RELATIVE TRACE IDEALS AND COHEN - MACAULAY QUOTIENTS OF MODULAR INVARIANT RINGS

PETER FLEISCHMANN

Abstract. Let G be a finite group, $\mathbb{F}$ a field whose characteristic p divides the order of G and A^G the invariant ring of a finite-dimensional $\mathbb{F}G$ - module V. In analogy to modular representation theory we define for any subgroup $H \leq G$ the (relative) trace - ideal $A^G_H \lhd A^G$ to be the image of the relative trace map $t^G_H: \ A^H \to A^G, \ f \mapsto \sum_{g\in[G:H]} g(f)$. Moreover, for any family $\mathcal{X}$ of subgroups of G, we define the **relative trace ideals** $A^G_{\mathcal{X}} := \sum_{X\in\mathcal{X}} A^G_X \lhd A^G$ and study their behaviour.

If $\mathcal{X}$ consists of proper subgroups of a Sylow p - group $P \leq G$, then $A^G_{\mathcal{X}}$ is always a proper ideal of A^G; in fact, we show that its height is bounded above by the codimension of the fixed point space V^P. But we also prove that if V is relatively $\mathcal{X}$ - projective, then $A^G_{\mathcal{X}}$ still contains all invariants of degree not divisible by p. If V is projective then this result applies in particular to the (absolute) trace ideal $A^G_{\{e\}}$.

We also give a 'geometric analysis' of trace ideals, in particular of the ideal $A^G_{<P} := \sum_{Q<P} A^P_Q$, and show that $\mathcal{I}^{G,P} := \sqrt{A^G_{<P}}$ is a prime ideal which has the geometric interpretation as 'vanishing ideal' of G - orbits with length coprime to p. It is shown that $A^G/\mathcal{I}^{G,P}$ is always a Cohen - Macaulay algebra, if the action of P is defined over the prime field. This generalizes a well known result of Hochster and Eagon for the case $P = 1$ (see [13]). Moreover we prove that if V is a direct summand of a permutation module (i.e. a 'trivial source module'), then the $A^G_{\mathcal{X}}$ are radical ideals and $A^G_{<P} = \mathcal{I}^{G,P}$. Hence in this case the ideal and the corresponding Cohen - Macaulay quotient can be constructed using relative trace maps.

1. Introduction

Let R be an arbitrary commutative ring, V a free R- module of finite rank, and G a finite group, acting linearly on V. Then G acts on $V^* := Hom_R(V, R)$ by the rule $gf := f \circ g^{-1}$. This action can be extended to the symmetric algebra $A = R[V] := S(V^*)$, on which G now acts as graded R - algebra automorphisms. We define $A^G = \oplus_{i\geq 0} A^G_i$ to be the corresponding graded ring of invariants.

Up to now, the relation between module theoretic properties of the RG - modules V or V^* and the ring and ideal structure of A^G does not seem to be well understood. In this paper we investigate these connections by utilizing techniques of modular representation theory to construct certain ideals of A^G. For any subgroup $H \leq G$ we define the (relative) trace - ideal $A^G_H \lhd A^G$ to be the image of the relative trace map $t^G_H: \ A^H \to A^G, \ f \mapsto \sum_{g\in[G:H]} g(f)$ (see sec.3). In classical invariant theory, where R is a field of characteristic zero, these trace maps, in particular the absolute trace map t^G_1, have always been a major tool in constructing invariants. This of course is due to the fact that if the index $|G : H|$ is invertible in R, then t^G_H is surjective. If R is a field of characteristic $p \mid |G : H|$, this is no longer true, as A^G_H cannot contain the constants. Notice that the condition $|G : H| \in R^*$ implies that all V and V^* are relatively H - projective in the sense of modular representation theory. So a natural

1991 *Mathematics Subject Classification.* Primary 20C20, 13A50, 13C14.

question is: How far away from being surjective are relative trace maps for subgroups H whose index in G is not a unit in R? We will investigate this question in a broader context, by defining trace ideals $A^G_{\mathcal{X}} := \sum_{H\in\mathcal{X}} A^G_H$, where $\mathcal{X}$ is an arbitrary family of subgroups of G. It turns out that, if $R = \mathbb{F}$ is a field of characteristic $p > 0$ and the elements of $\mathcal{X}$ intersect all Sylow p - subgroups of G properly, then the height of $A^G_{\mathcal{X}}$ is bounded above by the codimension of the fixed point space $\mathrm{codim}_{\mathbb{F}} V^P$ of a Sylow p - group P. Nevertheless we show, that if V (or V^*) is relatively H - projective (or $\mathcal{X}$ - projective), then the ideal A^G_H (or $A^G_{\mathcal{X}}$ respectively) is still large enough, as to contain iA^G_i for all $i \in \mathbb{N}_0$. If again R is a field of characteristic p, this implies that t^G_H is surjective in all degrees coprime to p; in particular this applies to projective and regular representations of G. (In the case of regular representations, this result has been obtained independently in [17].)

In the fourth section we investigate in more detail the situation of permutation modules. Generalizing a result of Shank and Wehlau [22] on $A^G_{\{e\}}$, we prove that in this case the $A^G_{\mathcal{X}}$ are always radical ideals. In particular if G is a p - group, the ideal $A^G_< := \sum_{U<P} A^G_U$ is prime and its quotient is a polynomial ring in the 'orbit products' of G on V^* (for the case of cyclic p - groups see also [18] and [20]). For the regular representation of the cyclic group of order p over a field of characteristic p we show that A^G can be generated by elements of a degree less than that given by Goebel's bound for general invariant rings of permutation modules.

In the last section we assume $R = \mathbb{F}$ to be a field of characteristic $p > 0$ and utilize the geometric interpretation of A^G as a ring of regular functions on the orbit variety V/G. For a fixed Sylow p - subgroup P of G we will define the relative trace ideal $A^G_{<P}$ and the prime ideal $\mathcal{I}^{G,P} = \sqrt{A^G_{<P}}$ in A^G. The latter one is the 'vanishing ideal' of G - orbits with length coprime to p. Using a result of Wilkerson [23], it will be shown that the quotient $A^G/\mathcal{I}^{G,P}$ is a Cohen - Macaulay ring of Krull dimension $\dim_{\mathbb{F}}\ V^P$, whenever the action of P on V is defined over the prime field of $\mathbb{F}$. Here V^P denotes the subspace of P - fixed points in V. If char $\mathbb{F}$ does not divide $|G|$, i.e. if $P = 1$, then $\mathcal{I}^{G,P} = (0)$ and we recover the well known fact, proved by Eagon and Hochster ([13]), that in this case A^G is Cohen - Macaulay.

All these constructions are viewed as efforts to link the module structure of V to the ring and ideal structure of A^G. In particular we show that if V is a 'trivial source module' (i.e. a direct summand of a permutation module), then all $A^G_{\mathcal{X}}$ are radical ideals and $A^G_{<P} = \mathcal{I}^{G,P}$ is prime (with CM - quotient $A^G/A^G_{<P}$ of dimension $\dim_{\mathbb{F}}\ V^P$).

For all facts about modular representations the reader might refer to [1], [6] or [16].

Througout the paper the groups considered will assumed to be finite.

2. Basic definitions

Definition 2.1. *Let R be a commutative ring, A an R - algebra and B an A - bimodule. We let $\mathcal{D}_R(A,B)$ denote the R - module of R - linear derivations from A to B, i.e.*

$$\mathcal{D}_R(A,B) := \{\partial \in \mathrm{Hom}_R(A,B) | \partial(aa') = a\partial(a') + \partial(a)a' \ \forall a, a' \in A\}.$$

With ${}_{RG}mod, mod_{RG}$ and ${}_{RG}mod_{RG}$ we denote the categories of RG - (bi) - modules V such that V is also a free R - module of finite rank.

If $Z(A)$ denotes the center of A, then $\mathcal{D}_R(A,B)$ is a $Z(A)$ - module with $(a\partial)(f) := a(\partial f)$ for $a \in Z(A)$, $f \in A$ and $\partial \in \mathcal{D}_R(A,B)$. If $B = {}_AA_A$, the regular bimodule, then $\mathcal{D}_R(A) := \mathcal{D}_R(A,A)$ is a $Z(A)$ - Lie - algebra. An ideal $I \triangleleft A$ will be called **derivation -**

stable if $\partial(i) \in I$ for all $i \in I$ and $\partial \in \mathcal{D}_R(A)$. For a group G, RG denotes the corresponding group ring.

Let $V \in {}_{RG}mod$. We define

$$T(V) := T_R(V) := \oplus_{i=0}^{\infty} V^{\otimes i}$$

to be the graded tensor algebra over V. The R - linear action of G on V together with the universal property of tensor products induces a representation of G as automorphisms of the (non - commutative) graded ring $T(V)$. For any $v_1 \otimes ... \otimes v_n$ and $g \in G$ we have: $g(v_1 \otimes ... \otimes v_n) = gv_1 \otimes ... \otimes gv_n$

If $A = T(V)$, $d := rank_R V$ and B is an $A - A$ bimodule then the map

$$c : \mathrm{Hom}_R(V, B) \cong B^d \to \mathcal{D}_R(A, B)$$

$$\alpha \mapsto d\alpha : (v_{i_1} \otimes ... \otimes v_{i_k} \mapsto \sum_j v_{i_1} \otimes ...\alpha(v_{i_j})... \otimes v_{i_k})$$

is an isomorphism of R - modules. In fact the functors $\mathrm{Hom}_R(V,\ )$, $(\ )^d$ and $\mathcal{D}_R(A,\ )$ from A - bimodules to R - modules are all isomorphic and exact. If $B = T(V)$, then each $\alpha \in \mathrm{Hom}_R(V, V) \subseteq \mathrm{Hom}_R(V, B)$ has the unique extension $d(\alpha) \in \mathcal{D}_R(A)$, thus we have the canonical embedding $d : \mathcal{E} := \mathrm{End}_R(V) \hookrightarrow \mathcal{D}_R(A)$; in particular $d(Id_V)(a) = deg(a) \cdot a$ for homogeneous $a \in T(V)$. Let $\mathcal{I}$ be a derivation - stable homogeneous ideal of A and $c_{\mathcal{I}} : A \to A/\mathcal{I}$ the canonical epimorphism. Then the exactness of $\mathcal{D}_R(A,\ )$ implies that for any $\partial \in \mathcal{D}_R(A, A/\mathcal{I})$ there is $\tilde{\partial} \in \mathcal{D}_R(A)$ such that $\partial = c_{\mathcal{I}} \circ \tilde{\partial}$. Hence $\partial(\mathcal{I}) = 0$ and $\mathcal{D}_R(A, A/\mathcal{I}) = \mathcal{D}_R(A/\mathcal{I}, A/\mathcal{I})$. Again we get isomorphisms

$$c : \mathrm{Hom}_R(V, A/\mathcal{I}) \to \mathcal{D}_R(A/\mathcal{I}, A/\mathcal{I}), \alpha \mapsto d\alpha.$$

If moreover $\mathcal{I} \cap V = 0$ then $V \subseteq A/\mathcal{I}$ and again each $\alpha \in \mathrm{End}_R(V)$ can be uniquely extended to an element $d\alpha \in \mathcal{D}_R(A/\mathcal{I})$ (i.e. satisfying $d\alpha(v) = \alpha(v)$ for $v \in V \subseteq (A/\mathcal{I})_1$) such that the element $\vartheta := d(Id_V)$ satisfies $\vartheta(f) = deg(f)f$ for each homogeneous $f \in A/\mathcal{I}$.

Now let $\mathcal{I}$ be the homogeneous ideal of $T(V)$, generated by the tensors $v \otimes w - w \otimes v$ for $v, w \in V$ and $S_R(V) = S(V)$ the symmetric algebra $S(V) := T(V)/\mathcal{I}$. Then

$$S(V) := \oplus_{i=1}^{\infty} S_i(V)$$

is also a graded R - algebra with $S_i(V) := < v_1 \otimes ... \otimes v_i + \mathcal{I} >_R$ and $v_i \in V$. We also define $\Lambda_R(V) = \Lambda(V)$ to be the exterior algebra $\Lambda(V) := T(V)/\mathcal{J}$ where $\mathcal{J}$ is the homogeneous ideal generated by the tensors $v \otimes v$ for $v \in V$ and denote $v_1 \wedge ... \wedge v_i := v_1 \otimes ... \otimes v_i + \mathcal{J}$. Notice that $v_1 \wedge v_2 = -v_2 \wedge v_1$.

Again

$$\Lambda(V) := \oplus_{i=1}^{\infty} \Lambda_i(V)$$

is a graded R - algebra with $\Lambda_i(V) := < v_1 \wedge ... \wedge v_i >_R$ and $v_i \in V$. In particular the ideals $\mathcal{I}$ and $\mathcal{J}$ are G - stable and hence we also have a natural action of G on $S(V)$ and $\Lambda(V)$ as automorphisms of graded R - algebras. Choosing an R - basis $\mathcal{B} := \{v_1, ..., v_d\}$ of V we can identify $S(V)$ with the polynomial ring $R[v_1, ..., v_d]$ in the indeterminates $v_1, ..., v_d$.

For $w_i := \sum_{j_i} w_{ij_i} v_{j_i}$ and $i = 1, ..., k$ we have

$$w_1 \wedge ... \wedge w_k = \sum_{j_1,...,j_k} w_{1j_1} \cdot ... \cdot w_{kj_k} v_{j_1} \wedge ... \wedge v_{j_k}.$$

If $k > d$ then in each sum the v_{j_r} can not be pairwise different, so $w_1 \wedge ... \wedge w_k = 0$. We see that each $\Lambda_i(V)$ has an R basis

$$\{v_{j_1} \wedge ... \wedge v_{j_i} \mid j_1 < j_2 < ... < j_i,\ 1 \leq j_k \leq d\}$$

of cardinality $\begin{pmatrix} d \\ i \end{pmatrix}$. In particular $\Lambda_i(V) = 0$ for $i > d$.

Let $v, w \in V$ then for any $\delta \in \mathcal{D}_R(A)$ we have $\delta(v \otimes w - w \otimes v) = \delta v \otimes w + v \otimes \delta w - \delta w \otimes v - w \otimes \delta v \in \mathcal{I}$ and $\delta(v \otimes v) = \delta v \otimes v + v \otimes \delta v = (\delta v + v) \otimes (\delta v + v) - \delta v \otimes \delta v - v \otimes v \in \mathcal{J}$. Hence $\mathcal{I}$ and $\mathcal{J}$ are derivation - stable.

If $A = \oplus_{i=1}^{\infty} S_i(V)$ then $\mathrm{Hom}_R(V, A) \cong A^d$ with $d = rank(V)$, hence $\mathcal{D}_R(A)$ is a free A - module of rank d. Now a straightforward induction, using the defining property of derivations (product rule), shows that c_I assigns to each vector $\alpha := (\alpha_1, ..., \alpha_n) \in A^n$ a unique derivation $d\alpha \in \mathcal{D}_R(A)$ given by the formula

$$d\alpha(f) = \sum_i \partial_{v_i}(f)\alpha_i \tag{1}$$

(Here ∂_{v_i} is a shorthand for the formal partial differential operator $\frac{\partial}{\partial_{v_i}}$.)

3. The trace map

We collect some further notation which will be used in the sequel: Let A be an arbitrary graded R - G - algebra i.e. a graded R - algebra on which G acts as graded algebra isomorphisms. For $W \in {}_{RG}mod$ and any subgroup $H \leq G$ we define $W^H := \{w \in W | hw = w\ \forall h \in H\}$ to be the R - submodule of H - fixed points in W. In particular A^H is the **algebra of H - invariants** in A. We have the following homomorphism of R - modules:

$$t_H^G : W^H \to W^G,\ w \mapsto \sum_{g \in [G:H]} gw,$$

where $[G : H]$ denotes an arbitrary cross section of (left) H - cosets in G such that $G = \biguplus_{g \in [G:H]} gH$. This is called the **(relative) trace homomorphism**, its image will be denoted with W_H^G. If the index $|G : H|$ is invertible in R, then t_H^G is an epimorphism; if $W = A = R[V] = S(V^*)$ then the trace homomorphism preserves degrees. For any $a, a' \in A^G$ and $b \in A^H$ we have $t_H^G(aba') = at_H^G(b)a'$, showing that the image $A_H^G := t_H^G(A^H)$ is an ideal in A^G. If $A = R[V]$ then A_U^H is a homogeneous ideal. If $\mathcal{X}$ is a set of subgroups of H, then we define $W_{\mathcal{X}}^H := \sum_{U \in \mathcal{X}} W_U^H$ and $W_<^H := \sum_{U < H} W_U^H$. Notice that again $A_{\mathcal{X}}^H$ and $A_<^H$ are ideals in A^H and homogenous if $A = R[V]$.

Let $U, H \leq G$ be subgroups and suppose we are given a cross section $[U : G : H]$ of U, H double - cosets with $G = \biguplus_{g \in [U:G:H]} UgH$. For each $g \in [U : G : H]$ the subset UgH is a disjoint union of H - cosets of the form ugH. Although UgH is not a subgroup in general, we can define $[UgH : H]$ to be a cross section in U of these H - cosets, i.e. $UgH = \biguplus_{t \in [UgH:H]} tgH$. Then $[UgH : H]$ is also a cross section $[U : U \cap {}^gH]$ of $U \cap {}^gH$ cosets in U, i.e. $U = \biguplus_{u \in [UgH:H]} u(U \cap {}^gH)$. Here, as in the following ${}^gH$ denotes gHg^{-1}. For each $a \in A^U$ and $b \in A^H$ we have the following 'Mackey - formulae':

$$t_H^G(b) = \sum_{g \in [U:G:H]} t_{U \cap {}^gH}^U(gb) \in A_{\mathcal{X}}^U \tag{1}$$

$$t_U^G(a) \cdot t_H^G(b) = \sum_{g \in [U:G:H]} t_{U \cap {}^gH}^G(agb) \in A_{\mathcal{X}}^G \tag{2}$$

where $\mathcal{X} := \{U \cap {}^xH \mid x \in G\}$ (e.g. see [3], [16]). The group G acts on $\mathcal{E}(V) := \mathrm{End}_R(V)$ by the rule $\alpha^g(v) = g^{-1}\alpha g(v)$ for $\alpha \in \mathcal{E}(V)$, $g \in G$ and $v \in V$. Notice that $\mathcal{E}(V)^G = \mathrm{End}_{RG}(V)$.

Now fix a set $\mathcal{X}$ of subgroups of G. In representation theory the module $V \in RG-mod$ is called **relatively $\mathcal{X}$ - projective** if and only if $\mathcal{E}(V)^G = \mathcal{E}(V)^G_{\mathcal{X}}$ or in other words, if the identity Id_V is contained in $\mathcal{E}(V)^G_{\mathcal{X}}$. Notice that V is relatively $\mathcal{X}$ - projective if and only if so is V^*. If $\mathcal{X} = \{H\}$ then the $\mathcal{X}$ - projective module V is called relatively H - projective (e.g. if the index $[G : H]$ is invertible in R). It is easy to see that $t^G_H(a) = t^G_{{}^gH}({}^ga)$ for any $g \in G$ and $a \in A^H$; hence $A^G_H = A^G_{{}^gH}$ and a module V is relatively H projective iff it is so for any G - conjugate of H. Notice that V is projective in ${}_{RG}mod$ if and only if it is relatively 1 - projective, i.e. with respect to the trivial subgroup ([2] 3.6.4).

Here we propose to extend the above machinery to graded R - G - algebras, where the endomorphism ring $\mathcal{E}(V)$ is replaced by the R - Lie - algebra of derivations and the identity element by the element $\vartheta = d(Id)$. So let A be an R - G - algebra and $U \le H \le G$ be subgroups. It is clear that G acts on $\mathcal{D}_R(A)$ by conjugation. Again we can define $\mathcal{D}_R(A)^H$ to be the Lie - subalgebra of H - fixed points and $\mathcal{D}_R(A)^H_U$ to be the image of the trace homomorphism

$$t^H_U: \ \mathcal{D}_R(A)^U \to \mathcal{D}_R(A)^H, \ \delta \mapsto \sum_{h \in [H:U]} h\delta h^{-1}.$$

As above we define: $\mathcal{D}_R(A)^H_< := \sum_{Y<H} \mathcal{D}_R(A)^H_Y$. Now $\mathcal{D}_R(A)^H_U$ and $\mathcal{D}_R(A)^H_<$ are **Lie - algebra ideals** in $\mathcal{D}_R(A)^H$. For any two subsets $D \subseteq \mathcal{D}_R(A)$ and $X \subseteq A$ we define $D \cdot X := \{\partial(b) \mid b \in X \text{ and } \partial \text{ in } D\}$. If $\delta = \sum_{g\in[G:H]} g\partial g^{-1}$ with $\partial \in \mathcal{D}_R(A)^H$, then we have for any $a \in A^H$, $h\partial(a) = \partial(ha) = \partial(a) \in A^H$. If moreover $a \in A^G \le A^H$, then

$$\delta(a) = \sum_{g\in[G:H]} g\partial g^{-1}(a) = \sum_{g\in[G:H]} g\partial(a) \in A^G_H$$

hence $\mathcal{D}_R(A)^G_H \cdot A^G \subseteq A^G_H$. From this we immediately get $\mathcal{D}_R(A)^G_{\mathcal{X}} \cdot A^G \subseteq A^G_{\mathcal{X}}$. We are now ready to prove:

Proposition 3.1. *Let G be a group, $\mathcal{X}$ a set of subgroups and $V \in {}_{RG}mod$. For a homogeneous derivation - stable and G - stable ideal $\mathcal{I}$ of $T(V)$ with $V \cap \mathcal{I} = 0$ we set $A := T(V)/\mathcal{I}$. Then the following are equivalent:*

(i)*: V is relatively $\mathcal{X}$ - projective;*

(ii)*: $\vartheta = d(Id_V) \in \mathcal{D}_R(A)^G_{\mathcal{X}}$;*

(iii)*: Each homogeneous component A_i with i invertible in R is relatively $\mathcal{X}$ - projective.*

If one of (i),(ii) *or* (iii) *holds, then we have the inclusions:*

$$\oplus_{i\in\mathbb{N}} i \cdot (A_i)^G \subseteq \mathcal{D}_R(A)^G_{\mathcal{X}} \cdot A^G \subseteq A^G_{\mathcal{X}}.$$

In particular if $R = \mathbb{F}$ is a field of characteristic $p > 0$ and V is relatively $\mathcal{X}$ - projective, then $A^G_{\mathcal{X}}$ contains all invariants of degree coprime to p.

Proof: The injection $d : \mathcal{E}(V) \to \mathcal{D}_R(A)$, $\alpha \mapsto d(\alpha)$ is an RG - module homomorphism, so $t^G_H(d(\alpha)) = d(t^G_H(\alpha))$. Moreover $Id_V \in \mathcal{E}(V)^G_{\mathcal{X}}$ implies that $Id_V = \sum_{H\in\mathcal{X}} t^G_H(\alpha_H)$ for some $\alpha_H \in \mathcal{E}(V)^H$, hence $\vartheta = d(Id_V) = \sum_{H\in\mathcal{X}} t^G_H(d(\alpha_H)) \in \mathcal{D}_R(A)^G_{\mathcal{X}}$. This proves '(i) $\Rightarrow$ (ii)'.

(ii) $\Rightarrow$ (iii): Suppose that $i \in R^*$ and $\vartheta \in \mathcal{D}_R(A)^G_{\mathcal{X}}$. Let π_i denote the projection of A onto A_i; then $\pi_i \in \mathrm{Hom}_{RG}(A, A_i)$ and for any $\partial \in \mathcal{D}_R(A)^H$ we have $\pi_i \circ \partial_{|A_i} \in \mathcal{E}(A_i)^H$. By assumption $\vartheta = d(Id_V) = \sum_{H\in\mathcal{X}} t_H^G(\partial_H)$ with $\partial_H \in \mathcal{D}_R(A)^H$. So we get

$$Id_{|A_i} = \pi_i(i^{-1}\vartheta_{|A_i}) = \sum_{H\in\mathcal{X}} \pi_i(t_H^G(i^{-1}\partial_H)) = \sum_{H\in\mathcal{X}} t_H^G(i^{-1}\pi_i\circ\partial_H) \in \mathcal{E}(A_i)_{\mathcal{X}}^G.$$

Hence A_i is relatively $\mathcal{X}$ - projective. The implication (iii) $\Rightarrow$ (i) is obvious, since $V \cong A_1$. If $\vartheta \in \mathcal{D}_R(A)_{\mathcal{X}}^G$ then for any $a \in (A_i)^G$ we get $ia = \vartheta a \in \mathcal{D}_R(A)_{\mathcal{X}}^G \cdot A^G \subseteq A_{\mathcal{X}}^G$.

$\circ$

Corollary 3.2. *If R is a field of characteristic p, dividing the group order $|G|$, V a projective module (e.g. the regular representation of G) and $A = R[V] = S(V^*)$, then the usual (absolute) trace map*

$$t_{<1>}^G : A \to A^G,\ a \mapsto \sum_{g\in G} g(a)$$

is surjective in all degrees $i \neq 0 \mod p$.

Remark 3.3. *For the case of the regular representation of G the result above also appears in* [17].

Corollary 3.4. *Let R be a commutative ring and $V \in {}_{RG}mod$ relatively H - projective for the subgroup $H \leq G$. Then for any $i \in \mathbb{N}$ which is invertible in R, the homogeneous components $S_i(V^*)$ and $\Lambda_i(V^*)$ of the symmetric or exterior tensors $S(V^*)$ respectively $\Lambda_i(V^*)$ are also relatively H - projective.*

Remark 3.5. *Suppose that R a field of characteristic p which divides $|G|$ and $A = R[V]$ with $V \in {}_{RG}mod$.*

a) Relative projectivity of V with respect to H will in general not imply relative projectivity of $S_{p\cdot i}(V^*)$ or $\Lambda_{p\cdot i}(V^*)$. If R is a field of characterisitic $p \mid |G|$ then each projective RG - module W has dimension divisible by p (because its restriction $W_{|P}$ to a Sylow p - subgroup P is a free RP - module; see [1]*). If V has dimension $d = p^k\ell$ with $\gcd(\ell,p) = 1$, then $S_{p^k}(V^*)$ has dimension $\begin{pmatrix} d+p^k-1 \\ p^k \end{pmatrix}$ which can be $\neq 0 \mod p$; hence $S_{p^k}(V^*)$ need not be projective. Similarly $\Lambda_{p^k}(V^*)$ has dimension $\begin{pmatrix} d \\ p^k \end{pmatrix} \neq 0 \mod p$ and $\Lambda(V^*)_{p^k}$ is not projective either.*

b) If P is a Sylow p - subgroup of G, then V is relatively P - projective. If moreover V is indecomposable, then there always exists a subgroup $Q \leq P$, unique up to conjugacy in G, which is minimal such that V is relatively Q - projective. In modular representation theory, such a subgroup is called a **vertex** *of V (e.g. see* [1] *pg. 67 ff). In this language, 3.1 says that the relative trace ideal A_Q^G of a vertex Q of V contains all invariants of degree coprime to p. This also holds for $A_{\mathcal{X}}^G$, whenever $\mathcal{X}$ contains a vertex of V.*

c) Due to a theorem of D.Higman ([2] *3.6.4) V is relatively H - projective if and only if V is a direct summand of an induced module $W^{\uparrow G}$, where $W \in {}_{RH}mod$ (e.g. $W = V_{|H}$). In this case A_H^G contains all invariants of degree coprime to p.*

4. Invariant rings of permutation modules

Now let G be a finite group, Ω a finite G - set and R^Ω the set of functions from Ω to R. We consider the permutation module $V_\Omega := R^\Omega \cong \oplus_{\omega\in\Omega} R\cdot e_\omega$, where $e_\omega(\omega') = 1$ if and only if $\omega = \omega'$ and zero otherwise. This module is selfdual, so $V_\Omega^* \cong V_\Omega$ and we can set

$A = A(R) := S_R(V_\Omega) = \oplus_{f\in\mathbb{N}_0^\Omega} R\pi(f)$, where for each $f \in \mathbb{N}_0^\Omega$ we define $\pi(f) := \prod_{\omega\in\Omega} e_\omega^{f(\omega)}$. Notice that for $\alpha, \beta \in \mathbb{N}_0^\Omega$ we have $\pi(\alpha)\cdot\pi(\beta) = \pi(\alpha+\beta)$ where $(\alpha+\beta)(\omega) = \alpha(\omega)+\beta(\omega)$, hence we can view $S(V_\Omega)$ as the semigroup - algebra $R[\mathbb{N}_0^\Omega]$.

The group Σ_Ω and its subgroup G act on A by permuting the monomials $\pi(f)$ according to the rule:

$$\sigma(\pi(f)) = \prod_{\omega\in\Omega} e_{\sigma(\omega)}^{f(\omega)} = \prod_{\omega\in\Omega} e_\omega^{f\circ\sigma^{-1}(\omega)} = \pi(f\circ\sigma^{-1}).$$

Hence the RG -module $S(V_\Omega)$ is isomorphic to the permutation module $V_{\mathbb{N}_0^\Omega}$ and in each degree i the module $S(V_\Omega)_i$ is isomorphic to the permutation module $V_{(\mathbb{N}_0^\Omega)_i}$, where $(\mathbb{N}_0^\Omega)_i := \{f \in \mathbb{N}_0^\Omega \mid \sum_\omega f(\omega) = i\}$ with the natural action of Σ_Ω and G. In particular the R - module A^G has a basis consisting of the orbit sums $\pi(f)^+ := \sum_{g\in[G:G_f]} g\pi(f) = t_{G_f}^G(\pi(f))$ for $f \in \mathbb{N}_0^\Omega$.

Now the Mackey - formula of section 3 describes the multiplication of orbit sums:

Lemma 4.1 ([9]). *For $\alpha,\ \beta \in \mathbb{N}_0^\Omega$ we have in $A(R)^G$:*

$$\pi(\alpha)^+\cdot\pi(\beta)^+ = \sum_{g\in[G_\alpha:G:G_\beta]} \frac{|G_{\alpha+g(\beta)}|}{|G_\alpha \cap\ {}^gG_\beta|}\pi(\alpha+g\beta)^+.$$

In particular if R has characteristic $p > 0$ and G is a p - group we get

$$\pi(\alpha)^+\cdot\pi(\beta)^+ = \sum_{g\in[G_\alpha:G:G_\beta],\ G_{\alpha+g(\beta)}=G_\alpha\cap\ {}^gG_\beta} \pi(\alpha+g\beta)^+.$$

The following is an extended version of a result of [10], where it has been used to obtain degree bounds for the generators of modular invariant rings of trivial source modules:

Theorem 4.2. *Let $\mathcal{X}$ be a set of subgroups of G, $A_\mathcal{X} := \sum_{H\in\mathcal{X}} t_H^G(A^H) \lhd A^G$ and $A_<^G := A_{\{H\ \mid\ H<G\}}^G$. Then:*

(i)*: $A_\mathcal{X}^G = \oplus_{\alpha\in\mathbb{N}_0^\Omega} Rg_\mathcal{X}(\alpha)\cdot\pi(\alpha)^+$, where $g_\mathcal{X}(\alpha) :=\ gcd(\frac{|G_\alpha|}{|H_\alpha|}\ \mid\ H \in \mathcal{X})\cdot 1_R$.*

(ii)*: For $T \subseteq \Omega$ let $\chi_T :\ \Omega \to \mathbb{N}_0$ be the characteristic function of T (i.e. 1 on T and 0 on $\Omega\backslash T$). Then $S := \langle\pi(\alpha)\ \mid\ \alpha \in \mathbb{N}_0^\Omega,\ G_\alpha = G\rangle_R \cong R[\pi(\chi_{\omega^G})\ \mid\ \omega^G \in \Omega/G]$ with algebraically independent elements $\pi(\chi_{\omega^G})$. In other words, S is a polynomial subalgebra of A^G in $|\Omega/G|$ many variables $\pi(\chi_{\omega^G})$.*

(iii)*: $A^G/A_<^G \cong S/(A_<^G \cap S)$ with $A_<^G \cap S = gcd(\frac{|G|}{|H|}\ \mid\ H < G)\cdot S$.*

Proof: (i): Let $H \in \mathcal{X}$ and $a \in A^H$, then $a = \sum_\alpha c_\alpha t_{H_\alpha}^H(\pi(\alpha))$.
Hence $t_H^G(a) = \sum_\alpha c_\alpha t_{H_\alpha}^G(\pi(\alpha)) = \sum_\alpha c_\alpha t_{G_\alpha}^G(t_{H_\alpha}^{G_\alpha}(\pi(\alpha))) = \sum_\alpha c_\alpha \frac{|G_\alpha|}{|H_\alpha|} t_{G_\alpha}^G(\pi(\alpha))$. This is contained in the set on the right hand side of (i). Let $g_\mathcal{X} = \sum_{H\in\mathcal{X}} \frac{|G_\alpha|}{|H_\alpha|} c_H$ with $c_H \in \mathbb{Z}$. Then $g_\mathcal{X}\pi(\alpha)^+ =$

$$\sum_{H\in\mathcal{X}} \frac{|G_\alpha|}{|H_\alpha|} c_H t_{G_\alpha}^G(\pi(\alpha)) = \sum_{H\in\mathcal{X}} c_H t_{G_\alpha}^G t_{H_\alpha}^{G_\alpha}(\pi(\alpha)) = \sum_{H\in\mathcal{X}} c_H t_H^G t_{H_\alpha}^H(\pi(\alpha)) \in A_\mathcal{X}^G.$$

(ii): If $\alpha \in \mathbb{N}_0^\Omega$ with $G_\alpha = G$, then α is constant on G - orbits in Ω. In this case α induces a function $\hat{\alpha} \in \mathbb{N}_0^{\Omega/G}$, $\omega^G \mapsto \alpha(\omega)$ and we get $\pi(\alpha) = \prod_{\omega^G\in\Omega/G}(\pi(\chi_{\omega^G})^{\hat{\alpha}(\omega^G)}$. Since different G - orbits are disjoint, the set $\{\pi(\chi_{\omega^G})\ \mid\ \omega \in \Omega/G\}$ is algebraically independent.

(iii): Obviously $A^G = A^G_< + S$ with $A^G_< \cap S = \oplus_{\alpha\in\mathbb{N}_0^\Omega,\ G_\alpha = G} Rg_<(\alpha)\cdot\pi(\alpha)^+ = gcd(\frac{|G|}{|H|} \mid H < G)\cdot S$. $\circ$

The next result shows, that in case of permutation modules over fields of positive characteristic, the $A^G_{\mathcal{X}}$ are always radical ideals. This generalizes Theorem 6.1 of [22]:

Proposition 4.3. *Let $\mathbb{F}$ be a field of characteristic $p > 0$, $G \le \Sigma_\Omega$ a permutation group, $\mathcal{X}$ a set of subgroups of G and $S_{\mathcal{X}} := \oplus_{\alpha\in\mathbb{N}_0^\Omega,\ p\ \mid\ |G_\alpha:H_\alpha|,\ \forall H\in\mathcal{X}} \mathbb{F}\pi(\alpha)^+$. Then $A^G = A^G_{\mathcal{X}} \oplus S_{\mathcal{X}}$ and*

$$\sqrt{A^G_{\mathcal{X}}} = A^G_{\mathcal{X}} = \oplus_{\alpha\in\mathbb{N}_0^\Omega,\ p \text{ does not divide } |G_\alpha:H_\alpha|, \text{ for some } H\in\mathcal{X}} \mathbb{F}\pi(\alpha)^+ .$$

Proof: Obviously $g_{\mathcal{X}}(\alpha) \ne 0$ if and only if p does not divide $|G_\alpha : H_\alpha|$ for some $H \in \mathcal{X}$ so we get $A^G = A^G_{\mathcal{X}} \oplus S_{\mathcal{X}}$ and the second equation for $A^G_{\mathcal{X}}$. Thus we only have to prove that $A^G_{\mathcal{X}}$ is a radical ideal. If this is false, we can assume without loss, that there is $0 \ne s \in S_{\mathcal{X}} \cap \sqrt{A^G_{\mathcal{X}}}$ with $s = c_1\pi(\alpha_1)^+ + \cdots + c_k\pi(\alpha_k)^+ \in S_{\mathcal{X}}$. Let $p^m\alpha_i$ denote the p^m - fold sum of α_i in the semigroup $\mathbb{N}_0^\Omega$. Then $(\pi(\alpha_i)^+)^{p^m} = (t^G_{G_{\alpha_i}}(\pi(\alpha_i)))^{p^m} = t^G_{G_{\alpha_i}}(\pi(p^m\alpha_i)) = t^G_{G_{p^m\alpha_i}}(\pi(p^m\alpha_i)) = \pi(p^m\alpha_i)^+$. Hence each s^{p^m} involves $\pi(p^m\alpha_i)$ with coefficient $c_i^{p^m}$. Since $G_{p^m\alpha_i} = G_{\alpha_i}$, we conclude $s^{p^m} \notin A^G_{\mathcal{X}}$ for all $m \in \mathbb{N}$, a contradiction. $\circ$

From 4.2 it is easy to see, that if $R = \mathbb{F}$ is a field of characteristic $p > 0$ and P is a p - group, then $A^G_< \cap S = 0$ and we get the following result:

Theorem 4.4. *If P is a p - group, $R = \mathbb{F}$ a field of characteristic $p > 0$ and $V = \mathbb{F}^\Omega$ is a permutation module, then the relative trace ideal $A^P_<$ of A^P is a prime ideal of height $|\Omega| - |\Omega/P|$; moreover A^P is generated by $A^P_<$ together with the subalgebra S of orbit products $\pi(\omega^G)$. In particular the quotient $A^P/A^P_< \cong S$ is a polynomial ring in $|\Omega/P|$ variables. This holds for any projective $\mathbb{F}P$ - module V, in particular for the regular representation $V \cong \mathbb{F}P$.*

Remark 4.5. *If $V = V_\Omega$ is a permutation module of the p - group P, then*

$$V = \oplus_{\omega^P\in\Omega/P} V_{\omega^P},$$

where the $V_{\omega^P} \cong R^{\uparrow P}_{P_\omega}$ are induced from the trivial module of the stabilizer P_ω, so they are P_ω - projective. The group P acts on Ω without fixed points (i.e. orbits of size 1) if and only if V_Ω is $\mathcal{X}$ - projective where $\mathcal{X}$ is the set of all proper subgroups of P. In this case 4.4 illustrates the statement of 3.1 that $A^P_<$ contains all invariants of degree coprime to p.

4.1. **The regular representation of $\mathbb{Z}_p$.** Now we consider more closely the groups of prime order. So let p be a prime and $P = \mathbb{Z}_p = < g >$. In this situation $A^P_<$ coincides with the absolute trace ideal $A^G_{\{e\}}$ of A^G. Hence 4.4 says that if R is a field of characteristic p, then for any permutation representation of $\mathbb{Z}_p$ on a set Ω, the trace ideal is prime with quotient being a polynomial algebra in as many variables as there are P orbits on Ω. We want to look at the regular representation $V = RP$ in more detail.

We can choose $g = (1\ 2\ \ldots\ p) \in \Sigma_p$ and view V as the natural permutation module $R^{\underline{p}}$ where $\underline{p} := \{1, 2, \ldots, p\}$; hence $A^G/A^G_< \cong R[N(e_1)]$ where $N(e_1)$ is the orbit product $e_1e_2\ldots e_p = e_1g(e_1)g^2(e_1)\ldots g^{p-1}(e_1)$. Due to a fundamental result of M. Göbel on invariant rings of permutation modules, the invariant ring A^P is generated by orbit sums of monomials whose degree is less than or equal to $\begin{pmatrix} p \\ 2 \end{pmatrix}$. In our special case we can do better:

Proposition 4.6. *Let R be an arbitrary commutative ring and $A = S(V)$ with $V = RP$ and $P \cong \mathbb{Z}_p$. Then A^G is generated by $N(e_1)$ together with elements of the trace ideal $A^P_{\{e\}}$ of degree less than or equal to $max\{3, \begin{pmatrix} p-1 \\ 2 \end{pmatrix} + 1\}$.*

Proof: For $p \in \{2,3\}$ the result can be verified easily. So we can assume that $p \geq 5$. As before we will identify the monomial $e_1^{f(1)} \dots e_p^{f(p)}$ with the exponent function $f \in \mathbb{N}_0^{\underline{p}}$. A homogeneous invariant $a \in A^P$ of degree i will be called **indecomposable** if it cannot be written as a sum of products of invariants of strictly lower degree; if it is not indecomposable it is called decomposable. Two homogeneous invariants of the same degree will be called **equivalent** if their difference is decomposable.

Following Göbel [12] we call an orbit sum $f^+ := \sum_{g \in P_f} g(f)$ 'special' if $f(\underline{p})$ is an interval containing 0. So $f = (0,1,2,2,3)$ is special while $h = (0,1,3,4,4)$ and $h' = (1,2,3,4,5)$ are not. In [12] it is proved that A^P is generated by special orbit sums, thus giving the degree bound $0 + 1 + \cdots + p - 1$. So we have to investigate the special orbit sums for indecomposability. Now the following lemma yields our result: $\circ$

Lemma 4.7. *Let $p \geq 5$ and $f^+ \in A^P_<$ be an indecomposable and special orbit sum. Then ($\#f^{-1}(\{0\}) \geq 2$) or ($\#f^{-1}(\{0\}) = 1$ and $\#f^{-1}(\{1\}) \geq 2$) or ($\#f^{-1}(\{0\}) = \#f^{-1}(\{1\}) = 1$ and $\#f^{-1}(\{2\}) = p-2$). In particular f^+ is of degree less or equal to $\leq 0 + 1 + 1 + 2 + 3 + \cdots + p - 2$.*

Proof: As f is special, $\#f^{-1}(\{0\})$ and $\#f^{-1}(\{1\}) \geq 1$ and we can assume that $\#f^{-1}(\{0\}) = \#f^{-1}(\{1\}) = 1$. Let $f(i_1) = 1$ and $f(i_0) = 0$ with $i_0 \equiv i_1 + c$ and $p > c \in \underline{p}$. (By cycling we can assume that $1 \leq i_1 < i_0 \leq p$). Then we get

$$f^+ = (f_1 - 1, f_2 - 1, \dots, 1, \dots, 0, \dots, f_p - 1)^+ \cdot (1,1,\dots,1,0,1,\dots,1,0,1,\dots,1)^+ -$$
$$\sum_{g^i \neq 1} [(f_1 - 1, f_2 - 1, \dots, 1, \dots, 0, \dots, f_p - 1) \cdot g^i((1,1,\dots,0,1,\dots,1,0,1,\dots,1))]^+.$$

All terms that have to be subtracted are decomposable, except for $i = c$, when $g^c((1,1,\dots,0,1,\dots,1,0,1,\dots,1))$ has its zero entries at $i_1 + c = i_0$ and $i_0 + c \neq i_1 \pmod p$. If $f(i_0 + c) > 2$, then

$$[(f_1 - 1, f_2 - 1, \dots, 1, \dots, 0, \dots, f_p - 1) \cdot g^c((1,1,\dots,0,1,\dots,1,0,1,\dots,1))]^+$$

has no entry 1, so is decomposable again; we conclude $f(i_0 + c) = 2$ and $-f^+$ is equivalent to

$$f'^+ := (f_1, f_2, \dots, 2, \dots, 0, \dots, 1, \dots, f_p)^+$$

with $f'(i_1) = 2$, $f'(i_0) = 0$, $f'(i_0 + c) = 1$ and $f'(j) = f(j)$ for all the other j's. Now consider $f'^+ =$

$$(f_1 - 1, f_2 - 1, \dots, 1, \dots, 0, \dots, 0, \dots, f_p - 1)^+ \cdot (1,1,\dots,1,\dots,0,\dots,1,\dots,1)^+ -$$
$$\sum_{g^i \neq 1} [(f_1 - 1, f_2 - 1, \dots, 1, \dots, 0, \dots, 0, \dots, f_p - 1) \cdot g^i((1,1,\dots,1,\dots,0,\dots,1,\dots,1))]^+.$$

A similar argument as before shows that $-f'^+$ is equivalent to

$$f''^+ := (f_1, f_2, \dots, 2, \dots, 1, \dots, 0, \dots, f_p)^+$$

with $f''(i_1) = 2$, $f''(i_0) = 1$, $f''(i_0 + c) = 0$ and $f''(j) = f(j)$ for all the other j's. Now iterating this process shows that $f(i_1 + \ell \cdot c) = 2$ for all $\ell \in \underline{p} \backslash \{1, p\}$ (modulo p). Hence $f(i) = 2$ for all $i \neq i_1, i_0$. $\circ$

Remark 4.8. *If $p = 5$ the bound in 4.6 gives 7, which in this case can be shown to be sharp by computer calculations (this has been verified by Gregor Kemper using a MAGMA implementation of his algorithms, see also* [14] *).*

5. A canonical trace ideal and Cohen-Macaulay quotients

In this section we will discuss a special feature of modular invariant theory over a field $\mathbb{F}$ of positive characteristic $0 < p \mid |G|$. As before let V be a finitely generated $\mathbb{F}G$ - module for the finite group G, $A := A(V) := \mathbb{F}[V]$ its symmetric algebra and A^G the invariant ring. Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$ and $\hat{A} := A \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Then clearly $\hat{V} := V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ can be viewed as an algebraic $\mathbb{F}$ - variety and the maximal ideal spectrum $max - spec\ \hat{A}$ is in bijection to the points in $\hat{V}$.

Now let $U, H \le X \le G$ be subgroups; notice that A is integral and finite over A^H and A as well as A^H are integrally closed in their quotient fields $Q(A)$ and $Q(A)^H$ respectively. Moreover $Q(A) : Q(A)^H$ is a Galois extension with Galois group H. The dual $i^* := i_H^*$ of the inclusion map $i_H : A^H \hookrightarrow A$ is the map $spec\ A \to spec\ A^H,\ m \mapsto m \cap A^H$. In this situation the classical 'lying over', 'going up', 'going down' and transitivity theorems of commutative algebra apply (see [3] Theorem 1.4.4. on pg. 7). Notice that $A^G \otimes_{\mathbb{F}} \overline{\mathbb{F}} = (A \otimes_{\mathbb{F}} \overline{\mathbb{F}})^G = \hat{A}^G$ and the above statements also hold for $\hat{A}$ instead of A. Hence $max - spec\ \hat{A}^H$ can be identified with the set $\hat{V}/H$ of H - orbits on $\hat{V}$ and $\hat{i}_H^*$, the dual of $\hat{i}_H := i_H \otimes_{\mathbb{F}} \overline{\mathbb{F}}$, can be identified with the orbit map $v \mapsto v^H$. Thus a strict geometric quotient $\hat{V}/H$ exists and the invariant ring $\hat{A}^H$ can be interpreted as the ring of regular functions $\mathcal{O}(\hat{V}/H)$ of the quotient variety (e.g. see [11] Theorem 5.52 pg. 187 or [3] pg.8). Notice that all H - orbits are Zariski - closed, since H is finite. The orbit map $\hat{i}_H^*$, being the comorphism of a finite embedding, is surjective and closed, i.e. sends closed irreducible subvarieties of $\hat{V}$ to closed irreducible subvarieties of $\hat{V}/H$.

The U - fixed points in $\hat{V}$ form a linear subspace, hence an irreducible closed $\mathbb{F}$ - subvariety $\hat{V}^U$ of $\hat{V}$. The image $\hat{i}_H^*(\hat{V}^U)$ is therefore a closed and irreducible subvariety of $\hat{V}/H$; it coincides with the set of orbits v^H such that $U \le_H G_v$. Here and in the following, $U \le_H X$ means that an H - conjugate ${}^hU = hUh^{-1}$ of U is contained in the subgroup $X \le G$.

In particular if $H = P$ is a Sylow p - subgroup of G, then

$$\hat{i}_G^*(\hat{V}^P) := \{v^G \mid p \not| \, |v^G|\} = \{v^G \mid P \le_G G_v\} \subseteq \hat{V}/G.$$

For any $v^H \in \hat{i}_H^*(\hat{V}^U)$, the fibre $(\hat{i}_H^*)^{-1}_{|\hat{V}^U}(v^H)$ consists of all $v' \in \hat{V}^U$ that are H - conjugate to v, hence it is a finite set and we conclude that the dimension of the variety $\hat{i}_H^*(\hat{V}^U) \subseteq \hat{V}/H$ is equal to the dimension of the $\overline{\mathbb{F}}$ - vector space $\hat{V}^U$ or of the $\mathbb{F}$ - vector space V^U. Notice that A and A^H still induce regular functions on V and V/H respectively, but the maps $A \to \mathcal{O}(V)$ and $A^H \to \mathcal{O}(V/H)$ are no longer injective and neither $\mathcal{O}(V)$ nor $\mathcal{O}(V/H)$ need to be integral domains.

Now let $\theta^* : \hat{Y} \hookrightarrow \hat{V}$ be the inclusion of a closed G - stable $\mathbb{F}$ - subvariety with $\mathcal{O}(\hat{Y}) = \hat{B} = B \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \theta(\hat{A}) \cong \hat{A}/\hat{\mathcal{I}}$ and $\hat{\mathcal{I}} := \ker \theta$. Again there is a strict quotient $\hat{Y}/G$ and $\hat{i}^* \circ \theta^* : \hat{Y} \to \hat{V}/G$ factors through the quotient map $j^* : \hat{Y} \to \hat{Y}/G$, which is the dual of the inclusion $\mathcal{O}(\hat{Y})^G \hookrightarrow \mathcal{O}(\hat{Y})$. Then $\hat{Z} := \hat{i}^* \circ \theta^*(\hat{Y})$ is a closed $\mathbb{F}$ - subvariety of $\hat{V}/G$

whose corresponding ring of regular functions $\mathcal{O}(\hat{Z}) =: \hat{C}$ is a certain quotient of $\hat{A}^G$. We summarize the situation in the following commutative diagrams:

$$\begin{array}{ccccc} \hat{Y}/G & \overset{\hat{u}^*}{\longrightarrow} & \hat{V}/G \\ & \searrow & & \nearrow & \\ \hat{j}^* \uparrow & & \hat{Z} & & \uparrow \hat{i}^* \\ & \nearrow & & & \\ \hat{Y} & \underset{\theta^*}{\hookrightarrow} & \hat{V} \end{array} \qquad \begin{array}{ccccc} \hat{B}^G & \overset{\hat{u}}{\longleftarrow} & \hat{A}^G \\ & \nwarrow & & \swarrow & \\ \hat{j} \downarrow & & \hat{C} & & \downarrow \hat{i} \\ & \swarrow & & & \\ \hat{B} & \underset{\theta}{\leftarrow} & \hat{A} \end{array}$$

Since the morphism $\omega^* = i^* \circ \theta^* : \hat{Y} \to \hat{Z}$ is surjective, its comorphism $\omega : \hat{C} \to \hat{B}$ is an injection and the kernel of $\phi : \hat{A}^G \to \hat{C}$ coincides with the kernel of $\hat{u}$, so $\hat{C} \cong \hat{A}^G/(\ker\ \theta\hat{i}_G) = \hat{A}^G/(\ker\ \theta \cap \hat{A}^G)$. Although the morphism $\gamma^* : \hat{Y}/G \to \hat{Z}$ is bijective by construction, it is in general **not** an isomorphism. Indeed, the comorphism $\gamma : \hat{C} \hookrightarrow \hat{B}^G$ makes $\hat{B}^G$ a purely inseparable, integral extension. This result can be found (without proof) in [21] pg. 51. For convenience we include a short proof here:

Lemma 5.1. *The injection $\gamma : A^G/(\ker\ \theta \cap A^G) \cong C \hookrightarrow B^G$ is a purely inseparable, integral ring extension.*

Proof: Let $b \in B^G$ and $a \in A$ with $\theta(a) = b$. Fix a Sylow p - group P of G and define $m := \prod_{h\in P} h(a) \in A^P$. Then $\theta(m) = \prod_{h\in P}(h\theta(a)) = b^{|P|}$. Hence $\frac{1}{|G:P|}\theta(t_P^G(m)) = \frac{1}{|G:P|}t_P^G(\theta(m)) = b^{|P|} \in \theta i_G(A^G) = \gamma(C)$. $\circ$

Remark 5.2. *Notice that we have not yet used the linear structure of $\hat{V}$, so all the statements above remain valid, if we start with an arbitrary affine $\mathbb{F}$ - variety $\hat{X}$ instead of the linear variety $\hat{V}$.*

We identify A and A^G with $A \otimes 1 \subseteq \hat{A}$ and $A^G \otimes 1 \subseteq \hat{A}^G$ respectively and define

$$\hat{\mathcal{I}}^{H,U} := \mathcal{I}(\hat{i}_H^*(\hat{V}^U)) = \{f \in \hat{A}^H \mid f(v^H) = 0,\ \forall\ v^H \in \hat{i}_H^*(\hat{V}^U)\}$$

i.e. the (prime) ideal in $\hat{A}^H$ corresponding to the irreducible and closed subset $\hat{i}_H^*(\hat{V}^U) \subseteq \hat{V}/H$; hence $\hat{A}^H/\hat{\mathcal{I}}^{H,U} \cong \mathcal{O}(\hat{i}_H^*(\hat{V}^U))$. We also define $\mathcal{I}^{H,U}$ to be the prime ideal $\hat{\mathcal{I}}^{H,U} \cap A \triangleleft A$. In particular we have $\mathcal{I}^{H,1} = 0$.

Lemma 5.3. *For subgroups $U, H \leq X \leq G$ we have $\mathcal{I}^{X,U} = \mathcal{I}^{H,U} \cap A^X$ and $\mathbb{F}[V^H]$ is a polynomial ring which is a finite integral and purely inseperable extension of $A^H/\mathcal{I}^{H,H}$.*

Proof: Notice that $\mathcal{I}^{X,U} = A^X \cap \mathcal{I}^{1,U} = A^X \cap A^H \cap \mathcal{I}^{1,U} = A^X \cap \mathcal{I}^{H,U}$. Now let $\theta^* : \hat{Y} := \hat{V}^H \hookrightarrow \hat{V}$ be the canonical injection. Clearly $\hat{Y} = \hat{Y}/H$ and we have $\hat{B} = \hat{B}^H \cong \hat{A}/\hat{\mathcal{I}}^{1,H}$ in our notation above. Since $\hat{Y}$ is a linear $\mathbb{F}$ - variety, both algebras $B = \mathbb{F}[V^H]$ and $\hat{B} \cong B \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ are polynomial subalgebras of A and $\hat{A}$ respectively. We also have $C \cong A^H/(A^H \cap \mathcal{I}^{1,H}) = A^H/\mathcal{I}^{H,H}$. From 5.1 we know that $B = B^H$ is a finite, integral and purely inseparable extension of C. $\circ$

The following example shows that $A^H/\mathcal{I}^{H,H}$ will in general be a proper subring of $\mathbb{F}[V^H]$.

Example 5.4. *We use the notation of section* 3. *Let* $V = \mathbb{F}^{\Omega} \cong V^*$ *be a permutation module and* P *a* p *- group. Then* $V^P = \langle v^+ \mid v \in V\rangle_{\mathbb{F}}$ *is the span of the orbit sums in* V. *If* $\delta_\omega \in V^*$ *denotes the element dual to* e_ω, *then* $\mathcal{I}^{1,P} = (\delta_\omega - \delta_{g(\omega)} \mid g \in P)$. *Hence* $\mathbb{F}[V^P] = \mathbb{F}[V]/\mathcal{I}^{1,P} \cong \mathbb{F}[\delta_{[\omega]} \mid [\omega] \in \Omega/P\}$. *On the other hand* $\mathcal{I}^{P,P} = \mathbb{F}[V]^P_<$ *(see 5.10), hence*

$$\mathbb{F}[V]^P/\mathcal{I}^{P,P} \cong \mathbb{F}[\pi(\omega^P) \mid \omega^P \in \Omega/P]$$

and its image in $\mathbb{F}[V^P]$ *coincides with the isomorphic but proper subalgebra* $\mathbb{F}[\delta_{[\omega]}^{|\omega^P|} \mid [\omega] \in \Omega/P\}$.

In the example above, $\mathbb{F}[V]^P/\mathcal{I}^{P,P}$ and $\mathbb{F}[V^P]$ are still abstractly isomorphic polynomial rings. As the next result shows, this is always true for finite groups whose action on V is defined over the prime field. This follows from results of Wilkerson on inseparable extensions of algebras over the Steenrod algebra [23]. At this point I like to thank David Wehlau and Jim Shank for pointing out a gap in an earlier version of 5.6 and also for putting my attention to the paper [23].

Let $\mathbb{F}_p$ denote the prime field of characteristic p, W be a finite - dimensional $\mathbb{F}_p$ - space and $\mathcal{A}_p$ the $mod(p)$ - Steenrod algebra. The following is part of Theorem II in [23]:

Theorem 5.5. *(Wilkerson) Let* R^* *be a sub -* $\mathcal{A}_p$ *- algebra of* $\mathbb{F}_p[W]$ *such that the graded field of quotients* $F\mathbb{F}_p[W]$ *is algebraic over* FR^*. *Define a filtration* $U_i(R^*)$ *of* W *by the rule that* $v \in U_i(R^*)$ *if and only if* v^{p^i} *is separable over* FR^*. *Let* $D^*(R^*)$ *be the 'diagonal' subalgebra of* $F\mathbb{F}_p[W]$ *generated by* $U_i(R^*)^{p^i}$.

Then $D^*(R^*)$ *is separable over* R^* *and* $D^*(R^*)$ *is isomorphic to*

$$\otimes_i(\mathbb{F}_p[U_i(R^*)/U_{i-1}(R^*)])^{p^i}$$

as $\mathcal{A}_p$ *- algebras.*

Proposition 5.6. *Let* G *be a finite group, and* $V \in \mathbb{F}G{-}mod$ *such that the* G *- action is defined over the prime field. Then* $\mathbb{F}[V]^G/\mathcal{I}^{G,G}$ *and* $\mathbb{F}[V^G]$ *are abstracly isomorphic polynomial rings.*

Proof: By flat base change we can assume that $\mathbb{F} = \mathbb{F}_p$ is the prime field. It is well known that the polynomial ring $\mathbb{F}[V]$, as well as its subring $\mathbb{F}[V]^G$ are unstable $\mathcal{A}_p$ - modules (e.g. see [19]). The ideal $\mathcal{I}^{1,G}$ is generated by the elements in $(V^G)^\perp$, so it is $\mathcal{A}_p$ - stable as well as $\mathcal{I}^{G,G} = \mathcal{I}^{1,G} \cap A^G$. Hence $\mathbb{F}[V]^G/\mathcal{I}^{G,G}$ is a sub - $\mathcal{A}_p$ - algebra of $\mathbb{F}[V^G]$ and, by 5.3, the extension is purely inseparable. Using the notation of 5.5, and the extension of this result to $\mathcal{A}_p$, we see that $\mathbb{F}[V]^G = D^*(\mathbb{F}[V]^G)$ is a polynomial ring. Since its transcendence degree is the same as that of $\mathbb{F}[V^G]$, both rings are isomorphic. ∘

Remark 5.7. *In* [23] *only the case* $\mathbb{F} = \mathbb{F}_p$ *is considered. Replacing the mod(p) - Steenrod algebra by the Steenrod algebra* $\mathcal{A}_q$ *with* $q = p^s$ *as in* [19] *chapter 11, we expect 5.5 and 5.6 to hold as well for* $\mathbb{F}_q$ *instead of* $\mathbb{F}_p$ *but leave the corresponding details to a subsequent paper.*

For any ideal $\mathcal{J} \triangleleft A^G$ let $\mathcal{V}(\mathcal{J}) := \{v^G \in V/G \mid f(v) = 0,\ \forall f \in \mathcal{J}\}$. In [22] it has been shown that $\mathcal{V}(A^G_{\{e\}})) = \{v^G \mid p \text{ divides } |G_v|\}$ and that $\sqrt{A^G_{\{e\}}} = \mathcal{P}_G \cap A^G$, where $\mathcal{P}_G$ is the ideal in A, generated by $(g-1)V^*$ for all elements $g \in G$ of order p. (Part of this result is also contained in [8]). We are, in some sense, looking at the other extreme, namely the subvariety of orbits whose stabilisers contain a Sylow p - group of G. Obviously we have

$$i^*(V^P) = \{v^G \mid |P| \le |G_v|\} = \{v^G \in V/G \mid p \text{ does not divide } |v^G|\},$$

where P denotes a fixed Sylow p - group of G. In order to make the corresponding ideal $\mathcal{I}^{G,P}$ computable, we like to describe it in terms of relative traces. To do this, we first go the other way round and give a geometrical description of relative trace ideals. Notice that, by Hilbert's Nullstellensatz, we have for any ideals $\mathcal{I}, \mathcal{J} \lhd A = A^{\{e\}}$, $\sqrt{\mathcal{I}} = \sqrt{\mathcal{J}}$ if and only if $\mathcal{V}(\mathcal{I}) = \mathcal{V}(\mathcal{J})$. Using the orbit map i_G^* one can easily see that the same is true for ideals in A^G. We need the following

Lemma 5.8. (i)*: For $U \leq G$, $f \in A^U$ and $v \in V^U$ we have*

$$t_U^G(f)(v) = f(t_U^G(v)).$$

(ii)*: For every $v \in V$ and every Sylow p - group Q of the stabilizer G_v, there is an element $f \in A^Q$ with $t_Q^G(f)(v) \neq 0$.*

Proof: (i): We have $t_U^G(f)(v) = \sum_{g \in [G:U]}(gf)(v)$, where $[G:U]$ denotes an arbitrary system of representatives for the right cosets of U in G (since $f \in A^U$ the value $t_U^G(f)$ is independent of the choice of coset representatives). Since in this situation we have to consider the G - action on V and on V^* simultaneously we take some care in choosing a system which is appropriate for our needs: it is well known that for any subgroup $U \leq G$ there always exists a system R of coset representatives for U in G which represents at the same time left - and right U - cosets, i.e. $G = \uplus_{x \in \mathrm{R}} xU = \uplus_{x \in \mathrm{R}} Ux = \uplus_{x \in \mathrm{R}} x^{-1}U$. This is a combinatorical consequence of Ph. Hall's 'marriage theorem'.

Now we get $t_U^G(f)(v) = \sum_{x \in \mathrm{R}}(xf)(v) = \sum_{x \in \mathrm{R}}(x^{-1}f)(v)$ (because of the special property of R) $= \sum_{x \in \mathrm{R}} f(xv) = f(t_U^G(v))$.

(ii): This can be seen in a similar way as in the proof of Theorem 2.1. of [22]: let $A = \mathbb{F}[X_1 \ldots, X_n]$, and for $V \ni w \neq v$ define $i(w) := \min\ \{i \mid X_i(w-v) \neq 0\}$ and $\tilde{f} := \prod_{w \in v^G,\ w \neq v}(X_{i(w)} - X_{i(w)}(w))$. Then $\tilde{f}(w) = 0$ for all $w \in v^G \backslash \{v\}$ and $\tilde{f}(v) \neq 0$. Now let $f := \prod_{q \in Q} q(\tilde{f}) \in A^Q$; then $f(v) = \prod_{q \in Q} \tilde{f}(q^{-1}v) = (\tilde{f}(v))^{|Q|} \neq 0$ and for every $w = gv \neq v$: $f(w) = \tilde{f}(w) \prod_{q \in Q \backslash \{1\}} \tilde{f}(q^{-1}w) = 0$. Hence $t_Q^G(f)(v) = f(t_Q^G(v)) = f(t_{G_v}^G(t_Q^{G_v}(v))) = |G_v : Q| \sum_{g \in [G:G_v]} f(gv) = |G_v : Q| \cdot f(v) \neq 0$. $\circ$

Theorem 5.9. *Let $\mathbb{F}$ be a field of characteristic $p > 0$ and $A := \mathbb{F}[V]$ for $V \in \mathbb{F}G - mod$. Let $\mathcal{X}$ be a set of subgroups of G, $P \in Syl_p(G)$, the set of Sylow p - subgroups of G and $\mathcal{X}_P := \{U \leq P \cap {}^gX \mid X \in \mathcal{X},\ g \in G\}$. Then we have for $A_{\mathcal{X}}^G := \sum_{X \in \mathcal{X}} t_X^G(A^X)$:*

(i)*:*

$$A_{\mathcal{X}}^G = A_{\mathcal{X}_P}^G.$$

(ii)*:*

$$\mathcal{V}(A_{\mathcal{X}}^G) = \{v^G \mid P_v \in Syl_p(G_v),\ P_v \cap {}^gX < P_v,\ \forall X \in \mathcal{X},\ g \in G\} =$$

$$\{v^G \mid p \mid |G_v : X_v|,\ \forall X \in \mathcal{X}_P\} = i_G^*(\cup_{Q \leq P,\ \text{minimal with } Q \notin \mathcal{X}_P} V^Q).$$

(iii)*:*

$$\sqrt{A_{\mathcal{X}}^G} = \cap_{Q \leq P,\ \text{minimal with } Q \notin \mathcal{X}_P} \mathcal{I}^{G,Q} =$$

$$(\cap_{Q \leq P,\ \text{minimal with } Q \notin \mathcal{X}_P} ((g-1)V^* \mid g \in Q)A) \cap A^G.$$

Proof: (i): Let $X \in \mathcal{X}$ and $a \in A^X$, then the first Mackey formula of section 3 yields: $t_X^G(a) = \sum_{g\in[P:G:X]} t^P_{P\cap\ {}^gX}(ga) \in A^P$. Hence

$$|G:P|t_X^G(a) = t_P^G(t_X^G(a)) = \sum_{g\in[P:G:X]} t^G_{P\cap\ {}^gX}(gb) \in A^G_{\mathcal{X}_P}.$$

This shows '$\subseteq$'. The other inclusion follows immediately from transitivity and conjugacy relations of trace maps.

(ii): The second equality is clear. So we only have to show the first and last ones. Let RHS denote the right hand side of the first equation. To show '$\subseteq$' we assume $v \in V$ such that $v^G \not\in$ RHS. By Sylow's theorem, all Sylow p - groups of G are conjugate, so any G - orbit v^G can be represented by a suitable $v \in V$, such that $Q := P_v$ is a Sylow p - group of G_v. Then there is $X \in \mathcal{X}$ and $g \in G$ with $Q = Q \cap\ {}^gX$. From 5.8 (ii) there is $f \in A^Q$ with $0 \neq t_Q^G(f)(v) = t^G_{{}^gX}(t_Q^{{}^gX}(f))(v)$. But for $\phi := t_Q^{{}^gX}(f) \in A^{{}^gX}$ we have ${}^{g^{-1}}\phi = t^X_{{}^{g^{-1}}Q}({}^{g^{-1}}f) \in A^X$ and $t_X^G({}^{g^{-1}}\phi) = t^G_{{}^gX}(\phi) = t^G_Q(f) \in A^G_{\mathcal{X}}$ and $v \not\in \mathcal{V}(A^G_{\mathcal{X}})$.

To prove '$\supseteq$' let $v^G \in$ RHS, $X \in \mathcal{X}$ and $f \in A^G_X$. Again we can assume that $Q := P_v \in Syl_p(G_v)$ and that $f = t_X^G(b)$ with $b \in A^X$. Then from the first Mackey formula we get $f = \sum_{g\in[Q:G:X]} t^Q_{Q\cap\ {}^gX}(gb)$ and $t^Q_{Q\cap\ {}^gX}(gb)(v) = (gb)(t^Q_{Q\cap\ {}^gX}(v)) = \frac{|Q|}{|Q\cap\ {}^gX|}(gb)(v) = 0$. Hence $v \in \mathcal{V}(A^G_{\mathcal{X}})$.

To prove the third equality notice that $\mathcal{X}_P$ is closed under taking subgroups, so $\mathcal{V}(A^G_{\mathcal{X}}) = \cup_{Q\le P,\ Q\not\in\mathcal{X}_P}\{v^G \mid Q \in Syl_p(G_v)\} = \cup_{Q\le P,\ Q\not\in\mathcal{X}_P} i_G^*(V^Q) = \cup_{Q\le P,\ \text{minimal with } Q\not\in\mathcal{X}_P} i_G^*(V^Q)$.

(iii): The first equality follows from (ii) and Hilbert's Nullstellensatz. To prove the second statement, notice that V^Q is the socle $soc(V_{|Q})$ of V restricted to Q and it is well known that $(soc(V_{|Q}))^\perp = Rad(\mathbb{F}[Q]) \cdot V^*_{|Q}$, where Rad denotes the Jacobson radical. In case of p - groups in characteristic $p > 0$, the Jacobson radical coincides with the augmentation ideal $\mathbb{F}[Q](h-1 \mid h \in Q)$, hence $(soc(V_{|Q}))^\perp = \langle (h-1)V^* \mid h \in Q\ \rangle_{\mathbb{F}}$. So we conclude that $\hat{\mathcal{I}}(V^Q) = \hat{\mathcal{I}}^{1,Q} = \hat{A}((h-1)V^* \mid h \in Q)$. Now the rest follows by intersection with A and A^G. $\circ$

The following consequences include a characterization of the ideal $\mathcal{I}^{G,P}$ in terms of relative traces:

Proposition 5.10. *Let P be a fixed Sylow p - group of G. Then the following hold:*

(i)*: If $\mathcal{X} = \{H\}$ with a normal subgroup $H \triangleleft G$, then $\mathcal{X}_P = \{U \le H \cap P\}$ and $\mathcal{V}(A^G_H) = i_G^*(\cup_{x\in P\setminus(H\cap P),\ x^p\in H\cap P}\ V^x)$ and*

$$\sqrt{A^G_H} = (\cap_{x\in P\setminus(H\cap P),\ x^p\in H\cap P}\ (x-1)V^*A)\ \cap\ A^G.$$

In particular we obtain Feshbach's theorem (see [8]*):*

$$\sqrt{A^G_1} = (\cap_{x\in P\setminus\{e\},\ x^p=1}\ (x-1)V^*A)\ \cap\ A^G.$$

(ii)*: For $\mathcal{X} := \{Q \mid Q < P\}$ and $A^G_{\mathcal{X}} = A^G_{<P}$ we have:*

$$\mathcal{V}(A^G_{<P}) = \mathcal{V}(\mathcal{I}^{G,P}) = \{v^G \in V/G \mid p \text{ does not divide } |v^G|\} = i_G^*(V^P).$$

$$\sqrt{A^G_{<P}} = \mathcal{I}^{G,P} = A^G \cap (hf - f \mid f \in A,\ h \in P)A.$$

In particular height $A^G_{<P} = \operatorname{codim}\ i^*(\hat{V}^P) = \operatorname{codim}_{\mathbb{F}}\ V^P$.

(iii)*: For every family $\mathcal{X}$ with $\mathcal{X}_P$ consisting of proper subgroups of P we have*

$$\text{height } A^G_{\mathcal{X}} \leq \text{codim}_{\mathbb{F}}\, V^P.$$

Proof: Most of these statements are immediate consequences of 5.9.

In (i) let $Q = \langle q_1, \dots, q_s \rangle$ be a subgroup of P minimal subject to the condition not lying in $\mathcal{X}_P$, i.e. not being a subgroup of $H \cap P$; suppose moreover that the q_i form a minimal generating set. Then some $q_i \notin H \cap P$, hence $Q = \langle q_i \rangle$ and $q_i^p \in H \cap P$.

(ii) is clear by 5.9 and the fact that height $A^G_{<P}$ = height $\sqrt{A^G_{<P}}$ = height $\mathcal{I}^{G,P}$ = codim $\mathcal{V}(\mathcal{I}^{G,P})$ = codim $i^*_G(V^P) = \text{codim}_{\mathbb{F}}(V^P)$.

(iii) follows from the fact $A^G_{\mathcal{X}} = A^G_{\mathcal{X}_P} \leq A^G_{<P}$. $\circ$

If V is projective relative to a proper subgroup of a Sylow p - group of G, and hence with respect to some $Q < P$, we know from 3.1, that $A^G_{<P}$ is 'large' in the sense that it contains all invariants of degree not divisible by p.

If G has a cyclic Sylow p - group P, then all trace ideals can be described in an easy way: notice that in this case, the lattice of subgroups of P is linearly ordered, so, for any family $\mathcal{X}$ there is a unique subgroup $U \leq P$ such that $\mathcal{X}_P = \{X \leq U\}$. If $U < P$ then there is also a unique subgroup $Q \leq P$ containing U as a maximal subgroup. So we get:

Corollary 5.11. *Let $P \in Syl_p(G)$ be cyclic, then every trace ideal $A^G_{\mathcal{X}}$ is either A^G (if $P \in \mathcal{X}_P$), or of the form $A^G_{<Q}$ for a subgroup $Q \leq P$. Moreover we have*

$$\mathcal{V}(A^G_{<Q}) = i^*_G(V^Q)$$

and $\sqrt{A^G_{<Q}}$ is the prime ideal $\mathcal{I}^{G,Q} = ((V^Q)^{\perp}A) \cap A^G$ of height $\text{codim}_{\mathbb{F}} V^Q$.

In the following we are going to construct certain Cohen - Macaulay quotients of modular invariant rings. Recall that a (positively) graded $\mathbb{F}$ - algebra R is Cohen - Macaulay (CM) if and only if it is free as a module over the subalgebra generated by a homogeneous system of parameters (h.s.o.p.) of R. In that case R is free over any such h.s.o.p..

Definition 5.12. *(see [4] pg.270) Let $S \leq R$ be a subring; a* **Reynolds operator** *for the pair (R, S) is a split epimorphism of S modules $\pi: \ _SR \to S$, i.e. with a right inverse $\rho: S \hookrightarrow R$. For example t^G_P is a Reynolds operator for the pair (A^P, A^G).*

Due to a theorem of Eagon and Hochster ([13]) it is well known that A^G is always a Cohen - Macaulay ring if char $\mathbb{F}$ is coprime to $|G|$. Using the Reynolds operator t^G_P this has been generalized by Campbell, Hughes and Pollack, who proved that if char $\mathbb{F} = p > 0$, the invariant ring A^G is still Cohen - Macaulay provided A^P is ([5]). The next proposition will be a further generalization thereof.

Crucial for all these results is the following theorem of Hochster-Eagon (see [4] 6.4.5):

Theorem 5.13. *Suppose S is a subring of R such that R is integral over S and π is a Reynolds operator for (R, S). Then, if R is CM, so is S.*

First notice that if $\mathcal{I} \triangleleft A^H$ is an ideal then, as $A^G \leq A^H$ we have $t^G_H(\mathcal{I})A^G = t^G_H(\mathcal{I}A^G) \subseteq t^G_H(\mathcal{I})$; hence $t^G_H(\mathcal{I})$ is an ideal of A^G.

Proposition 5.14. *Let $\mathbb{F}$ be a field of characteristic p and P a Sylow p - subgroup of G. Suppose that $\mathcal{I} \triangleleft A^P$ is an ideal satisfying*

$$t^P_{P\cap\ {}^gP}(g\mathcal{I}) \subseteq \mathcal{I} \quad (*)$$

for all $g \in G$. Then $t^G_P(\mathcal{I}) = \mathcal{I} \cap A^G \triangleleft A^G$ and t^G_P induces a Reynolds operator for the pair $(A^P/\mathcal{I}, A^G/(\mathcal{I} \cap A^G))$. In particular if $A^P/\mathcal{I}$ is Cohen - Macaulay so is $A^G/(\mathcal{I} \cap A^G)$.

Proof: Let $a \in \mathcal{I} \cap A^G$, then $|G : P|a = t^G_P(a)$, hence $a \in t^G_P(\mathcal{I})$. Let $b \in t^G_P(\mathcal{I})$; then, by the Mackey - formula, $b = t^G_P(i) = \sum_{g\in[P:G:P]} t^P_{P\cap\ {}^gP}(gi) \subseteq \mathcal{I} \cap A^G$. Now we have a surjection $t^G_P :\ A^P/\mathcal{I} \to A^G/t^G_P(\mathcal{I}) = A^G/(\mathcal{I}\cap A^G)$ and a canonical embedding $\rho : A^G/\mathcal{I}\cap A^G \hookrightarrow A^P/\mathcal{I}$; both are homomorphisms of $A^G/(\mathcal{I} \cap A^G)$ - modules with $t^G_P \circ \rho = |G : P| \cdot Id$ invertible. Hence t^G_P induces a Reynolds operator for the pair $(A^P/\mathcal{I}, (A^G/\mathcal{I} \cap A^G))$. Clearly $A^P/\mathcal{I}$ is integral over $A^G/(\mathcal{I} \cap A^G)$ and the claim follows from 5.13. ∘

Remark 5.15. (i) *Notice that for $\mathcal{I} = 0$ we recover the above mentioned result of Campbell, Hughes and Pollack.*

(ii) *The condition $(*)$ above is satisfied whenever $g\mathcal{I} = \mathcal{I}$ for all g in the normalizer $N_G(P)$ and $A^P_< \leq \mathcal{I}$.*

(iii) *Let $\mathcal{Y}$ be a family of subgroups of $P \in Syl_p(G)$ such that $Y \in \mathcal{Y} \Rightarrow P\cap\ {}^gY \in \mathcal{Y}$ for every $g \in G$, then condition $(*)$ of 5.14 is satisfied for $\mathcal{I} = A^P_{\mathcal{Y}}$. Indeed, for $Y \in \mathcal{Y}$, $a \in A^Y$ we have $t^P_{P\cap\ {}^gP}(gt^P_Y(a)) = t^P_{P\cap\ {}^gP}(t^{{}^gP}_{{}^gY}(ga)) = \sum_{h\in[P\cap\ {}^gP:{}^gP:{}^gY]} t^P_{P\cap\ {}^gP}(t^{P\cap\ {}^gP}_{P\cap\ {}^gP\cap\ {}^{hg}Y}(hga)) = \sum_{h\in[P\cap\ {}^gP:{}^gP:{}^gY]} t^P_{P\cap\ {}^gP\cap\ {}^{hg}Y}(hga) = \sum_{h\in[P\cap\ {}^gP:{}^gP:{}^gY]} t^P_{P\cap\ {}^{hg}Y}(t^{P\cap\ {}^{hg}P}_{P\cap\ {}^gP\cap\ {}^{hg}Y}(hga)) \in A^P_{\mathcal{Y}}$.*

Now we can prove the following theorem on Cohen - Macaulay quotients of modular invariant rings, which generalizes the theorem of Hochster and Eagon on the CM - property of non-modular invariant rings:

Theorem 5.16. *Let* char $\mathbb{F} = p$ *and P a Sylow p - subgroup of G. Suppose that the action of P on V is defined over the prime field. Then $\mathcal{I}^{G,P} = t^G_P(\mathcal{I}^{P,P}) = \mathcal{I}^{P,P} \cap A^G$ and $A^G/\mathcal{I}^{G,P}$ is a Cohen - Macaulay ring.*

Proof: From 5.10 we see that $A^P_< \leq \mathcal{I}^{P,P}$. Clearly for any $g \in N_G(P)$ we have $g\mathcal{I}^P_P = \mathcal{I}^P_P$, so $\mathcal{I}^{P,P}$ satisifies the assumption of 5.14. By 5.6 $A^P/\mathcal{I}^P_P$ is CM, so the result follows from 5.14. Notice that if $P = 1$ we get $\mathcal{I}^P_P = \mathcal{I}(V^*) = (0) \triangleleft A$ and A^G is CM. ∘

We consider the ideals $A^G_{<P}$ and $\mathcal{I}^{G,P}$ as links between the $\mathbb{F}G$ - module structure of V and the ideal structure of $\mathbb{F}[V]^G$ and therefore propose to study conditions on V, implying the equality

$$A^G_{<P} = \mathcal{I}^{G,P} \ (*),$$

which, by 5.10, is equivalent to the primality of $A^G_{<P}$. From the point of view of **constructive** modular invariant theory the analysis of $\mathbb{F}[V]^G$ might benefit in this case from the fact that $A^G_{<P}$ is constructible using relative trace maps and the quotient can be handled by methods, developed for Cohen - Macaulay rings.

By the results above, to prove equality $(*)$ it suffices to consider the case where G is a p - group: in fact $A^P_< = \mathcal{I}^{P,P}$ implies that $\mathcal{I}^{G,P} = t^G_P(\mathcal{I}^{P,P}) = t^G_P(A^P_<) = A^G_{<P}$. But the following example shows, that for p - groups in general, $A^G_{<P}$ can be stricly smaller than $\sqrt{A^G_{<P}}$:

Example 5.17. *(Kathrin Kuhnigk* [15]*) Let $p > 2$, $\mathbb{F}$ of characteristic p and $G = \langle g\rangle \cong \mathbb{Z}_p$, acting on $\mathbb{F}[X,Y]$ by $g^i(X) = X + iY$ and $g^i(Y) = Y$ for every i. Then, since $t_1^G(X^p) = t_1^G(X)^p$, it is clear that A_1^G is generated by $t_1^G(X^i)$ for $i = 1,\dots,p-1$. We get $t_1^G(X^i) = \sum_{\ell=0}^{p-1}(X+\ell Y)^i = \sum_{\ell=0}^{p-1}\sum_{j=0}^{i}\begin{pmatrix} i \\ j\end{pmatrix} X^{i-j}Y^j\ell^j = \sum_{j=0}^{i}\begin{pmatrix} i \\ j\end{pmatrix} X^{i-j}Y^j\ (\sum_{\ell=0}^{p-1}\ell^j)$. Whenever $0 < j < p-1$, the subset $\{\ell^j \mid 0 < \ell < p\}$ of $\mathbb{F}_p^*$ is a nontrivial multiplicative subgroup S. If S^+ is the sum of its elements, then $(s-1)S^+ = S^+$ for $1 \neq s \in S$, hence $S^+ = 0 = \sum_{\ell=0}^{p-1}\ell^j$. This last equation also holds for $j = 0$, while for $j = p-1$ we get $\sum_{\ell=0}^{p-1}\ell^j = -1 \in \mathbb{F}$. Hence $t_1^G(X^i) = -Y^{p-1}$ for $i = p-1$ and zero otherwise. We conclude that*

$$A^G_{<P} = (Y^{p-1}) < (Y) = \sqrt{A^G_{<P}},$$

We have already seen in 4.3 that all relative trace ideals $A^G_{\mathcal{X}}$ are radical ideals if V is a permutation representation. In the following theorem we will prove that this is still true for direct summands of permutation modules, in particular for projective modules.

Theorem 5.18. *If $V \in {}_{FG}mod$ is a direct summand of a permutation module, then all (relative) trace ideals $A^G_{\mathcal{X}}$ are radical ideals. In particular if $P \in Syl_p(G)$, then*

$$A^G_{<P} = \mathcal{I}^{G,P}.$$

is a prime ideal of height codim V^P *and the quotient $A^G/A^G_{<P}$ is a Cohen - Macaulay ring of Krull dimension* dim V^P.

If moreover P is cyclic, then each trace ideal $A^G_{\mathcal{X}}$ with $P \notin \mathcal{X}_P$ is a prime ideal of the form $\mathcal{I}^{G,Q}$ with height $\mathrm{codim}_{\mathbb{F}}\ V^Q$ *and $Q \leq P$ minimal subject to ${}^gX \cap P < Q$ for all $X \in \mathcal{X}$ and $g \in G$.*

Proof: From 5.9 and the remark 5.15 (iii) we see that

$$A^G_{\mathcal{X}} = A^G_{\mathcal{X}_P} = t_P^G(A^P_{\mathcal{X}_P}) = A^P_{\mathcal{X}_P} \cap A^G.$$

So, together with 'lying over' and 'going up and down' for prime ideals and integral extensions, we can restrict to the Sylow p - group P, and hence assume right away that G is a p - group. For p - groups in characteristic p any transitive permutation module is indecomposable and so, by the Krull - Schmidt theorem, any direct summand of a permutation module is again a permutation module (see [7] Chapter IX, 3.4.). Hence we can assume that V is a permutation module and in this case we know the results already from 4.3 and 4.4.

The last statement is an immediate consequence of 5.11. $\circ$

Remark 5.19. (i) *Notice that the proof of 5.18 is independent of 5.6 and hence of the result 5.5 in* [23].

(ii) *The last statement of 5.18 generalizes a result of Shank and Wehlau* [22] *Corollary* 6.2.

(iii) *Let V be indecomposable with vertex $Q \leq P$, then V is a direct summand of $W^{\uparrow G}$ where W can be chosen to be an indecomposable summand of $V_{|Q}$ and then is unique up to isomorphism and conjugation. In modular representation theory such an $\mathbb{F}Q$ - module W is called a* **source** *of V. It is well known (and proved by 'Mackey decomposition') that V is a direct summand of a permutation module if and only if the trivial module $\mathbb{F}_Q$ is a source of V (see e.g.* [16], *chapter II,* 12.5*). Thus we have shown that $A^G_{<P} = \mathcal{I}^{G,P}$ for 'trivial source modules'. In particular if $Q < P$ (which is equivalent to the fact that the trivial $\mathbb{F}P$ - module is not a direct summand of V_P), then A^G is generated by $A^G_{<P}$ together with preimages of*

generators for the CM - algebra $A^G/A^G_{<P}$; these latter ones can all be chosen to be of degree divisible by p.

Acknowledgements: I thank Jacques Alev (Reims), David Green (Wuppertal) and Gregor Kemper (Heidelberg) for stimulating and fruitful discussions during the process of writing this paper. I also thank David Wehlau and Jim Shank (Kingston, Ontario) for their very valuable and constructive criticism.

References

[1] J. Alperin, *Local Representation Theory*, Cambridge Univ. Press 1986.
[2] D. Benson, *Representations and Cohomology I*, Cambridge Univ. Press 1991.
[3] D. Benson, *Polynomial Invariants of Finite Groups*, Cambridge Univ. Press 1993.
[4] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge Univ. Press 1993.
[5] H.E.A. Campbell, I. Hughes, R.D. Pollack, *Rings of Invariants and p - Sylow Subgroups*, Canad. Math. Bull. **34**(1), 42-47, (1991).
[6] C.W. Curtis, I. Reiner, *Methods of Representation theory I,II*, Wiley Interscience 1981, 1987.
[7] W. Feit, *The Representation Theory of Finite Groups*, North-Holland (1982).
[8] M. Feshbach, *p - subgroups of compact Lie groups and torsion of infinite height in* $H^*(BG;\mathbb{F}_p)$, Mich. Math.J.**29**, 299-306, (1982).
[9] P. Fleischmann, *A new degree bound for Vector Invariants of Symmetric Groups*, to appear in Transactions of the AMS http://www.exp-math.uni.essen.de/ peter
[10] P. Fleischmann, W. Lempken, *On generators of modular invariant rings of finite groups*, Bull. London Math. Soc. **29**, (585-591), (1997). http://www.exp-math.uni.essen.de/ peter
[11] J. Fogarty, *Invariant theory*, Benjamin (1969).
[12] M. Göbel, *Computing bases for rings of permutation- invariant polynomials*, J. Symb. Comp. **19**, 285-291, (1995).
[13] M. Hochster, J.A. Eagon, *Cohen - Macaulay rings, invariant theory and the generic perfection of determinantal loci*, Amer. J. Math. **93**, 1020-1058, (1971).
[14] G. Kemper, *Calculating invariant rings of finite groups over arbitrary fields*, J. of Symbolic Computation, (1995).
[15] K. Kuhnigk, *Der Transfer in der modularen Invariantentheorie*, Diplomarbeit, Universität Göttingen, (1998).
[16] P. Landrock, *Finite Group Algebras and their Modules*, LMS Lecture Note Series, Cambridge University Press (1984).
[17] M. Neusel, *Mme Bertin's* $\mathbb{Z}/4$ *revisited*, preprint (1997).
[18] M. Neusel, *The transfer in the invariant theory of modular representations*, preprint, (1997).
[19] L. Smith, *Polynomial Invariants of Finite Groups*, A.K. Peters Ltd., (1995).
[20] L. Smith, *Modular vector invariants of cyclic permutation representations*, preprint (1997).
[21] J.P. Serre, *Algebraic groups and Class Fields*, Springer, (1975).
[22] R.J. Shank, D.L. Wehlau, *The transfer in modular invariant theory*, preprint, http://www.mast.queensu.ca/ shank/transfer.dvi, (1997).
[23] C. Wilkerson, *Rings of invariants and inseparable forms of algebras over the Steenrod algebra*, preprint http://hopf.math.purdue.edu/pub/hopf.html, (1979).

Institute for Experimental Mathematics, University of Essen, Ellernstr. 29, 45326 Essen, Germany

E-mail address: peter @ exp-math.uni-essen.de