Perfect p' - semiregular Groups: The separable case

Now let $G = G'$ be p' - semiregular over $\mathbb{C}$. By 1.3 (i) it is clear that $G/O_p(G)$ is p' - semiregular over a suitable field $\mathbb{F}$ of characteristic p with a faithful module V. Hence $G/O_p(G)$ appears in the list of 4.1.

First we rule out the Suzuki groups:

Lemma 5.1 *Let $q := 2^{2m+1}$ and let $G = Sz(q) = {}^2B_2(q)$ be the Suzuki group of order $(q^2+1)q^2(q-1)$. Then G is not p' - semiregular over $\mathbb{C}$ for any prime p.*

Proof: Since the Sylow 2 - groups of G are neither quaternion nor cyclic, the only candidate for p' - semiregularity is $p = 2$. We have to show that every (irreducible) complex representation has g - fixed points for some $2'$ - element g. We will show that for every $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ there is a nontrivial cyclic $2'$ - group H such that $(\chi|_H, 1_H) > 0$:

We consider G as the fixed - point group in $Sp_4(\bar{\mathbb{F}}_q)$ of a generalized Frobenius - endomorphism, which interchanges long and short roots. Let $r := 2^m$. Note that G has three 'maximal tori': the 'split torus' $T_0 \cong \mathbb{Z}_{q-1}$ and the tori $T_1 \cong \mathbb{Z}_{q-2r+1}$ and $T_2 \cong \mathbb{Z}_{q+2r+1}$ satisfying $C_G(T_i) = T_i$. Let $B := UT_0$ be the Borel - subgroup of G with $U \in \mathrm{Syl}_2(G)$. It is well known that each semisimple (here $2'$ -) element of G can be conjugated into a 'maximal torus' of G (here T_i for $i \in \{0, 1, 2\}$). Since all tori of G are self - centralizing, each element of G is either unipotent, hence conjugate to an element of U, or semisimple, hence conjugate to an element of T_i. It is easy to see that $\gcd(|T_i|, |T_j|) = 1$ for $i \neq j$, so the conjugates of $U\backslash\{1\}$ and the $T_i\backslash\{1\}$ form a partition of $G\backslash\{1\}$, which is of course well known.

By Harish - Chandra theory ([5]) we know that

$$\mathrm{Irr}_{\mathbb{C}}(G) = \{1, St, \tilde{\lambda}_i^G \mid \lambda_i \in \mathrm{Irr}_{\mathbb{C}}(T_0)\backslash\{1\}\} \cup \{\chi \mid \chi \ cuspidal\}.$$

Here $\tilde{\lambda}$ denotes the inflation of λ to a character of the Borel subgroup; the first subset consists of the 'principal series' and the second of the 'discrete series' of characters. In particular $\tilde{\lambda}_i^G = \tilde{\lambda}_j^G \iff \lambda_i = \overline{\lambda}_j$. So we will assume that the $\lambda_1, ..., \lambda_{\frac{q-2}{2}}$ are pairwise different and not inverse.
We have $N_G(T_0)/T_0 = \langle w_0 \rangle \cong \mathbb{Z}_2$ with w_0 the longest element in the Weyl - group B_2 and ${}^w t = t^{-1}$ for all $t \in T_0$. ¿From this we see immediately:

$$\tilde{\lambda}_i^G(t_0) = \lambda_i(t_0) + \overline{\lambda}_i(t_0) \ \forall \ t_0 \in T_0\backslash\{1\}.$$

Moreover $\tilde{\lambda}_i^G(t_i) = 0 \ \forall t_i \in T_i\backslash\{1\} \ i = 1, 2$. In particular we conclude

$$(\tilde{\lambda}_i^G|_{T_i}, 1_{T_i}) = \frac{q^2+1}{|T_i|} \ for \ i = 1, 2.$$

Now the second orthogonality relation of irreducible characters yields for any $t \in T_0$: $q - 1 = |C_G(t)| = \sum_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} \chi(h)\overline{\chi}(h) =$

$$2 + 2 \cdot \frac{q-2}{2} + \sum_{i=1,\ldots,\frac{q-2}{2}} (\lambda_i^2(h) + (\overline{\lambda})^2(h)) + \sum_{\chi \ cuspidal} \chi(h)\overline{\chi}(h) =$$

$$q - 1 + \sum_{\chi \ cuspidal} \chi(h)\overline{\chi}(h).$$

Hence $\chi(h) = 0$ for all $h \in T_0\backslash\{1\}$ and cuspidal χ. We conclude $(\chi|_{T_0}, 1|_{T_0}) = \frac{\chi(1)}{q-1} > 0$ for each cuspidal χ. Clearly $(St|_B, 1|_B) = (St, 1_B^G) = 1 > 0$, hence $(St|_{T_0}, 1|_{T_0}) > 0$. This finishes the proof. $\circ$

Proposition 5.2 *Assume that G is a faithful, separably p' - semiregular group with $G = G'$ and $O_p(G) = 1$. Then one of the following holds.*

(i) *$G \cong SL_2(5)$.*

(ii) *$G \cong SL_2(r)$ with $r \in \mathcal{R} \cup \{7, 9, 17\}$, and $p = 3$.*
Conversely, if (G, p) satisfies any of the conditions (i), (ii)*, then there exists a faithful G-module V in characteristic* 0 *such that G acts p' - semiregularly on V.*

Proof: 1) As $SL_2(5)$ is semiregular, this group is separably p' - semiregular for all primes p. Furthermore, if $(G,p) = (SL_2(r), 3)$ and $r \in \mathcal{R} \cup \{17\}$, then the 3-Brauer characters of degree $(r-1)/2$ are liftable to characteristic zero, which implies that G is separably $3'$ - semiregular. Also, the 3-Brauer characters of degree 6 of $SL_2(7)$ are liftable, therefore $SL_2(7)$ is separably $3'$ - semiregular.

¿From now on we suppose that ψ is a separably p' - semiregular character of G. Consider an irreducible constituent χ of ψ. Then χ also is p' - semiregular. Furthermore, Ker χ is a p-group, and so it is trivial. Thus, χ is a faithful irreducible p' - semiregular character of G.

2) Next we check $G = SL_2(q)$, $q = 2^\gamma$, $\gamma \geq 2$, for separably $2'$ - semiregularity using again the notation of [6]. Consider the cyclic subgroup $A = \langle a \rangle$ of order $q-1$. Direct computation shows that $(\chi|_A, 1_A)_A = 2$, or 1, according to as χ is the Steinberg character, or one of the remaining irreducible characters, a contradiction.

3) Here we suppose that $G = SL_2(q)$, $q = p^\gamma \geq 5$, $p > 2$. We use the notation for characters and conjugacy classes of G used in [6]. As χ is faithful, χ is one of the characters χ_i, θ_j, i,j odd, ξ_s, η_s, $s = 1,2$. Let A denote the cyclic subgroup $\langle b^2 \rangle$ of order $(q+1)/2$ if $q \equiv 1(\mod 4)$, and the cyclic subgroup $\langle a^2 \rangle$ of order $(q-1)/2$ if $q \equiv -1(\mod 4)$. If $\chi = \chi_i$, θ_j, or ξ_s, then $(\chi|_A, 1_A)_A = 2$, a contradiction. Therefore $\chi = \eta_s$ (and $q \equiv 1(\mathrm{mod} 4)$). Suppose $q-1$ has an odd prime divisor t. Then consider the p'-element $x = a^l$ with $l = (q-1)/t$ and the cyclic subgroup C generated by x. It is clear that $(\eta_s|_C, 1_C)_C = (q-1)/2t > 0$. In particular, Spec$(x,\chi) \ni 1$, a contradiction. Hence, $q - 1 = 2^\alpha$ for some $\alpha \geq 2$. Recall that $q \equiv 1(\mathrm{mod} 4)$. So, $q+1$ has an odd prime divisor t. Consider the p'-element $y = b^m$ with $m = (q+1)/t$ and the cyclic subgroup D generated by y. Here we have $(\eta_s|_D, 1_D)_D = (q+1-2t)/t$, which implies that $2^{\alpha-1}+1 = (q+1)/2 = t$ is prime. In particular, $\alpha = 2^\beta + 1$ for some $\beta \geq 0$. It is easy to see that the equality $p^\gamma = 2^{2^\beta+1} + 1$ is impossible for $\beta \geq 2$. If $\beta = 0$, then $q = 5$. If $\beta = 1$, then $q = 9$, $(q+1)/2 = 5$, and we obtain that $SL_2(9)$ is separably $3'$ - semiregular.

We have shown that $SL_2(p^\gamma)$ is separably p' - semiregular if and only if $p^\gamma = 5$ or 9. Together with 5.1 the proof is finished. $\circ$

We shall need some auxiliary statements concerning $SL_2(4) \cong A_5$.

Lemma 5.3 (i) *Let V_4 denote the 4-dimensional absolutely irreducible A_5-module in characteristic 2. Then $H^i(A_5, V_4) = 0$ for $i = 1,2$.*

(ii) *Assume that H is a finite group such that $H/O_2(H) \cong A_5$ and elements of order 5 in H act fixed-point-freely on $O_2(H) \neq 1$. Then H splits over $O_2(H)$.*

Proof: (i) Clear, because V_4 is the (projective) Steinberg module for $SL_2(4)$.
(ii) See [15]. $\circ$

We also need the following simple arithmetic observation.

Lemma 5.4 *Let $H \in \{L_2(q), {}^2B_2(q) \mid q = 2^a \neq 2, 4, 16, 256, 65536\}$. Then H contains elements of some prime order r with $r \neq 2$ and r is not a Fermat prime.*

Proof: First consider the case where a has an odd prime divisor t. Then $|H|$ is divisible by $2^t - 1$, and $2^t - 1$ is coprime to every Fermat prime. So we can choose r to be any prime divisor of $2^t - 1$. Now suppose that $a = 2^b$. Then $H = L_2(q)$, and $b \geq 5$. In this case, $q^2 - 1 = 2^{2^{b+1}} - 1$ is divisible by $2^{2^5} + 1$ and so by 641. Set then $r = 641$. $\circ$

Proposition 5.5 *Assume that (G, V) is a separably p' - semiregular pair such that V is irreducible, $G = G'$, $O_p(G)$ is not cyclic, and $G/O_p(G) \in \{SL_2(q), {}^2B_2(2^{2a+1})\}$. Then $p = 2$ and $G = ES$ is the uniquely determined extension of $E = O_2(G) \cong 2_-^{1+4}$ and $S \cong SL_2(5)$ and $E \cap S = Z(E) = Z(S) = Z(G) \cong \mathbb{Z}_2$. Conversely, this extension is a perfect separably $2'$ - semiregular group.*

Proof: Put $M := \mathcal{M}_{p'}(G)$ (as introduced in 1.8) as well as $P := O_p(G)$. The hypothesis on the structure of G/P clearly implies that $G = PM$. Since both P and M are normal in G and since $G = G'$ we get $G = M$. Moreover, application of 1.9 now yields $P = E * Z$ with an extraspecial p - group E and a cyclic group Z such that $Z(E) = \Omega_1(Z) \leq Z = Z(P) \leq Z(G)$.
Consider the normal subgroup $R := PC_G(P)$. Since $R^\infty = C_G(P)^\infty$ and $G = G'$ as well as $P \not\leq Z(G)$, R is a proper subgroup of G contained in $O_{\{p,2\}}(G)$. Again we apply 1.9 to see that $p = 2$ and thus $R = P$ as well as $C_G(P) = Z$; in particular any odd prime divisor of $|G|$ is a Fermat prime. Hence, by 5.4, $G/P \cong SL_2(q)$ with $q \in \{2^2, 2^4, 2^8, 2^{16}\}$.

Write $E = 2_\pm^{1+2\ell}$ for a suitable $\ell \geq 1$ and put $\bar{G} = G/Z$. Note that $\bar{P} = O_2(\bar{G})$ is a 2ℓ - dimensional module over $\mathbb{F}_2$ for $L := \bar{G}/\bar{P}$, say with Brauer character φ. At this point we prove:

$$q = 4 \text{ and } \varphi = (2\ell - 4) \cdot \vartheta_1 + \vartheta_4,$$

where $\vartheta_i \in \mathrm{IBr}_2(L)$ and $\deg \vartheta_i = i$, $i = 1, 4$.
For this purpose, we first show that

$$\dim \mathrm{Ker}(\bar{g} - 1) = 2\ell - 2^{c+1} \tag{1}$$

for $g \in G$ with $\bar{g}$ of prime order $2^{2^c} + 1$ for some $c \geq 0$. Without loss we may suppose that $|g| = |\bar{g}|$. Denote by χ the G-character afforded by V. Then 1.9 implies that $\chi|_P = m\psi$, where ψ is a faithful irreducible character of P (of degree 2^ℓ). It is clear that ψ is extendible to an irreducible character $\tilde{\psi}$ of $\tilde{P} = P \cdot \langle g \rangle$. Then,

$$\chi|_{\tilde{P}} = \sum_{i=1}^{m} \tilde{\psi}\alpha_i, \tag{2}$$

where $\alpha_i \in \mathrm{Irr}_{\mathbb{C}}(\langle g \rangle)$. By 1.7, one may choose $\tilde{\psi}$ such that

$$\tilde{\psi}(g) = \pm 2^{\frac{1}{2}\dim \mathrm{Ker}(\bar{g}-1)}. \tag{3}$$

Let ω denote a $|g|$th primitive root of 1. From (3) it follows that $\mathrm{Spec}(g, \tilde{\psi}) \ni \omega^j$ for any $\omega^j \neq 1$. But $\mathrm{Spec}(g, \chi) \not\ni 1$, therefore in (2) we must have $\alpha_i(g) = 1$ for all i. In particular, $\chi(g) \in \mathbb{Z}$. But $\mathrm{Spec}(g, \chi) \not\ni 1$, hence $\chi(g) = -\chi(1)/2^{2^c}$. Using (2) and (3) again, we see that $\dim \mathrm{Ker}(\bar{g} - 1) = 2\ell - 2^{c+1}$, as claimed in (1).

Assume $q = 2^{2^b} \geq 16$. Then $L \cong L_2(q) = L_2(16^{2^{b-2}})$ contains a subgroup $K \cong L_2(16)$. We shall apply (1) to elements $g \in K$ of order 3, 5, and 17. Let $\theta_{i,j}$ denote the 2-Brauer characters of K of degree i, $i = 1, 2, 4, 8, 16$, $j = 1, 2, \ldots$ (Here we name the $\theta_{i,j}$ in the same order as they are listed in [14]). Writing

$$\varphi = a_{1,1}\theta_{1,1} + \sum_{j=1}^{4} a_{2,j}\theta_{2,j} + \sum_{j=1}^{6} a_{4,j}\theta_{4,j} + \sum_{j=1}^{4} a_{8,j}\theta_{8,j} + a_{16,1}\theta_{16,1}, \tag{4}$$

one has $\sum_{i,j} i a_{i,j} = 2\ell$. Applying (1) and (4) to elements g of order 3, we get

$$a_{1,1} + 2\sum_j a_{4,j} + 2\sum_j a_{8,j} + 6a_{16,1} = 2\ell - 2.$$

In particular, $a_{8,j} = 0$, $a_{16,1} = 0$, and $\sum_j a_{2,j} + \sum_j a_{4,j} = 1$. Applying (2) to elements g of order 5, we obtain

$$a_{1,1} + 2a_{4,1} + 2a_{4,2} = 2\ell - 4.$$

Hence, $a_{4,1} = a_{4,2} = a_{2,j} = 0$, $\sum_{j=3}^{6} a_{4,j} = 1$, and $a_{1,1} = 2\ell - 4$. The last equality implies that $\dim \mathrm{Ker}(\bar{g} - 1) \geq 2\ell - 4$ for any $g \in G$. But this contradicts (1) applied to elements of order 17.
So now $q = 4$ and $L \cong L_2(4)$. Calculations with the Brauer characters of $L_2(4)$ (similar to those we have performed for K) now yield $\varphi = (2\ell - 4) \cdot \vartheta_1 + \vartheta_4$, as claimed.

Since $\bar{G}$ is perfect and ϑ_4 is projective, we get a decomposition $\bar{P} = \bar{P}_1 \oplus \bar{P}_4$ such that $\bar{P}_1 = C_{\bar{P}}(\bar{G})$ corresponds to $(2\ell - 4)\vartheta_1$ and $\bar{P}_4 = [\bar{P}, \bar{G}]$ corresponds to ϑ_4. Since $\dim \bar{P}_1 = 2\ell - 4$ is even and since L has Schur multiplier isomorphic to $\mathbb{Z}_2$, the fact that $\bar{G}$ is perfect now implies that $2\ell = 4$ and hence $\bar{P} = \bar{P}_4$. Moreover, by 5.3, $\bar{G}$ splits over $\bar{P}$. So there exists a supplement S of P in G; without loss of generality we may assume that S is perfect. As $Z \leq Z(G)$ and L has Schur multiplier $\mathbb{Z}_2$, $S_0 := S \cap Z \leq \Omega_1(Z)$ and $S/S_0 \cong L$. Also note that G cannot contain subgroups isomorphic to A_5; hence $S_0 \cong \mathbb{Z}_2$ and $S \cong SL_2(5)$.

If $|Z| \geq 4$, we apply similar arguments as above to the group G/Z^2 ; using the fact that ϑ_4 is projective we derive the contradiction $G' < G$. Therefore $Z = Z(E) \cong \mathbb{Z}_2$ and so $P = E$ as well as $G = PS$ with $P \cap S = Z$. As A_5 is not involved in $\mathrm{Out}(2_+^{1+4}) = O_4(2)$, we even have $P \cong 2_-^{1+4}$.

Conversely, embed the extraspecial 2-group $E = 2_-^{1+4}$ in $GL_4(\mathbb{C})$ via its irreducible 4-dimensional representation. Then the normalizer of E in $GL_4(\mathbb{C})$ contains a perfect subgroup $G = E \cdot A_5$. We verify that the corresponding 4-dimensional representation Φ of G is $2'$ - semiregular. If $g \in G$ is of order 3, then one can identify $\langle g \rangle$ with the subgroup $\Omega_2^-(2)$ in

$$2_-^{1+2} \cdot \Omega_2^-(2) * 2_+^{1+2} \cdot \Omega_2^+(2) \leq G.$$

This identification shows that every g-eigensubspace is either 0 or a faithful 2_+^{1+2}-module, and so has dimension 0, 2, or 4. Also, g is real, therefore g acts fixed-point-freely in Φ. This conclusion implies that the reduction mod 2 of Φ is a sum of two 2-dimensional irreducible $\mathbb{F}_2 A_5$-modules. From this it follows that every element of order 5 acts fixed-point-freely in Φ. To prove the uniqueness statement in the theorem, see [9]. $\circ$

Now we are in a position to prove the main result of this section:

Theorem 5.6 *Assume that G is a (nontrivial) perfect finite group which is p' - semiregular over $\mathbb{C}$. Then one of the following holds.*

(i) $p = 2$ *and G is a perfect group admitting a collection $K_1, \ldots, K_m$ of normal 2-subgroups with the following properties:*

a) $\cap_{i=1}^m K_i = 1$;

b) Each G/K_i is isomorphic either to $SL_2(5)$ or to $2_-^{1+4} \cdot A_5$;

c) If $G/K_i \cong G/K_j \cong SL_2(5)$ then $K_i = K_j$.

(ii) $p = 3$ *and* $G \cong SL_2(r)$ *with* $r \in \mathcal{R} \cup \{7, 9, 17\}$.

(iii) $p \geq 5$ *and* $G \cong SL_2(5)$.

Conversely, if (G,p) satisfies any of the conditions (i) – (iii), *then there exists a faithful G-module V in characteristic 0 such that G acts p' - semiregularly on V.*

Proof: Let (G,V) be a faithful separably p' - semiregular pair with $G = G'$. In view of 5.2 we may assume that $P := O_p(G) \neq 1$. Clearly G/P is a perfect p' - semiregular group in characteristic p, and thus is determined by 4.1. Similarly as in the proof of 5.5 we verify that $G = \mathcal{M}_{p'}(G)$.
First we study the case where V is irreducible; so in particular $Z(G)$ is cyclic. First assume $P \leq Z(G)$. The structure of G/P then immediately implies that $P = Z(G)$ if $p = 2$ and that $Z(G)/P \cong \mathbb{Z}_2$ if $p \neq 2$. So in any case G is a covering group of $\bar{G} := G/Z(G)$ and $Z(G)$ is isomorphic to a quotient of the Schur multiplier of $\bar{G}$ of order divisible by p. The relevant Schur multipliers are all known; inspection of the cases (i) (ii) and (iv) in 4.1 then leaves the following possibilities for (G,p):

$$(2SL_2(4), 2),\ (3L_2(9), 3),\ (6L_2(9), 3),\ (2\ {}^2B_2(8), 2).$$

A direct check now reveals that among these four only the group $2SL_2(4) \cong SL_2(5)$ survives. This in turn implies that in our situation G/P cannot satisfy condition (iii) of 4.1.
Henceforth we may assume that $P \not\leq Z(G)$. Thus, by 1.9 $P = E * Z$ with an extraspecial p - group E and a cyclic p - group Z such that $Z = Z(P) \leq Z(G)$. Application of 5.5 in

combination with 4.1 now shows that $p = 2$ and $G \cong 2_-^{1+4}A_5$.
So the claimed results hold in case V is irreducible.

In the next step we suppose that that the character χ of V is a sum $\sum_{i=1}^{m} \chi_i$ with $\chi_i \in \mathrm{Irr}_{\mathbb{C}}(G)$ and $m > 1$. Put $K_i = \mathrm{Ker}\,\chi_i$ and note that $K_i \leq P$, and that G/K_i is a faithful irreducible separably p' - semiregular group; moreover, as χ is faithful, $\cap_{i=1}^{m} K_i = 1$. Note that we can apply the results of this proof obtained so far to the groups G/K_i. If $p > 2$ then $O_p(G/K_i) = 1$ and thus $P = K_i$ for any i, but then $P = \cap K_i = 1$, a contradiction. Therefore $p = 2$, and condititions $a)$ and $b)$ of (i) hold. Studying the natural embedding $G/(K_1 \cap K_2) \hookrightarrow G/K_1 \times G/K_2$ and using again that A_5 has Schur multiplier isomorphic to $\mathbb{Z}_2$ the claim in $c)$ follows because G is perfect. Hence case (i) holds.

Finally suppose (G,p) is as listed in our theorem. In the cases (ii), (iii), G is separably p' - semiregular by 5.2. Consider case (i). Then G/K_i has a faithful irreducible $2'$ - semiregular character, say, χ_i. Clearly, $\sum_{i=1}^{m} \chi_i$ is a faithful $2'$ - semiregular G-character. $\circ$

Remark 5.7 *It is easy to construct groups G satisfying part* (i) *of the preceding theorem such that for the relevant $2'$ - semiregular pair (G,V) the G - module V is not irreducible.*
(i) *By [9] ((6.7) pg. 124) there exists a group $G = E : S$ where $S \cong SL_2(5)$ normalizes $E \cong 2_-^{1+4}$ and $Z(G) = Z(E) \times Z(S)$. Clearly, G is separably $2'$ - semiregular over a 6 - dimensional space V being a sum of two irreducibles of dimension 2 and 4, respectively.*
(ii) *Putting $G = (E_1 \times E_2)S$ such that $E_i \cong 2_-^{1+4}$ is normalized by $S \cong SL_2(5)$ and $Z(S) < Z(G) = Z(E_1) \times Z(E_2)$ as well as $Z(S) \neq Z(E_i)$ for $i = 1,2$ we easily verify that G is separably $2'$ - semiregular with an 8 - dimensional space V which is the direct sum of 2 irreducibles of dimension 4.*

6 Non - solvable p' - semiregular Groups

In this final chapter we generalize the results of chapters 4 and 5 to describe the nonsolvable p' - semiregular groups. For this purpose we assume that G is a nonsolvable finite group and V is a faithful G - module such that the pair (G,V) is p' - semiregular. In order to simplify the description of G we may and will assume that $G = O^p(G)$ (see 1.13). Moreover, if (G,V) is inseparable we even may assume without loss that $O_p(G) = 1$ (see 1.3).
Notice that by 4.1 and 5.6 the possible structures for the group G^∞ are known. For the sake of convenience we put $K_1 = C_G(G^\infty)$ as well as $G_1 = G^\infty K_1$. So $G/G_1 = \mathrm{Out}_G(G^\infty)$, the group of outer automorphisms of G^∞, induced by elements of G.

Assume first that $p \neq 2$ and let $S \in \mathrm{Syl}_2(G)$ as well as $Z := Z(G^\infty)$. By 4.1 and 5.6 we have $G^\infty \cong SL_2(q)$ for a suitable odd q; in particular, $Z = G^\infty \cap K_1 \cong \mathbb{Z}_2$ and S is quaternion. Clearly, $Z \in \mathrm{Syl}_2(K_1)$ and thus $K_1 = Z \times K_0$ where $K_0 = O(K_1) = O(G)$;

moreover, $G_1 = G^\infty \times K_0$, $\pi(G^\infty) \cap \pi(K_0) \subseteq \{p\}$ and K_0 is p' - semiregular and solvable. Suppose now that G_1 is a proper subgroup of G. If q is a prime then $\mathrm{Aut}(G^\infty) \cong PGL_2(q)$ and so $|G : G_1| = 2$; furthermore, $G_0 := G^\infty S = G^\infty\langle\delta\rangle$ contains G^∞ with index 2 such that δ induces a diagonal automorphism on G^∞. Clearly, as S is quaternion, G_0 cannot split over G^∞; it is easy to verify that such an extension G_0 exists within the group $SL_2(q^2)$ and that it is unique up to isomorphism.
Conversely, by 1.6 and 1.9, we can see that the groups $G_1 = G^\infty \times O(G)$, $G_0 = G^\infty S$ and $G = O(G) : G_0$ are indeed p' - semiregular.

We are left to consider the case $q = p^a$ with $a \geq 2$. Recall that now $\mathrm{Out}(G^\infty) = \langle\delta\rangle \times \langle\phi\rangle$ where δ is a diagonal automorphism of order 2 and where ϕ is the Frobenius automorphism of order a. Since S' is cyclic of index 4 in S and since $S_\infty := S \cap G^\infty$ is quaternion, we get $|S : S_\infty| = |G : G_1|_2 \leq 2$. Therefore $G = G_1\langle\beta\rangle$ such that $c := |G : G_1| = 2^e b$ with $0 \leq e \leq 1$, $b|a$ and $(b, 2p) = 1$.

Suppose now that r is a prime dividing b such that $r \notin \pi(K_0)$. Considering the subgroup H of G containing G_1 with index r an easy inspection of the structure of a Sylow p - normalizer of H/K_0 then shows that H splits over G_1. Therefore G contains an element ρ of order r inducing a nontrivial field automorphism on G^∞; now ρ also acts nontrivially on the diagonal torus of G^∞. The presence of Zsigmondy primes shows that G does not satisfy the $\mathcal{Z}_{p'}$ - property; in particular, G cannot be p' - semiregular. This contradiction shows that all odd prime divisors of b are already contained in $\pi(K_0)$; in particular, $(b, |G^\infty|) = 1$.

Putting $G_2 = G_1\langle\beta^2\rangle$ we then have $G_2 = G^\infty : K_2$ with $K_0 \triangleleft K_2$ and $K_2/K_0 \cong G_2/G_1 \cong \mathbb{Z}_b$; moreover, K_2 is a p' - semiregular group of odd order with $\pi(K_2/K_0) \subseteq \pi(K_0)$ and $\mathcal{M}_{p'}(K_2) \leq K_0$. So G_2 is a p' - semiregular group with $|G : G_2| = |S : S_\infty| = 2^e \leq 2$.

Henceforth we may assume that $|G : G_2| = 2$ and thus $G = G_2\langle\tau\rangle$ with $\langle\tau\rangle = O_2(\langle\beta\rangle)$ and $S = S_\infty\langle\tau\rangle$.
If τ induces a diagonal automorphism on G^∞, then the nonsplit extension $G_0 = G^\infty S = G^\infty\langle\tau\rangle$ of G^∞ can be realized within $SL_2(q^2)$ and is unique up to isomorphism for the same reasons as above. So we may assume that a is even and that τ induces ϕ_2 or $\delta\phi_2$ on G^∞ where $\phi_2 = \phi^{a/2}$ and δ is the diagonal automorphism on G^∞ induced by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
As $q = p^a$ with a even, 4 divides $q - 1$; therefore without loss $S_\infty = \langle z, y\rangle$ where $y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ with $\langle\epsilon\rangle = O_2(\mathbb{F}_q^*)$.

Clearly, S_∞ is τ - invariant and τ^2 acts trivially on S_∞; so $\tau^2 \in Z(S_\infty)$. As S is quaternion, we even must have $\tau^2 \neq 1$, i.e. $\tau^2 = y^2$.
If τ is induced by ϕ_2, then $y^\tau = y$ and so the order of $y\tau$ is two. If τ is induced by $\delta\phi_2$, then τ centralizes $x = z_4\tau$ where $z_4 \in \langle z\rangle$ has order 4; but now $o(x\tau) = 2$. So in any of these two cases S splits over S_∞, a contradiction. This completes the analysis of the case $p \neq 2$. Finally we observe that conversely, by 1.6 and 1.9, the groups G_1, G_2, G_0 and G

as described above are p' - semiregular, indeed.

Now we are left to investigate the case when $p = 2$. Suppose first that the pair (G, V) is inseparable and thus $O_2(G) = 1 = Z(G^\infty)$ (see 4.1). Clearly, now $\pi(G^\infty) \cap \pi(K_1) \subseteq \{2\}$ and so $G_1 = G^\infty \times K_1$ with a $2'$ - semiregular solvable group K_1. Moreover, as $G/G_1 = \mathrm{Out}_G(G^\infty)$ is abelian and as $G = O^2(G)$, G/G_1 is a cyclic group of odd order d, say. If $G = G_1$ we are done; so assume that G_1 is a proper subgroup of G. Suppose that $G^\infty = A_1 \times A_2$ where $A_1 \cong {}^2B_2(2^{a_1})$ and $A_2 \cong SL_2(2^{a_2})$ with odd $a_1, a_2 \geq 3$ such that $(a_1, a_2) = 1$. Recall that $\mathrm{Aut}(A_i)/A_i \cong \mathbb{Z}_{a_i}$; an inspection of the local structure of a Sylow 2 - normalizer of $\mathrm{Aut}(A_i)$ now reveals that $\mathrm{Aut}(A_i)$ splits over A_i for $i = 1, 2$.
Since $\mathrm{Aut}(G^\infty) \cong \mathrm{Aut}(A_1) \times \mathrm{Aut}(A_2)$ and $(a_1, a_2) = 1$, we get $G = (A_1 \times A_2 \times K_1)\langle\varphi\rangle$ where $\varphi = \varphi_1\varphi_2 = \varphi_2\varphi_1$ such that φ_i induces an automorphism of order d_i on A_i and acts trivially on A_j with $i \neq j$; in particular $|G/G_1| = d_1 d_2 = d$.
Since G satisfies the $\mathcal{Z}_{2'}$ - property, we deduce that $(|A_i|, d_j) = 1$ as well as $\pi(G/G_1) \subseteq \pi(K_1)$. Therefore $G = G^\infty : K$ with $K_1 \leq K$ and $K/K_1 \cong G/G_1 \cong \mathbb{Z}_d$; moreover $\mathcal{M}_{2'}(K) = \mathcal{M}_{2'}(K_1) \leq K_1$ and K is a solvable $2'$ - semiregular group with $\pi(K) \cap \pi(G^\infty) \subseteq \{2\}$.

Clearly, if $G^\infty \cong SL_2(2^a)$ or $G^\infty \cong {}^2B_2(2^{2b+1})$ with $a \geq 2$, $b \geq 1$ then the arguments just presented can be modified to yield the analogous results.
Also observe that conversely, by 1.6 and 1.9, any of the groups G described above is $2'$ - semiregular.

So, finally we can assume that the pair (G, V) is separable and thus $G^\infty/Q \cong A_5$ with $Q := O_2(G^\infty) \neq 1$, i.e. case (i) of 5.6 holds for G^∞. As $G = O^2(G)$ and $\mathrm{Aut}(A_5) \cong \Sigma_5$, we get $G = G^\infty K$ where $K = C_G(G^\infty/K)$ is a solvable $2'$ - semiregular $\{3,5\}'$ - group with $\pi(G^\infty) \cap \pi(K) \subseteq \{2\}$ and $G^\infty \cap K = Q$.
Conversely, by 3.4, we easily verify that any group $G = G^\infty K$ with K as described above is $2'$ - semiregular.

So we have obtained the following:

Theorem 6.1 *Let (G, V) be a p' - semiregular pair for some prime p with a faithful G - module V and $O^p(G) = G$. If (G, V) is inseparable assume in addition that $O_p(G) = 1$. Then G^∞ is as described in 4.1 or 5.6; furthermore one of the following holds:*

(I) *$p \geq 3$ and $G^\infty \cong SL_2(q)$ for some suitable odd prime power q such that*

a) $S \in \mathrm{Syl}_2(G)$ is (gen.) quaternion and $Z := Z(G^\infty) \cong \mathbb{Z}_2$.

b) $C_G(G^\infty) = Z \times O(G)$ and $G_1 := G^\infty C_G(G^\infty) = G^\infty \times O(G)$ are p' - semiregular.

c) If q is a prime then either $G = G_1$ or $G = G_1\langle\delta\rangle$ where δ induces a diagonal automorphism of order 2 on G^∞.

d) If q is not a prime then $q = p^a$ with $a \geq 2$ and G contains a subgroup G_2 of index at most 2 such that $G_2 = G_1\langle\beta\rangle = G^\infty : K$ and $K = O(G)\langle\beta\rangle$ with β inducing a field automorphism on G^∞ of odd order b dividing a; also $G_2/G_1 \cong K/O(G) \cong \mathbb{Z}_b$ with $\pi(K/O(G)) \subseteq \pi(O(G))$ and $\mathcal{M}_{p'}(K) \leq O(G)$.
Furthermore if $|G/G_2| = 2$, then $G = G_2\langle\delta\rangle$ with δ as described in c) and $G_0 = G^\infty\langle\delta\rangle \cong 2PGL_2(q)$ with $S \in \mathrm{Syl}_2(G_0)$.

(II) *$p = 2$ and one of the following holds:*

a) $G^\infty \cong SL_2(2^n)$ or ${}^2B_2(2^n)$ with $n \geq 2$ or $n = 2m+1 \geq 3$, respectively. Moreover, $G = G^\infty : K$ where K is a solvable $2'$ - semiregular group such that $K_1 := C_G(G^\infty) \leq K = K_1\langle\varphi\rangle$ with φ inducing a field automorphism on G^∞ of odd order f dividing n; furthermore $\pi(K/K_1) \subseteq \pi(K_1)$ and $\pi(K_1) \cap \pi(G^\infty) \subseteq \{2\}$ as well as $\mathcal{M}_{2'}(K) \leq K_1$.

b) $G^\infty = A_1 \times A_2$ where $A_1 \cong {}^2B_2(2^{a_1})$ and $A_2 \cong SL_2(2^{a_2})$ with odd $a_1, a_2 \geq 3$ such that $\gcd(a_1, a_2) = 1$. Moreover, $G = G^\infty : K$ where K is a solvable $2'$ - semiregular group such that $K_1 := C_G(G^\infty) \leq K = K_1\langle\varphi_1\varphi_2\rangle$ with φ_i inducing a field automorphism of order f_i on A_i and acting trivially on A_j, $j \neq i$; in particular, $G/(A_1 \times A_2 \times K_1) \cong K/K_1 \cong \mathbb{Z}_f$ with $f = f_1 f_2$ and $f_i | a_i$. Moreover, $\pi(K/K_1) \subseteq \pi(K_1)$ and $\pi(K_1) \cap \pi(G^\infty) \subseteq \{2\}$ as well as $\mathcal{M}_{2'}(K) \leq K_1$.

c) $Q := O_2(G^\infty) \neq 1$ and $G^\infty/Q \cong A_5$ as well as $G = G^\infty K$ where $K = C_G(G^\infty/Q)$ is a solvable $2'$ - semiregular $\{3,5\}'$ - group with $K \cap G^\infty = Q$, $\pi(K) \cap \pi(G^\infty) \subseteq \{2\}$ and $\mathcal{M}_{2'}(K)$ being a cyclic subgroup of $C_G(G^\infty)$.

*Conversely, any of the groups G satisfying the properties described in (*I*) or (*II*) is p' - semiregular.*

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References

[1] M. Aschbacher, *Thin finite simple groups*, J. Algebra 54 (1978), 50 – 152.

[2] H. Bender, *Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt*, J. Algebra 17 (1971), 527 – 554.

[3] R. Burkhardt, *Die Zerlegungsmatrizen der Gruppen $PSL(2, p^f)$*, J. Algebra 40 (1976), 75 – 96.

[4] C. W. Curtis, I. Reiner, *Methods of Representation Theory I*, J. Wiley and Sons, 1981.

[5] C.W. Curtis, I. Reiner, *Methods of Representation Theory II*, J. Wiley and Sons, 1987.

[6] L. Dornhoff, *Group Representation Theory*, Dekker, New York, 1971.

[7] D. Gorenstein, *Finite Groups*, Harper and Row, 1968.

[8] R. Guralnick, R. Wiegand, *Galois groups and the multiplicative structure of field extensions*, Trans. Amer. Math. Soc. 331 (1992), 563 – 574.

[9] D. F. Holt, W. Plesken, *Perfect Groups*, Clarendon Press, Oxford, 1989.

[10] B. Huppert, N. Blackburn, *Finite Groups II*, Springer, 1982.

[11] B. Huppert, N. Blackburn, *Finite Groups III*, Springer, 1982.

[12] B. Huppert, *Endliche Gruppen*, Springer, 1967.

[13] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, 1976.

[14] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Oxford University Press 1995.

[15] A.R. Prince, *On 2 - Groups Admitting A_5 or A_6 with an Element of Order 5 Acting Fixed Point Freely*, J. Algebra 49 (1977), 374 – 386.

[16] R. Steinberg, *Lectures on Chevalley Groups*, Yale, 1968.

[17] H. Zassenhaus, *Über endliche Fastkörper*, Abhandl. Math. Sem. Hamburg. Univ. 11 (1936), 187 – 220.

[18] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monath. Math. Phys. 3 (1892), 265–284.