Polynomial Identities for Orbit Numbers of General Linear and Unitary Groups over Finite Fields

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1 Introduction

For a natural number $n \in \mathbf{N}$ let $\mathcal{P}(n)$ be the set of all partitions of n and denote an element $c = 1^{c_1}2^{c_2}...n^{c_n} \in \mathcal{P}(n)$ by the sequence $(c_1, c_2, ...c_n) \in \mathbf{N}_0^n$ with $\sum_{i=1,\dots,n} c_i \cdot i = n$.
For $n \in \mathbf{N}$ and $\epsilon \in \{0, \pm 1\}$ consider the polynomial

$$\mathcal{F}_{n,\epsilon}(X) = \sum_{(c_k)\in\mathcal{P}(n)} \prod_{(k=1,..,n \ \ c_k>0)} X^{c_k-1} \cdot (X-\epsilon).$$

Then $\mathcal{F}_{n,1}(q)$ is equal to the number of conjugacy classes in the general linear group $GL_n(q)$ over a finite field with q elements, $\mathcal{F}_{n,-1}(q)$ is equal to the number of conjugacy classes in the finite unitary group $U_n(q^2)$ and $\mathcal{F}_{n,0}(q)$ gives the number of $GL_n(q)$ orbits (i.e. similarity classes) of matrices in $Mat_n(\mathbf{F}_q)$ as well as the number of adjoint $U_n(q^2)$ - orbits on its Lie algebra

$$\mathcal{U}_n(q^2) = \{m \in Mat_n(\mathbf{F}_{q^2}) \mid m = (-m^{tr})^{(q)} \}.$$

Here m^{tr} denotes the transpose of m and $m^{(q)}$ the matrix obtained from m by raising each entry to the q th power.
Let $\mathcal{C}l_{n,\epsilon}(q)$ denote these orbit numbers in the various cases. The formula for conjugacy classes can be found for example in Wall's papers [4],[5] or in a paper of Macdonald [2] but I do not know a reference for the case of Lie algebras.

One result of this paper consists in a unified proof of the formula

$$\mathcal{C}l_{n,\epsilon}(q) = \mathcal{F}_{n,\epsilon}(q)$$

together with a polynomial identity for $\mathcal{F}_{n,\epsilon}(X)$, involving partitions and 'multipartitions' of n. To state the result we need the following notation:

Definition 1.1 *Let n be a natural number.*
a) An $n \times n$ - matrix $(c_i^j)_{i,j=1,\dots,n}$ with entries in $\mathbf{N}_0$ is called a **multipartition** *of n if: $\sum_{i,j=1,\dots,n} c_i^j \cdot i \cdot j = n$. The set of all multipartitions of n will be denoted by $\mathcal{MP}(n)$.*
b) With $\binom{r}{r_1,\dots,r_k}$ we denote the number $\frac{r\cdot(r-1)\dots(r-(\sum_j r_j)+1)}{r_1!\cdot\dots r_k!}$.
c) For $\epsilon \in \{0, \pm 1\}$ and $i \in \mathbf{N}$ we define the polynomial

$$\Psi_i^\epsilon(X) := 1/i \sum_{d|i} \mu(d)(X^{i/d} - \epsilon^{i/d}) \in \mathbf{Q}[X].$$

Here μ is the Möbius function of elementary number theory.

Let $G_n^\epsilon(q)$ denote the group $GL_n(q)$ if $\epsilon = 1$ and $U_n(q^2)$ if $\epsilon = -1$. It is well known that each multipartition (c_i^j) of n gives rise to a unique $G_n^\epsilon(q)$ - conjugacy class $\mathcal{C}((c_i^j))(q)$ of subgroups which are isomorphic to

$$\prod_{i,j=1}^{n} (G_j^{\epsilon^i}(q^i))^{c_i^j},$$

where $G_j^{\epsilon^i}(q^i)) \cong GL_j(q^i)$, if $\epsilon = 1$ or $i \equiv 0 \bmod 2$ and $G_j^{\epsilon^i}(q^i) \cong U_j(q^{2i})$ otherwise.
These subgroups appear as centralizers of semisimple elements in $G_n^\epsilon(q)$ or the corresponding Lie algebra, whenever q is 'large enough' (precisely if and only if $\Psi_i^\epsilon(q) > \sum_j c_i^j$ for all $1 \le i \le n$). On the other hand, each centralizer of a semisimple element in $G_n^\epsilon(q)$ or the corresponding Lie algebra is $G_n^\epsilon(q)$ - conjugate to an element in a suitable $\mathcal{C}((c_i^j))(q)$.
Two semisimple $G_n^\epsilon(q)$ - classes or orbits in the Lie algebra are called of the same **genus** $(c_i^j) \in \mathcal{MP}(n)$ if and only if the centralizers of representatives are $G_n^\epsilon(q)$ - conjugate and hence lie in the same class $\mathcal{C}((c_i^j))(q)$.
Thus, if q is large enough, then multipartitions parametrize the semisimple genera of $G_n^\epsilon(q)$. Using these notations our result is the following:

Theorem 1.2 *For $\epsilon = \pm 1$ let $\mathcal{C}l_n^\epsilon(q)$ be the number of conjugacy classes of $G_n^\epsilon(q)$ and $\mathcal{C}l_n^0(q)$ the number of adjoint $G_n^\epsilon(q)$ - orbits in the corresponding Lie algebras $\mathcal{G}\ell_n(q)$ and $\mathcal{U}_n(q^2)$ (see the definition in section 2).*

Then we have the following polynomial identities:

$$\mathcal{C}l_n^\epsilon(q) = \sum_{(c_i^j)\in\mathcal{MP}(n)} \prod_{i=1,\dots,n} \binom{\Psi_i^\epsilon(q)}{c_i^1, c_i^2, \dots, c_i^n} \cdot |\mathcal{P}(1)|^{c_i^1} \cdot |\mathcal{P}(2)|^{c_i^2} \cdot \dots |\mathcal{P}(n)|^{c_i^n} =$$

$$\sum_{(c_k)\in\mathcal{P}(n)} \prod_{(k=1,..,n \;\; c_k>0)} q^{c_k - 1} \cdot (q - \epsilon). \tag{1.1}$$

Moreover

$$\prod_{i=1,\dots,n} \binom{\Psi_i^\epsilon(q)}{c_i^1, c_i^2, \dots, c_i^n}$$

is the total number of semisimple $G_n^\epsilon(q)$ classes (if $\epsilon = \pm 1$) and orbits (if $\epsilon = 0$) of genus $c = (c_i^j) \in \mathcal{MP}(n)$ with centralizer lying in $\mathcal{C}((c_i^j))(q)$.

We also give a formula for the number of semisimple classes in $G_n^\epsilon(q)$ of given genus and given order of elements.

In case of GL_n the identity (1.1) follows from a general fact on semisimple orbits in finite dimensional algebras over arbitrary fields, under the action of its unit groups (see 3.1 for the definition of semisimple elements).
In fact if B is such an algebra with unit group B^* and Jacobson radical $Rad\ B$, then 3.3 states that the number of semisimple B^* orbits in B equals the number of semisimple $(B/Rad\ B)^*$ orbits in $B/Rad\ B$.

Since this result does not cover the unitary cases of (1.1) we also give a unified proof for (1.1) and all relevant ϵ, using the theory of connected affine algebraic groups.

2 Basic notation

Let G be a group acting on a set Ω, then let ω^G or $[\omega]$ denote the G - orbit of ω and G_ω its stabiliser in G. The set of G - orbits in Ω will be denoted by $\Omega\backslash G$. As usual a map $\phi : \Omega_1 \to \Omega_2$ between G - sets, satisfying $\phi({}^g\omega) = {}^g\phi(\omega)$ is called a G - map. Moreover for each $H \leq G$ we define $\Omega_H := \{\omega \in \Omega | H \leq G_\omega\}$.

We start with the following observation which is useful for the parametrization of orbits in G - sets:

Suppose Ω, Ω_1, Ω_2 are G-sets. Furthermore let $\jmath : \Omega \to \Omega_1 \times \Omega_2$ be an injective G - map. For any $t \in \Omega_1$ let $\Omega_2(t) := \{v \in \Omega_2 \mid \exists\ \omega \in \Omega \text{ with } \jmath(\omega) = (t, v)\}$. Then $\Omega_2(t)$ is a G_t - set and for any system $\Omega_1 : G$ of representatives for $\Omega_1\backslash G$, there is a bijection:

$$\Psi : \Omega\backslash G \longrightarrow \biguplus_{t\in\Omega_1:G} \Omega_2(t)\backslash G_t$$

with $\Psi(\omega^G) = (pr_{\Omega_2}(\jmath(\omega)))^{G_t}$. Here pr_{Ω_i} is the i-th projection of $\Omega_1 \times \Omega_2$ and $\omega \in \omega^G$ is chosen such that $pr_{\Omega_1}(\jmath(\omega)) = t \in \Omega_1 : G$. (We consider $\Omega_2(t)$ and $\Omega_2(t')$ as disjoint sets whenever $t \neq t'$.)

If for example G is a finite group and p a prime dividing $|G|$, then we can take G as Ω, the set $G_{p'}$ of elements having an order prime to p as Ω_1 and the set of p - elements (having order a power of p) as Ω_2. Now we define $\jmath : G \to G_{p'} \times G_p, g \mapsto (g_{p'}, g_p)$, where $g = g_{p'}g_p = g_pg_{p'}$ is the well known unique decomposition of g into commuting p' - and p - parts.

If $\mathcal{C}l(G) = |G\backslash G|$ is the number of conjugacy classes of G, $\mathcal{C}l_p(G) = |G_p\backslash G|$ the number of conjugacy classes of p - elements and $\mathcal{C}l_{p'}(G) = |G_{p'}\backslash G|$ the number of p' - classes, we get the following identity for $\mathcal{C}l(G)$:

$$\mathcal{C}l(G) = \sum_{s^G \in G_{p'}\backslash G} \mathcal{C}l_p(G_s) = \sum_{u^G \in G_p\backslash G} \mathcal{C}l_{p'}(G_u/O_p(G_u)). \tag{2.1}$$

This is due to the fact that by a classical theorem of Brauer, the number of different non - isomorphic irreducible p - modular representations of a finite group H over an algebraically closed field of characteristic p equals the number of p' - conjugacy classes of H. Each of these representations has the maximal normal p - subgroup $O_p(H)$ in its kernel. Thus $\mathcal{C}l_{p'}(H) = \mathcal{C}l_{p'}(H/O_p(H))$. Now the identity easily follows from the previous lemma.

A similar situation is the following: Let G be a connected algebraic group acting on its Lie algebra $Lie(G)$ by the adjoint representation. Suppose G and $Lie(G)$ are defined over the finite field $\mathbf{F}_q$ with compatible Frobenius endomorphisms $F : G \longrightarrow G$ and $\tilde{F} : Lie(G) \longrightarrow Lie(G)$ satisfying $\tilde{F}(ad(g)(\gamma)) = ad(F(g))(\tilde{F}(\gamma))$. Then the finite group of fixed points G_F acts on the finite Lie algebra $Lie(G)_{\tilde{F}}$. By $(\mathcal{E})_{ss}$ we denote the set of semisimple elements of a subset $\mathcal{E}$ of G or $Lie(G)$ and by $(\mathcal{E})_{uni}$ or $(\mathcal{E})_{nil}$ the corresponding set of unipotent or nilpotent elements. For any $\gamma \in Lie(G)$ let $Lie(G)_\gamma$ be the centralizer algebra of γ, i.e. the set $\{\nu \in Lie(G) \mid [\nu, \gamma] = 0\}$.

Using the unique Jordan decomposition of elements in $Lie(G)$ and $Lie(G)_{\tilde{F}}$ we get the following identity for the number $\mathcal{C}l(G_F, Lie(G)_{\tilde{F}})$ of G_F orbits on $Lie(G)_{\tilde{F}}$:

$$\mathcal{C}l(G_F, Lie(G)_{\tilde{F}}) = \sum_{\gamma^{G_F} \in (Lie(G)_{\tilde{F}})_{ss} \backslash G_F} |((Lie(G)_\gamma)_{\tilde{F}})_{nil} \backslash (G_F)_\gamma| =$$

$$\sum_{\nu^{G_F} \in (Lie(G)_{\tilde{F}})_{nil} \backslash G_F} |((Lie(G)_\nu)_{\tilde{F}})_{ss} \backslash (G_F)_\nu| \qquad (2.2).$$

3 Semisimple orbits

The identities (2.1) and (2.2) are the source of equation (1.1) in 1.2. To exploit them, we need some information on the semisimple orbits of centralizers of unipotent or nilpotent elements. For this purpose we include the following elementary considerations on semisimple orbits in an algebra, under the action of its unit group.

Notice that in any finite dimensional algebra B with 1 over a field K, conjugation of 'semisimple elements' under the group of units B^* is already controlled inside the semisimple quotient $B/Rad\ B$ and its groups of units. Here $Rad\ B$ denotes the Jacobson radical of B. To make the statement more precise we need:

Definition 3.1 *Let B be a finite dimensional algebra over the field K. We call the element $a \in B$ semisimple if and only if left multiplication $\lambda(a) \in End_K(B)$, $a' \mapsto a \cdot a'$ is a semisimple endomorphism in the sense that its minimal polynomial has no multiple roots in an algebraic closure $\bar{K}$ of K.*

Notice that a is semisimple if and only if the subalgebra $K[a] \leq B$ is semisimple, justifiying the term 'semisimple' without any reference to the side on which the multiplication acts.

Lemma 3.2 *Let $I \triangleleft B$ be an ideal and $a, b \in B$ two semisimple elements of B. Then a and b are conjugate mod I by a unit $u \in B^*$ if and only if they are conjugate mod I^2 by some unit $v \in B^*$.*

Proof: Let $\overline{B} := B/I^2$, then $(\overline{I})^2 = 0$, and as I is stable under the conjugation action of B^*, the units of B also act on $\overline{B}$. It is clear that for any semisimple element $s \in B$, the image $\overline{s}$ is again a semisimple element in $\overline{B}$.
Suppose now we are given $u \in B^*$ such that $u \cdot \overline{a} \cdot u^{-1} = \overline{b} + \overline{i}$ with $i \in I$. We are finished if we find an element $v \in B^*$ such that $v \cdot \overline{b} \cdot v^{-1} = \overline{b} + \overline{i}$, since then $v^{-1} u \cdot a \cdot u^{-1} v = b \ mod \ (I^2)$. Consider the endomorphism $[\ ,\overline{b}] : \overline{I} \longrightarrow \overline{I}$, $[\overline{i}, \overline{b}] = \overline{i}\overline{b} - \overline{b}\overline{i}$. Since left and right multiplication by $\overline{b}$ are commuting semisimple endomorphisms of $\overline{I}$, $[\ ,\overline{b}]$ is semisimple as well. Now for any vector space V over K and an element $\sigma \in End_K(V)$ such that $K[\sigma]$ is a semisimple subalgebra of $End_K(V)$, there is a decomposition $V = ker\ \sigma \oplus \sigma(V)$: clearly we have a decomposition $V = ker\ \sigma \oplus C$ with a σ - invariant complement C; but $\sigma(V) = \sigma(C) \subseteq C$ and $dim\ \sigma(V) = dim\ V - dim\ ker(\sigma) = dim\ C$; hence $\sigma(V) = C$. If $Z_{\overline{I}}(\overline{b})$ denotes the centralizer algebra $\{ \ \overline{i} \in \overline{I} \mid \overline{b} \cdot \overline{i} = \overline{i} \cdot \overline{b} \ \}$ then this yields in our situation:

$$\overline{I} = Z_{\overline{I}}(\overline{b}) \oplus [\overline{I}, \overline{b}].$$

Let $\overline{i} = [\overline{i}_0, \overline{b}] + \overline{i}_1$ with $\overline{i}_1 \in Z_{\overline{I}}(\overline{b})$, then $(1 + \overline{i}_0) \cdot \overline{b} \cdot (1 - \overline{i}_0) = \overline{b} + \overline{i} - \overline{i}_1$. Notice that the elements $\overline{b} + \overline{i}$ and $\overline{i}_1$ commute. Hence $\overline{i}_1 = (\overline{b} + \overline{i}) - (1 + \overline{i}_0) \cdot \overline{b} \cdot (1 - \overline{i}_0)$ is a difference of commuting semisimple elements, and at the same time nilpotent. We conclude $\overline{i}_1 = 0$, thus finishing the proof. $\circ$

The fact that the Jacobson radical of a finite dimensional algebra is a nilpotent ideal immediately gives:

Corollary 3.3 *Let B be a finite dimensional algebra over the field K. If B_{ss} denotes the set of semisimple elements in B, then*

$$B_{ss} \backslash B^* \cong (B/Rad\ B)_{ss} \backslash (B/Rad\ B)^*.$$

If K is in fact algebraically closed, then B can be viewed as a linear variety and the Zariski - open subset B^* as a connected algebraic group. In this situation the following is easily established, using basic results on algebraic groups:

Lemma 3.4 *Let $1 = \sum_{i=1,\ldots,s} e_i$ be a decomposition of 1 into orthogonal primitive idempotents. Then we get for the group B^* of units in B:*

1) The set $T = \{\sum_{i=1,\ldots,s} \lambda_i e_i \mid \lambda_i \in K^*,\ i = 1, \ldots, s\ \}$ is a maximal torus of B^*.

2) The Lie algebra $Lie(B^*)$ coincides with the bracket algebra $B_{[\ ,\ ]}$ where $[a,b] = ab - ba$ for $a, b \in B$. The unipotent radical $R_u(B^*)$ is equal to the normal subgroup $1_B + Rad(B)$ of B^* and its Lie algebra $Lie(1_B + Rad(B))$ is equal to $Rad(B)_{[\ ,\ ]}$.

3) The group $(B/Rad(B))^* \cong B^*/R_u(B^*)$ is a connected reductive algebraic group with Lie algebra $B/Rad(B)_{[\ ,\ ]}$.

Now let G be any connected linear algebraic group defined over the algebraically closed field K. Then each semisimple element $s \in G$ lies in some maximal torus T and all maximal tori are conjugate. By [1] 11.8, pg.150 the same holds for semisimple elements in the Lie algebra $Lie(G)$: each semisimple element $\delta \in Lie(G)$ is contained in a maximal **toral Lie algebra**, i.e. the subalgebra $Lie(T) \subseteq Lie(G)$ of a maximal torus T of G. From these facts one can deduce that the semisimple classes of G are in bijection with the set $T \backslash W$ and the semisimple adjoint orbits of $Lie(G)$ are in bijection with the set of orbits $Lie(T)\backslash W$. In both cases $W = N_G(T)/C_G(T)$ is the Weyl group of G with respect to the maximal torus T of G.

It is well known that the canonical epimorphism $c : G \longrightarrow G/R_u(G) =: \overline{G}$ induces an isomorphism of pairs (W,T) and $(\overline{W},\overline{T})$. This means $W \cong \overline{W}$, $T \cong \overline{T}$ and $\overline{{}^w t} = {}^{\overline{w}}\,\overline{t}$ for all $t \in T$ and $w \in W$. Of course $\overline{T}$ is a maximal torus of $\overline{G}$ and $\overline{W} = W(\overline{G},\overline{T})$ the corresponding Weyl group.

We have for the sets of semisisimple elements in G and $Lie(G)$:

$$(G_{ss})_T = T, \ (Lie(G)_{ss})_T = Lie(G_T)_{ss} = Lie(T).$$

Since G_T acts trivially on $Lie(T)$ we obtain the following lemma which is an analogue of 3.3 for connected linear algebraic groups:

Lemma 3.5 *Let G_{ss} and $Lie(G)_{ss}$ be the sets of semisimple elements in the connected linear algebraic group G and its Lie algebra $Lie(G)$ and let T be a maximal torus of G. Then:*

$$G_{ss}\backslash G \cong T\backslash W(G,T) \cong (G/R_u(G))_{ss}\backslash(G/R_u(G));$$
$$Lie(G)_{ss}\backslash G \cong Lie(T)\backslash W(G,T) \cong (Lie(G/R_u(G)))_{ss}\backslash(G/R_u(G)).$$

It is useful to know that centralizers in G of semisimple elements stay under control if one factors out a normal connected unipotent subgroup. This result is well known in the case of groups but I do not know a reference in the case of Lie algebras, so I give a proof. First we need a lemma.

Lemma 3.6 *Let G be a linear algebraic group, T a torus of G and $U \leq G$ a connected abelian unipotent subgroup of G which is normalized by T. Then for any $t \in T$ and $\delta \in Lie(T)$ the group U_t is connected and we have:*

$$U = U_t \times [t,U];$$

$$Lie(U)^+ = (Lie(U))_\delta \times [\delta,U].$$

Here $Lie(U)^+$ denotes $Lie(U)$, viewed as an additive linear algebraic group and $[t,U]$ denotes the group generated by (here in fact equal to) the set of commutators $[t,u] = t^{-1}u^{-1}tu$ with $u \in U$, $[\delta,U] := \{-\delta + {}^u\delta \mid u \in U\ \}$, which is a closed irreducible additive subgroup of $(Lie(U))^+$ and $(Lie(U))_\delta := \{\mu \in Lie(U) \mid [\mu,\delta] = 0\ \}$.

Proof: We consider the Lie algebras which occur as subalgebras of $Lie(G)$. We fix a $t \in T$. Consider the morphism $\psi : U \longrightarrow [t,U]$, $u \mapsto t^{-1}\ {}^u t$. Since U is abelian we get:

$\psi(u)\psi(v) = t^{-1}\ {}^ut \cdot t^{-1}\ {}^vt = t^{-1}\ {}^ut \cdot {}^u(t^{-1})\ {}^{uv}t = t^{-1}\ {}^{uv}t = \psi(uv)$, showing that $[t,U]$ is a closed connected subgroup of U. Since the kernel of ψ is equal to U_t, its image $[t,U]$ has dimension $dim\ U - dim\ U_t$. Moreover let $v \in U_t \cap [t,U]$. Then $v = t^{-1}\ {}^ut$ for some $u \in U$ and ${}^ut = t \cdot v$ which implies $v = 1$ by Jordan decomposition. Now we deduce $U = (U_t)^0 \times [t,U]$ since the right hand side is a closed connected subgroup of dimension $dim\ U$.
For any $u \in U$ consider the morphism $\psi_u : T \longrightarrow U$, $t \mapsto t^{-1} \cdot int(u)(t) = [t, u^{-1}] \in U$. Differentiating we get: $d\psi_u : Lie(T) \longrightarrow Lie(U)$, $\delta \mapsto {}^u\delta - \delta$, where ${}^u\delta$ denotes the adjoint action.
Now we fix $\delta \in Lie(T)$ and consider $c_\delta : U \longrightarrow Lie(U)$, $u \mapsto d\psi_u(\delta) = {}^u\delta - \delta$, which is clearly a morphism of varieties. Since U is abelian it is also a homomorphism of algebraic groups: $c_\delta(uu') = {}^{uu'}\delta - \delta = {}^{uu'}\delta - {}^u\delta + {}^u\delta - \delta = {}^u({}^{u'}\delta - \delta) + {}^u\delta - \delta = {}^{u'}\delta - \delta + {}^u\delta - \delta = c_\delta(u') + c_\delta(u)$. The kernel of c_δ is the subgroup U_δ. Hence the image $c_\delta(U)$ is a connected algebraic subgroup of $(Lie(U))^+$ of dimension $dim\ U - dim\ U_\delta$. Since δ is semisimple we know $Lie(U)_\delta = Lie(U_\delta)$. Suppose now $\mu \in c_\delta(U) \cap (Lie(U))_\delta$ then $\mu = {}^u\delta - \delta$, which implies ${}^u\delta = \mu + \delta$ with $[\mu, \delta] = 0$. From Jordan decomposition for Lie algebras of algebraic groups we conclude: $\mu = 0$ because ${}^u\delta$ is semisimple. Now the claim again follows by dimension reasons. $\circ$

Theorem 3.7 *Let G be a linear algebraic group, $U \triangleleft G$ a closed connected normal unipotent subgroup of G and T a torus in G. Let $G \longrightarrow \overline{G} = G/U$ be the canonical epimorphism and identify T with $\overline{T}$ and $Lie(T)$ with $Lie(\overline{T}) \leq Lie(\overline{G}) = Lie(G)/Lie(U)$. Then we get for any $t \in T$ or $\delta \in Lie(T)$:*

$$\overline{G_t} = (\overline{G})_t \text{ and } \overline{G_\delta} = (\overline{G})_\delta.$$

Proof: As soon as this is proved for abelian U, the general case will follow by induction on dimensions:
There always exists the nontrivial characteristic subgroup $Z := Z(U)^0$ of U. By induction $(\overline{G})_x = ((G/Z)/(U/Z))_x = ((G/Z)_x \cdot (U/Z))/(U/Z) = ((G_x \cdot Z/Z) \cdot (U/Z))/(U/Z) = G_x \cdot U/U = \overline{(G_x)}$, for $x = t$ or δ.
So we can assume U to be abelian. Let $\overline{g} \in (\overline{G})_t$ then by the previous lemma ${}^gt = t\ u_1u_0$ with $u_0 \in U_t$ and $u_1 = t^{-1}\ {}^{\tilde{u}}t$ for some $\tilde{u} \in U$. Hence ${}^gt = {}^{\tilde{u}}tu_0$ and since u_0 commutes with $\tilde{u}$ and t it must be 1. So $\overline{g} = \overline{\tilde{u}^{-1}g} \in \overline{G_t}$. The case of Lie algebras can now be settled in a completely similar way: For $\overline{g} \in (\overline{G})_\delta$ we have ${}^g\delta = \delta + \mu_0 + {}^v\delta - \delta$ with $\mu_0 \in (Lie(U))_\delta$ and $v \in U$. Hence ${}^g\delta - \mu_0 = {}^v\delta$ with $[\mu_0, {}^g\delta] = 0$. Again $\mu_0 = 0$ by Jordan decompostion and we finish as we did before. $\circ$

Suppose G and $Lie(G)$ are defined over the finite field $\mathbf{F}_q$ with corresponding Frobenius endomorphisms $F : G \longrightarrow G$, $\tilde{F} : Lie(G) \longrightarrow Lie(G)$ such that $\tilde{F}(ad(g)\delta) = ad(F(g))\tilde{F}(\delta)$ for all $g \in G$ and $\delta \in Lie(G)$. Then it is a consequence of Lang's theorem ([Sp] 3.3.16), that for any element $x \in G_F$ or $Lie(G)_F$ such that the (adjoint) centralizer G_x is connected, the set $(x^G)_F$ forms a single G_F - class. On the other hand any F - stable G - orbit contains an F - stable element. By 3.7 connected centralizers in G of semisimple elements stay connected if we factor out the unipotent radical of G. Thus we conclude:

Theorem 3.8 *Let G, $Lie(G)$ and F be as above and T be an F - stable maximal torus of G. If all centralizers of semisimple elements in G, respectively $Lie(G)$ are connected, then the following holds:*

$$(G_F)_{ss}\backslash G_F \cong (T\backslash W(G,T))_F \cong ((G/R_u(G))_F)_{ss}\backslash(G/R_u(G))_F;$$

respectively

$$(Lie(G))_{\tilde{F}})_{ss}\backslash G_F \cong (Lie(T)\backslash W(G,T))_{\tilde{F}} \cong (Lie(G/R_u(G))_{\tilde{F}})_{ss}\backslash(G/R_u(G))_F.$$

4 Orbit numbers of general linear and unitary groups

Conjugacy classes of GL_n are described using the theory of normal forms of matrices, which is a special case of the theory of finitely generated modules over principal ideal domains. Let R be such a PID and P be a full cross section of associativity classes of primes in R. Then any finitely generated R-torsion module M is isomorphic to a direct sum of cyclic R - modules with prime-power annihilators. Collecting all summands whose annihilators are powers of the same prime $\pi \in P$ we get the π - **primary component** M_π of M.

We are interested in the endomorphism ring $End_R M$ and its group of units $Aut_R M$. It is easy to see that

$$End_R M \;\cong \prod_{\pi\in P} End_R M_\pi \text{ and } Aut_R M \;\cong \prod_{\pi\in P} Aut_R M_\pi.$$

Thus we can assume that M consists of a single primary component. The next result follows easily from Fitting's theorem on endomorphism rings:

Theorem 4.1 *Let R be a PID and $\pi \in R$ a prime in R. Suppose we are given natural numbers $0 < a_1 < a_2 < .... < a_k$ and $b_1, b_2, ..., b_k \in \mathbf{N}$, as well as an R - module $M = \bigoplus_{i=1,...k} M_i$ such that $M_i \cong R/(\pi^{a_i}) \oplus \,...\, \oplus R/(\pi^{a_i})$ with b_i many summands. Then we have the following isomorphisms:*

$$End_R M/Rad(End_R M) \cong \prod_{i=1...k} End_R(M_i/Rad\ M_i) \cong \prod_{i=1...k} Mat_{b_i}(R/(\pi));$$

$$Aut_R M/(Id_M + Rad(End_R M)) \cong \prod_{i=1...k} Aut_R(M_i/Rad\ M_i) \cong \prod_{i=1...k} GL_{b_i}(R/(\pi)).$$

The general linear group $GL_n(q)$ over a finite field $\mathbf{F}_q$ as well as the unitary group $U_n(q^2)$ are special series of finite groups of Lie type. The group $GL_n(q)$ is the group of fixed points $(GL_n(\overline{\mathbf{F}}_q))_{F_q}$ of the Frobenius endomorphism

$$F_q : GL_n(\overline{\mathbf{F}}_q) \longrightarrow GL_n(\overline{\mathbf{F}}_q),\ (g_{ij}) \mapsto ({g_{ij}}^{(q)}),$$

whereas the **unitary group** $U_n(q^2)$ is the group of fixed points $(GL_n(\overline{\mathbf{F}}_q))_{F_{-q}}$ of the generalized Frobenius endomorphism

$$F_{-q} : GL_n(\overline{\mathbf{F}}_q) \longrightarrow GL_n(\overline{\mathbf{F}}_q),\ g \mapsto F_q(g^{-tr}),$$

where tr denotes transposition of matrices.

Its description as classical groups is the following:
consider the involution $\tau : \mathbf{F}_{q^2} \longrightarrow \mathbf{F}_{q^2},\ c \mapsto c^q$ and the hermitian form on the $\mathbf{F}_{q^2}$ - vector space $(\mathbf{F}_{q^2})^n$ given by $< v, w >:= \sum_{i=1,...,n} v_i \cdot\ {w_i}^q$, for all $v = (v_1, ..., v_n),\ w = (w_1, ..., w_n) \in (\mathbf{F}_{q^2})^n$. Then $U_n(q^2)$ is the subgroup $\{g \in GL_n(q^2) |\ < g(v), g(w) >=< v, w > \forall\ v, w \in \mathbf{F}_{q^2}^n\}$.

The Lie algebra $\mathcal{G}\ell_n(\overline{\mathbf{F}}_q) := Lie(GL_n(\overline{\mathbf{F}}_q))$ is the 'bracket-algebra' $Mat_n(\overline{\mathbf{F}}_q)_{[\ ,\ ]}$ where $[\ ,\ ]$ is the commutator of matrices. The endomorphisms

$$F_q : \mathcal{G}\ell_n(\overline{\mathbf{F}}_q) \longrightarrow \mathcal{G}\ell(\overline{\mathbf{F}}_q),\ (m_{ij}) \mapsto ({m_{ij}}^{(q)}).$$

and

$$F_{-q} : \mathcal{G}\ell_n(\overline{\mathbf{F}}_q) \longrightarrow \mathcal{G}\ell(\overline{\mathbf{F}}_q),\ m \mapsto F_q(-m^{tr})$$

are compatible in an obvious sense with the adjoint action of $GL_n(\overline{\mathbf{F}}_q)$ on $\mathcal{G}\ell_n(\overline{\mathbf{F}}_q)$. We denote

$$\mathcal{G}\ell_n(q) := \mathcal{G}\ell_n(\overline{\mathbf{F}}_q)_{F_q} = Mat_n(\mathbf{F}_q)_{[\ ,\ ]}$$

and

$$\mathcal{U}_n(q^2) := \mathcal{G}\ell_n(\overline{\mathbf{F}}_q)_{F_{-q}} = \{m \in Mat_n(\mathbf{F}_{q^2})\ |\ m = -F_q(m^{tr})\ \}.$$

For all $g \in GL_n(\overline{\mathbf{F}}_q)$ the centralizer groups are exactly the (Zariski - open) groups of units in the the corresponding centralizer algebras, hence they are connected. It is well known that in this situation the conjugacy classes of $GL_n(q)$ and $U_n(q^2)$ are in bijection to the F_q - stable, respectively F_{-q} - stable conjugacy classes of $GL_n(\overline{\mathbf{F}}_q)$. The same is true for the adjoint orbits of $GL_n(q)$ and $U_n(q^2)$ on $\mathcal{G}\ell_n(q)$ respectively $\mathcal{U}_n(q^2)$.

Let V be an n - dimensional vector space over the perfect field K and $\alpha \in End_K V$. We can view V as a $K[X]-$ or $K[\alpha]-$ module via the K - algebra homomorphism $c_\alpha : K[X] \longrightarrow K[\alpha] \subseteq End_K V,\ f \mapsto f(\alpha)$. Then $ker\ c_\alpha = (f_\alpha) \neq 0$, with the **minimal polynomial** $f_\alpha \in K[X]$ of α. As $K[\alpha]$ - module, V splits into the direct sum of $r-$ primary components V_r. Here r runs through the set of monic irreducible divisors of f_α over K. We have

$$V_r \cong \bigoplus_{i=1,...,k(r)} (K[X]/(r^{a_i(r)}))^{b_i(r)}$$

with $0 < a_1(r) < ... < a_{k(r)}(r)$, if r divides f_α.

Let $Irr_{\leq n}K[X]$ be the set of all monic irreducible polynomials in $K[X]$ with degree less than or equal to n. Then there is an obvious bijection between the set of conjugacy classes under $GL(V)$ in $End_K V$ and the set of parameters $(S, ((a_i(r), b_i(r))_{i=1,...,k(r)})_{r\in S})$ consisting of a subset $S \subseteq Irr_{\leq n}K[X]$ and numbers $k(r), a_i(r), b_i(r) \in \mathbf{N}$, for any $r \in S$ subject to the conditions: $a_{k(r)}(r) > ... > a_1(r) > 0, b_i(r) > 0$ and $\sum_{r\in S} deg(r) \sum_{i=1,...,k(r)} a_i(r) \cdot$

$b_i(r) = n$. If α belongs to a class with parameters $(S, ((a_i(r), b_i(r))_{i=1,\ldots,k(r)})_{r\in S})$, then the minimal polynomial is $f_\alpha = \prod_{r\in S} r^{a_{k(r)}(r)}$ and the characteristic polynomial is $cp(\alpha) = \prod_{r\in S} r^{s(r)}$ with $s(r) = \sum_{i=1,\ldots,k(r)} a_i(r)\cdot b_i(r)$.

Using 4.1 we see: α is semisimple $\iff a_1(r) = k(r) = 1$ for all irreducible factors r of $f_\alpha \iff End_{K[\alpha]}V$ is a semisimple ring $\iff V_{K[\alpha]}$ is a semisimple $K[\alpha]$ - module. Similarly we see that two semisimple elements α and β are conjugate under $GL(V)$ if and only if they have the same characteristic polynomial.

If $End_K(V)_{ss}$ denotes the set of semisimple elements in $End_K V$ then we get:

The map $\alpha \mapsto cp(\alpha)$ is a bijection

$$cp : End_K(V)_{ss}\backslash GL(V) \longrightarrow \{f \in K[X]|\ f \textit{ monic with degree } n\}$$

respectively

$$cp| : GL(V)_{ss}\backslash GL(V) \longrightarrow \{f \in K[X]|\ f \textit{ monic with degree } n \textit{ and } f(0) \neq 0\}.$$

If $\alpha \in GL_n(\overline{\mathbf{F}}_q)$ is a semisimple element, then the corresponding $GL_n(\overline{\mathbf{F}}_q)$ conjugacy class is F_q - resp. F_{-q} - stable, if and only if the set of roots of $cp(\alpha)$ is invariant under the automorphism $q : \overline{\mathbf{F}}_q \longrightarrow \overline{\mathbf{F}}_q$, $\lambda \mapsto \lambda^q$ respectively $q^- : \overline{\mathbf{F}}_q{}^* \longrightarrow \overline{\mathbf{F}}_q{}^*$, $\lambda \mapsto \lambda^{-q}$.

The same is true for semisimple elements in $\mathcal{G}\ell_n(\overline{\mathbf{F}}_q)$ in case of F_q and for the endomorphism F_{-q} one simply has to replace q^- by the automorphism $-q : \overline{\mathbf{F}}_q^+ \longrightarrow \overline{\mathbf{F}}_q^+$, $\lambda \mapsto -\lambda^q$.

Lemma 4.2 *Let $\epsilon \in \{\pm 1\}$. We consider the groups $\overline{\mathbf{F}}_q^*$ and $\overline{\mathbf{F}}_q^+$ as $\mathbf{Z}$ - sets via: ${}^z t = t^{(\epsilon q)^z}$ for $t \in \overline{\mathbf{F}}_q^*$ and ${}^z s = (\epsilon^z s)^{q^z}$ for $s \in \overline{\mathbf{F}}_q^+$ and $z \in \mathbf{Z}$.*
We define

$$\mathcal{O}^*_{i,q,\epsilon} := \{t^{\mathbf{Z}} \in \overline{\mathbf{F}}_q^*\backslash\mathbf{Z}|\ \ |t^{\mathbf{Z}}| = i\ \},\quad \mathcal{O}^+_{i,q,\epsilon} := \{s^{\mathbf{Z}} \in \overline{\mathbf{F}}_q^+\backslash\mathbf{Z}|\ \ |s^{\mathbf{Z}}| = i\ \},$$

and for $m \in \mathbf{N}$ with $(m,q) = 1$,

$$\mathcal{O}^*_{i,m,q,\epsilon} := \{t^{\mathbf{Z}} \in \mathcal{O}^*_{i,q,\epsilon}|\ t^m = 1\ \},\quad \mathcal{O}^{*,m}_{i,q,\epsilon} := \{t^{\mathbf{Z}} \in \mathcal{O}^*_{i,q,\epsilon}|\ ord(t) = m\ \}$$

Then we have:

$$|\mathcal{O}^*_{i,q,\epsilon}| = \Psi^\epsilon_i(q);\quad |\mathcal{O}^+_{i,q,\epsilon}| = \Psi^0_i(q);\quad |\mathcal{O}^{*,m}_{i,q,\epsilon}| = \delta_{i,ord_{mod(m)}(\epsilon q)}\Phi(m)/i,$$

and the identity:

$$|\mathcal{O}^*_{i,m,q,\epsilon}| =: \Psi^\epsilon_{i,m,q} = \sum_{z|m,ord_{mod(z)}(\epsilon q)=i} \Phi(z) = 1/i \sum_{d|i} \mu(d)\gcd(q^{i/d} - \epsilon^{i/d}, m).$$

Here Φ is Euler's function.

Proof: Let Ω be a $\mathbf{Z}$ - set such that for all $i \in \mathbf{N}$ the set of fixed points $\Omega_{i\mathbf{Z}}$ is finite. For $\mathcal{O}_i := \{\omega^{\mathbf{Z}} \mid |\omega^{\mathbf{Z}}| = i\}$, we have: $\biguplus_{d|i} \biguplus_{o \in \mathcal{O}_d} o = \Omega_{i\mathbf{Z}}$ and Möbius inversion yields:

$$|\mathcal{O}_i| = 1/i \sum_{d|i} \mu(d) |\Omega_{(i/d)\mathbf{Z}}|.$$

Now we get for the various $\mathbf{Z}$ - actions mentioned in the lemma and $\Omega = \overline{\mathbf{F}}_q^*$, $i \in \mathbf{N}$:

$$|\Omega_{i\mathbf{Z}}| = |\{t \in \overline{\mathbf{F}}_q^* | t^{(\epsilon q)^i - 1} = 1\}| = q^i - \epsilon^i;$$

for $\Omega = \{t \in \overline{\mathbf{F}}_q^* | t^m = 1\}$:

$$|\Omega_{i\mathbf{Z}}| = |\{t \in \overline{\mathbf{F}}_q^* | t^{(\epsilon q)^i - 1} = 1 = t^m\}| = \gcd(q^i - \epsilon^i, m);$$

for $\Omega = \overline{\mathbf{F}}_q^+$:

$$|\Omega_{i\mathbf{Z}}| = |\{s \in \overline{\mathbf{F}}_q^+ | \epsilon^i s^{q^i} = s\}| = q^i.$$

Möbius inversion now gives the corresponding cardinalities of orbit sets.
Let $t \in \overline{\mathbf{F}}_q^*$ be an element of order m. We have for $k \geq 0$: $t^{(\epsilon q)^k} = t \iff m \mid (\epsilon q)^k - 1 \iff ord_{mod(m)}(\epsilon q) \mid k$. Hence $\mathcal{O}_{i,q,\epsilon}^{*,m} \neq \emptyset \iff i = ord_{mod(m)}(\epsilon q)$. If $i = ord_{mod(m)}(\epsilon q)$, then the disjoint union of all orbits in $\mathcal{O}_{i,q,\epsilon}^{*,m}$ consists of the $\Phi(m)$ many primitive m - th roots of unity in $\overline{\mathbf{F}}_q$. $\circ$

The roots of an irreducible polynomial $f \in \mathbf{F}_q[X]$ form a $\mathbf{Z}$ - orbit in $\overline{\mathbf{F}}_q$ under the action, mentioned in 4.2 for $\epsilon = 1$, whose length coincides with the degree d_f of f. Hence, for $X = q$, the polynomial $\Psi_i^0(X)$ counts the number of irreducible polynomials over $\mathbf{F}_q$ with degree i, and $\Psi_i^1(q)$ is the number of those with nonvanishing constant term.

Now consider the map

$$cp : \mathcal{G}\ell_n(q)_{ss} \backslash GL_n(q) \longrightarrow \mathcal{P}(n), \ \alpha^{GL_n(q)} \mapsto (c_i),$$

where c_i denotes the number of all $\mathbf{F}_q$ - irreducible divisors of $cp(\alpha)$ with degree i. If $c_i^j(\alpha)$ is the number of different irreducible polynomials in $\mathbf{F}_q[X]$ with degree i, that appear in the factorization of $cp(\alpha)$ with multiplicity j, then the assignment $\alpha^{GL_n(q)} \mapsto (c_i^j(\alpha))_{i,j=1,\dots,n}$ is a map

$$m : \mathcal{G}\ell_n(q)_{ss} \backslash GL_n(q) \longrightarrow \mathcal{MP}(n)$$

respectively

$$m| : GL_n(q)_{ss} \backslash GL_n(q) \longrightarrow \mathcal{MP}(n),$$

which is surjective if q is large enough.

In case of $\mathcal{U}_n(q^2)$ we replace the irreducible polynomials of degree i by $\mathbf{Z}$ - orbits in $\mathcal{O}_{i,q,-1}^+$ and in case of $U_n(q^2)$ by $\mathbf{Z}$ - orbits in $\mathcal{O}_{i,q,-1}^*$ (both times with $\epsilon = -1$) to get similar maps:

$$m : \mathcal{U}_n(q^2)_{ss} \backslash U_n(q^2) \longrightarrow \mathcal{MP}(n)$$

respectively

$$m| : U_n(q^2)_{ss}\backslash U_n(q^2) \longrightarrow \mathcal{MP}(n)$$

which we will also denote by m for simplicity.

Now we get for the fibres of m:

$$|m^{-1}((c_i^j))| = \prod_{i=1,\ldots,n} \binom{\Psi_i{}^{\epsilon}(q)}{c_i^1, c_i^2, \ldots, c_i^n}.$$

Here $\epsilon = 0$ in case of Lie algebras, $\epsilon = 1$ for $GL_n(q)$ and $\epsilon = -1$ for $U_n(q^2)$:

Theorem 4.3 *Keep the notation of 1.2. For $\epsilon \neq 0$ let $\mathcal{C}l_{n,ss}^{\epsilon}(q)$ be the number of semisimple conjugacy classes of $G_n^{\epsilon}(q)$ and let $\mathcal{C}l_{n,ss}^0(q)$ be the number of adjoint $G_n^{\epsilon}(q)$ - orbits of semisimple elements in the corresponding Lie algebra.*

Then these numbers satisfy the following polynomial identity:

$$\mathcal{C}l_{n,ss}^{\epsilon}(q) = \sum_{(c_i^j)\in\mathcal{MP}(n)} \prod_{i=1,\ldots,n} \binom{\Psi_i^{\epsilon}(q)}{c_i^1, c_i^2, \ldots, c_i^n} = (q-\epsilon)\cdot q^{n-1}.$$

Proof: The first equation follows immediately from the remarks above. The second one follows from the bijections:

$$\mathcal{G}\ell_n(q)_{ss}\backslash GL_n(q) \cong \{\ f \in \mathbf{F}_q[X] \mid f \text{ monic},\ deg(f) = n\ \},$$

$$GL_n(q)_{ss}\backslash GL_n(q) \cong \{\ f \in \mathbf{F}_q[X] \mid f \text{ monic},\ deg(f) = n,\ f(0) \neq 0\ \},$$

$$\mathcal{U}_n(q^2)_{ss}\backslash U_n(q^2) \cong \{\ f \in \overline{\mathbf{F}}_q[X] \mid f \text{ monic},\ deg(f) = n, f^{(q)}[X] = (-1)^n f[-X]\},$$

and $U_n(q^2)_{ss}\backslash U_n(q^2) \cong$

$$\{f \in \overline{\mathbf{F}}_q[X] \mid f \text{ monic},\ deg(f) = n,\ f(0) \neq 0,\ f^{(q)}[X] = f(0)^{-1}X^n f(X^{-1})\}.$$

Here $f^{(q)}[X] = X^n + f_{n-1}^q X^{n-1} + \ldots + f_0^q$ if $f[X] = X^n + f_{n-1}X^{n-1} + \ldots + f_0$. The right hand side of the equations above follows by counting these sets of polynomials. $\circ$

If $\alpha \in \alpha^{GL_n(q)} \in m^{-1}((c_i^j))$, then the corresponding parameters of the conjugacy class are: $(S,\ (\ldots,(1,j)_{p_{i_1}},\ldots,(1,j)_{p_{i_{c_i^j}}},\ldots))$, where the p_{i_k} are pairwise different irreducible polynomials $\in S$ with degree $d_{(p_{i_k})} = i$.

The corresponding $GL_n(q)$ - conjugacy class of centralizer rings and centralizers satisfies:

$$Z_{Mat_n(F)}(\alpha) \cong \prod_{i,j=1,\ldots,n} \prod_{k=1,\ldots,c_i^j} Mat_j(\mathbf{F}_q[X]/(p_{i_k})),$$

respectively

$$Z_{GL_n(q)}(\alpha) \cong \prod_{i,j=1,\ldots,n} \prod_{k=1,\ldots,c_i^j} GL_j(\mathbf{F}_q[X]/(p_{i_k})).$$

The structure of finite extension fields over finite fields only depends on its degree, so:

$$Z_{GL_n(q)}(\alpha) \cong \prod_{i,j=1,\dots,n} (GL_j(\mathbf{F}_{q^i}))^{c_i^j}.$$

In particular the isomorphism type of centralizers only depends on the multipartition $m(\alpha^{GL_n(q)})$. The same holds for the ambient centralizer ring.

Notice that for $GL_n(q)$ the identities (1.1) ($\epsilon = 1$ or 0) could be proved using 3.3. Now we give a unified argument valid for the unitary cases as well.
Recall that Lang's theorem states that for any Frobenius endomorphism $F : G \to G$ of a connected algebraic group G, the Lang map $\mathcal{L}_F : G \to G, g \mapsto g^{-1}F(g)$ is surjective.

We start with an explicit description of the centralizers of semisimple elements in $G_n^\epsilon(q)$: Let $\alpha \in \alpha^{G_n^\epsilon(q)} \in m^{-1}((c_i^j))$ and $b_k^{(i)} = \{\lambda, \lambda^{\epsilon q}, \dots\}$ be one of the $c_i^j(\alpha)$ many orbits in $\mathcal{O}_{i,q,\epsilon}^*$, occuring with multiplicity j among the roots of $cp(\alpha)$. We define the diagonal matrix:

$$b_k^{(i,j)} = \begin{pmatrix} \tilde{\lambda} & 0\dots & 0 \\ 0 & \tilde{\lambda}^{\epsilon q} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \tilde{\lambda}^{(\epsilon q)^{i-1}} \end{pmatrix} \in GL_{ij}(\overline{\mathbf{F}}_q)$$

where $\tilde{\lambda} = diag(\lambda, \dots, \lambda) \in GL_j(\overline{\mathbf{F}}_q)$. Furthermore we define a Frobenius endomorphism on $GL_{ij}(\overline{\mathbf{F}}_q)$ by $F_{\epsilon q}^{ij} : GL_{ij}(\overline{\mathbf{F}}_q) \longrightarrow GL_{ij}(\overline{\mathbf{F}}_q)$, $(g_{\mu\nu}) \mapsto (g_{\mu\nu}^{(q)})$ $(if\ \epsilon = 1)$ $resp.$ $(g_{\mu\nu}^{(q)})^{-tr}$ $(if\ \epsilon = -1)$. If $E(j)$ denotes the unit matrix in $GL_j(\overline{\mathbf{F}}_q)$ and

$$\pi^{(i,j)} = \begin{pmatrix} 0 & E(j) & 0 \\ \dots & \dots & \dots \\ \dots & \dots & E(j) \\ E(j) & \dots & 0 \end{pmatrix} \in GL_{ij}(\overline{\mathbf{F}}_q)$$

the block permutation matrix corresponding to an i - cycle then:

$$^{F_{\epsilon q}^{ij}}(b_k^{(i,j)}) = {}^{\pi^{(i,j)}}(b_k^{(i,j)}).$$

By the theorem of Lang, we can find $v^{(i,j)} \in GL_{ij}(\overline{\mathbf{F}}_q)$ with $\mathcal{L}_{F_{\epsilon q}^{ij}}(v^{(i,j)}) = (v^{(i,j)})^{-1} \cdot F_{\epsilon q}^{ij}(v^{(i,j)}) = {\pi^{(i,j)}}^{-1}$. Now we get ${}^{v^{(i,j)}}(b_k^{(i,j)}) \in GL_{ij}(\overline{\mathbf{F}}_q)_{F_{\epsilon q}^{ij}}$ and the block matrices

$$b = \begin{pmatrix} \dots & & \dots & \dots & \dots \\ \dots & b_1^{(i,j)} & \dots & \dots & \dots \\ \dots & \dots & \dots & & \dots \\ \dots & \dots & \dots & b_{c_i^j}^{(i,j)} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in GL_n(\overline{\mathbf{F}}_q)$$

and

$$v = \begin{pmatrix} \dots & & \dots & \dots & \dots \\ \dots & v^{(i,j)} & \dots & \dots & \dots \\ \dots & \dots & \dots & & \dots \\ \dots & \dots & \dots & v^{(i,j)} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in GL_n(\overline{\mathbf{F}}_q)$$

satisfy ${}^{F_{\epsilon q}}({}^{v}b) = {}^{v}b$, in other words the matrix $\tilde{\alpha} = {}^{v}b$ is a representative of $\alpha^{G^{\epsilon}_n(q)}$ and we get the following commutative diagram of centralizers ($int(x)$ denotes left conjugation by x):

$$\begin{array}{ccc}
Z_{GL_{ij}(\overline{\mathbf{F}}_q)}(b_k^{(i,j)}) & \xrightarrow{int(v^{(i,j)})} & Z_{GL_{ij}(\overline{\mathbf{F}}_q)}({}^{v^{(i,j)}}(b_k^{(i,j)})) \\
\downarrow {}^{\pi^{(i,j)\,-1}}F^{ij}_{\epsilon q} & & \downarrow F^{ij}_{\epsilon q} \\
Z_{GL_{ij}(\overline{\mathbf{F}}_q)}(b_k^{(i,j)}) & \xrightarrow[int(v^{(i,j)})]{} & Z_{GL_{ij}(\overline{\mathbf{F}}_q)}({}^{v^{(i,j)}}(b_k^{(i,j)}))
\end{array}$$

and

$$\begin{array}{ccc}
Z_{GL_{ij}(\overline{\mathbf{F}}_q)}(b_k^{(i,j)}) & \xrightarrow{\cong} & GL_j(\overline{\mathbf{F}}_q)^{(1)} \times ... \times GL_j(\overline{\mathbf{F}}_q)^{(i)} \\
\downarrow {}^{\pi^{(i,j)\,-1}}F^{ij}_{\epsilon q} & & \downarrow \Pi(\epsilon q) \\
Z_{GL_{ij}(\overline{\mathbf{F}}_q)}(b_k^{(i,j)}) & \xrightarrow[\cong]{} & GL_j(\overline{\mathbf{F}}_q)^{(1)} \times ... \times GL_j(\overline{\mathbf{F}}_q)^{(i)}
\end{array}$$

where $\Pi(\epsilon q)(g_1, g_2, ..., g_i) = (F^j_{\epsilon q}(g_i), F^j_{\epsilon q}(g_1), ..., F^j_{\epsilon q}(g_{i-1}))$ and $F^j_{\epsilon q} : GL_j(\overline{\mathbf{F}}_q) \longrightarrow GL_j(\overline{\mathbf{F}}_q)$ is a Frobenius endomorphism.
Hence we get for the groups of fixed points:

$$Z_{GL_{ij}(\overline{\mathbf{F}}_q)}({}^{v^{(i,j)}}(b_k^{(i,j)}))_{F^{ij}_{\epsilon q}} \cong GL_j(\overline{\mathbf{F}}_q)_{(F^j_{\epsilon q})^i}.$$

The group on the right hand side is isomorphic to $G^{\epsilon^i}_j(q^i)$, i.e. to $GL_j(q^i)$ if $\epsilon = 1$ or if $\epsilon = -1$ and $i = 0(mod\ 2)$ and to $U_j(q^{2i})$ otherwise. The centralizer in $G^{\epsilon}_n(q)$ of $\tilde{\alpha} = {}^{v}b$ is a direct product of these groups. Now we get:

Theorem 4.4 *The genera of semisimple conjugacy classes of $G^{\epsilon}_n(q)$ or the semisimple $G^{\epsilon}_n(q)$ - orbits in the corresponding Lie algebra are parametrized by elements of the set $\mathcal{MP}(n)$ of multipartitions of n. (This is a bijection if q is suitably large.)*

The number of semisimple classes of genus (c_i^j) whose elements have order m in the finite group is

$$\sum_{d|m} \mu(d) \prod_{i=1,\dots,n} \binom{\Psi^{\epsilon}_{i,m/d,q}}{c_i^1, c_i^2, ..., c_i^n}.$$

The total number of semisimple classes of genus (c_i^j) is

$$\prod_{i=1,\dots,n} \binom{\Psi^{\epsilon}_i(q)}{c_i^1, c_i^2, ..., c_i^n}.$$

The corresponding centralizers are isomorphic to the direct product

$$\prod_{i=1,j}^{n} (G^{\epsilon^i}_j(q^i))^{c_i^j},$$

where $G^{\epsilon^i}_j(q^i)) \cong GL_j(q^i)$, if $\epsilon = 1$ or $i \equiv 0 \bmod 2$ and $G^{\epsilon^i}_j(q^i) \cong U_j(q^{2i})$ otherwise.

Proof: Each semisimple class $\alpha^{G^\epsilon_n(q)} \in m^{-1}((c_i^j))$ can be represented by an element of the form $\tilde{\alpha} = {}^v b$. Moreover the centralizers of b in $GL_n(\overline{\mathbf{F}}_q)$, as well as the element v only depend on c_i^j. Hence with this choice of representatives for the classes corresponding to a given multipartition, the centralizers in $G^\epsilon_n(q)$ of the different $\tilde{\alpha}$ coincide. Obviously different multipartitions yield non - isomorphic centralizers and we get the claim about semisimple genera. The isomorphism type follows from the remarks above.
The number of classes with given genus whose order of elements divides m follows easily from the arguments above if one replaces $\Psi^\epsilon_i(q)$ by $\Psi^\epsilon_{i,m,q}$. Now Möbius inversion finishes the proof. $\circ$

Now let K be an arbitrary field, V an n- dimensional K- vector space and β a nilpotent element in $End_K V$. The elements in $GL(V)$ of the form $Id_V + \beta$ are exactly the unipotent ones.

The parameters of the classes $\beta^{GL(V)}$ respectively $(Id_V + \beta)^{GL(V)}$ are of the form $(S = \{X\}, (a_t(X), b_t(X))_{t=1,\dots,k(X)}),\ a_{k(X)}(X) > \dots > a_1(X)$, respectively $X - 1$ in place of X. If $c_i(\beta)$ (resp. $c_i(Id_V + \beta)$) is the number of exponents i that occur, i.e. $c_i(\beta) = b_t(X)$ if $i = a_t(X)$ and zero if i does not occur among the $a_t(X)$, then: $(c_i(\nu))_{i=1,\dots,n} \in \mathcal{P}(n)$ for all nil- or unipotent elements ν.

Lemma 4.5 *The assignment $\nu \mapsto (c_i(\nu))_{i=1,\dots,n}$ yields bijections, called λ:*

$$\lambda : End_K(V)_{nil}\backslash GL(V) \longrightarrow \mathcal{P}(n),\ \ GL(V)_{uni}\backslash GL(V) \longrightarrow \mathcal{P}(n).$$

Here we have for nilpotent ν:

$$c_i(\nu) = rank(\nu^{i+1}) + rank(\nu^{i-1}) - 2 \cdot rank(\nu^i).$$

Proof: The Jordan canonical form of a nilpotent matrix easily gives: $rank(\nu^k) = \sum_{i=1,\dots,n} max(i-k, 0)c_i$. $\circ$

From 4.1 the centralizer in $GL(V)$ of a nilpotent or unipotent element ν, whose conjugacy class corresponds to (c_i) is given by

$$Z_{GL(V)}(\nu)/(Id_V + Rad(End_{K[\nu]}V)) \cong \prod_{i=1,\dots,n} GL_{c_i}(K).$$

If $u = 1 + \nu \in GL_n(\overline{\mathbf{F}}_q)$ and ν is nilpotent in Jordan form, then $F_{-q}(u) = (1+\nu)^{-tr} = 1 - \nu^{tr}(1+\nu)^{-tr}$, where $rank(((\nu)^{tr}(1+\nu)^{-tr})^k) = rank((\nu^{tr})^k) = rank(\nu^k)$. Hence u and $F_{-q}(u)$ are conjugate in $GL_n(\overline{\mathbf{F}}_q)$ and all unipotent conjugacy classes are stable under $F_{\epsilon q}$. Since centralizers are connected, the bijection λ carries over to the finite groups $G^\epsilon_n(q)$ and the corresponding Lie algebras:

Theorem 4.6 *The set of unipotent classes in $G^\epsilon_n(q)$ or of nilpotent orbits in the corresponding Lie algebra is in bijection with the set $\mathcal{P}(n)$ of all partitions of n. Any centralizer $Z_{G^\epsilon_n(q)}((c_i)_{i=1,\dots,n})$ corresponding to a partition $(c_i)_{i=1,\dots,n}$ satisfies:*

$$Z_{G^\epsilon_n(q)}((c_i)_{i=1,\dots,n})/O_p(Z_{G^\epsilon_n(q)}((c_i)_{i=1,\dots,n})) \cong \prod_{i=1}^{n} G^\epsilon_{c_i}(q).$$

Proof: The proof is similar to the case of semisimple classes, so we omit the details. ∘

Now we can prove the statement in 1.2:

We apply (2.1) and (2.2) in two different ways: The first equation follows from counting semisimple classes first and unipotent or nilpotent classes in the centralizer or centralizerrings of semisimple elements. The second equation follows the other way round: first counting unipotent or nilpotent classes, and then semisimple classes or orbits in the (finite) centralizers of uni- or nilpotent elements. Here we use 3.8. In the case of Lie algebras we also use the well known fact that, if $G = GL_n(\overline{\mathbf{F}}_q)$ and $\delta \in Lie(G)$, then $Lie(G)_\delta = Lie(G_\delta)$. Notice that the numbers of semisimple classes in $(G_\delta)_F$ stay the same if we move to the Frobenius fixed point groups of reductive quotients and their corresponding Lie algebras.

The following gives a sample of the class and orbit numbers $\mathcal{C}l_{n,\epsilon}(q)$ in 1.2:

$\epsilon = 0$	$\mathcal{C}l_{n,0}(q)$
$n = 1:$	q
$n = 2:$	$q^2 + q$
$n = 3:$	$q^3 + q^2 + q$
$n = 4:$	$q^4 + q^3 + 2q^2 + q$
$n = 5:$	$q^5 + q^4 + 2q^3 + 2q^2 + q$
$n = 6:$	$q^6 + q^5 + 2q^4 + 3q^3 + 3q^2 + q$
$n = 7:$	$q^7 + q^6 + 2q^5 + 3q^4 + 4q^3 + 3q^2 + q$
$n = 8:$	$q^8 + q^7 + 2q^6 + 3q^5 + 5q^4 + 5q^3 + 4q^2 + q$
$n = 9:$	$q^9 + q^8 + 2q^7 + 3q^6 + 5q^5 + 6q^4 + 7q^3 + 4q^2 + q$
$n = 10:$	$q^{10} + q^9 + 2q^8 + 3q^7 + 5q^6 + 7q^5 + 9q^4 + 8q^3 + 5q^2 + q$
$n = 11:$	$q^{11} + q^{10} + 2q^9 + 3q^8 + 5q^7 + 7q^6 + 10q^5 + 11q^4 + 10q^3 + 5q^2 + q$
$n = 12:$	$q^{12} + q^{11} + 2q^{10} + 3q^9 + 5q^8 + 7q^7 + 11q^6 + 13q^5 + 15q^4 + 12q^3 + 6q^2 + q$
$n = 13:$	$q^{13} + q^{12} + 2q^{11} + 3q^{10} + 5q^9 + 7q^8 + 11q^7 + 14q^6 + 18q^5 + 18q^4 + 14q^3 + 6q^2 + q$
$\epsilon = 1$	$\mathcal{C}l_{n,1}(q)$
$n = 1:$	$q - 1$
$n = 2:$	$q^2 - 1$
$n = 3:$	$q^3 - q$
$n = 4:$	$q^4 - q$
$n = 5:$	$q^5 - q^2 - q + 1$
$n = 6:$	$q^6 - q^2$
$n = 7:$	$q^7 - q^3 - q^2 + 1$
$n = 8:$	$q^8 - q^3 - q^2 + q$
$n = 9:$	$q^9 - q^4 - q^3 + q$
$n = 10:$	$q^{10} - q^4 - q^3 + q$
$n = 11:$	$q^{11} - q^5 - q^4 - q^3 + q^2 + q$
$n = 12:$	$q^{12} - q^5 - q^4 + q^2 + q - 1$
$n = 13:$	$q^{13} - q^6 - q^5 - q^4 + 2q^2$
$\epsilon = -1$	$\mathcal{C}l_{n,-1}(q)$
$n = 1:$	$q + 1$
$n = 2:$	$q^2 + 2q + 1$
$n = 3:$	$q^3 + 2q^2 + 3q + 2$
$n = 4:$	$q^4 + 2q^3 + 4q^2 + 5q + 2$
$n = 5:$	$q^5 + 2q^4 + 4q^3 + 7q^2 + 7q + 3$
$n = 6:$	$q^6 + 2q^5 + 4q^4 + 8q^3 + 11q^2 + 10q + 4$
$n = 7:$	$q^7 + 2q^6 + 4q^5 + 8q^4 + 13q^3 + 17q^2 + 14q + 5$
$n = 8:$	$q^8 + 2q^7 + 4q^6 + 8q^5 + 14q^4 + 21q^3 + 25q^2 + 19q + 6$
$n = 9:$	$q^9 + 2q^8 + 4q^7 + 8q^6 + 14q^5 + 23q^4 + 33q^3 + 36q^2 + 25q + 8$
$n = 10:$	$q^{10} + 2q^9 + 4q^8 + 8q^7 + 14q^6 + 24q^5 + 37q^4 + 49q^3 + 50q^2 + 33q + 10$
$n = 11:$	$q^{11} + 2q^{10} + 4q^9 + 8q^8 + 14q^7 + 24q^6 + 39q^5 + 57q^4 + 71q^3 + 69q^2 + 43q + 12$
$n = 12:$	$q^{12} + 2q^{11} + 4q^{10} + 8q^9 + 14q^8 + 24q^7 + 40q^6 + 61q^5 + 85q^4 + 102q^3 + 93q^2 + 55q + 15$
$n = 13:$	$q^{13} + 2q^{12} + 4q^{11} + 8q^{10} + 14q^9 + 24q^8 + 40q^7 + 63q^6 + 93q^5 + 125q^4 + 142q^3 + 124q^2 + 70q + 18$

Notice that the second table for $n = 1, \ldots, 12$ already appears in [2]. Similarly the first ten rows of the third table can be obtained from table 5 in [2] by multiplication with $q + 1$.

Final remarks: There are nice formulae and interpretations of the coefficients of $\mathcal{F}_{n,\epsilon}(X)$ in the cases $\epsilon = 0$ and $\epsilon = 1$.

In the former case, the defining formula for $\mathcal{F}_{n,0}(X)$ shows that the coefficient at X^k equals the number of partitions of n into k parts. Morever in [2], Macdonald proves the identity

$$1 + \sum_{n=1}^{\infty} \mathcal{F}_{n,1}(X)Y^n = \sum_{r=0}^{\infty} X^r Y^r \prod_{k>r}(1 - Y^k).$$

A comparison of coefficients at $X^k Y^n$ shows that the coefficient at X^k in $\mathcal{F}_{n,1}(X)$ is given as:

$$\#\{\text{ partitions of } n \text{ into an odd number of distinct parts, smallest being } k\} -$$

$$\#\{\text{ partitions of } n \text{ into an even number of distinct parts, smallest being } k\},$$

except that if $k = 0$, odd and even are interchanged.

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