Finite Groups Have Local Non - Schur Centralizers

Peter Fleischmann
Ingo Janiszczak
Wolfgang Lempken

Introduction

The main purpose in this paper is to prove the following

Theorem: *If G is a finite group of order divisible by the prime p then G contains a p - singular element g whose p - part is not contained in the commutator subgroup of $C_G(g)$.*

Apart from purely group theoretical interest this investigation has also been motivated by recent work on consequences of the Donald-Flanigan conjecture in [GG]. This conjecture claims the existence of a semisimple deformation of $\overline{F}_pG$ for any finite group G of order divisible by p.
In [GG] M.Gerstenhaber and D.Green observed that the presence of such a deformation forces G to have an element g such that $C_G(g)$ contains a normal subgroup of index p. Note that the existence of such an element g is an easy consequence of our result.
Finally note that the proof of the Theorem makes use of the classification of the finite simple groups.

1. Reduction to simple groups

Throughout this text we use standard notation for finite groups as for example in [G], [H] or [A]. In particular we use the following:
$|G|$ denotes the order of the group G;
$|G|_p$ denotes the maximal p - power dividing $|G|$ for a prime number p;
$\pi(G)$ is the set of prime divisors of $|G|$;
$G' = [G, G]$ is the commutator subgroup and $Z(G)$ is the centre of G;
for $x \in G$ and $N \lhd G$ the subgroup $\{g \in G | x^g \in xN\}$ is denoted by $C_G(x \ mod \ N)$.
$H^2(G, \mathbf{C}^*)$ denotes the Schur multiplier of G.

For the sake of convenience we make the following

Definition 1.1: *Let G be a finite group of order divisible by the prime p.*

(1) G satisfies the **strong Non - Schur property S(p)** *if there exists a p - element $g \in G$ such that g is not contained in $C_G(g)'$*

(2) G satisfies the **weak Non - Schur property W(p)** *if there exists an element $g \in G$ such that the p - part g_p of g is not contained in $C_G(g)'$.*

(3) G satisfies the **Commutator index property C(p)** *if there exists an element $g \in G$ such that p divides the index $|C_G(g) : C_G(g)'|$.*

We start with a few elementary observations :

For any finite group G and Sylow p - subgroup P we consider the transfer map $V_{G \to P}$ as defined in [H,Chap.5]. It is well known that the kernel of this homomorphism from G to P/P' is the minimal normal subgroup $G'(p)$ of G with abelian p - factor group. The intersection $G'(p) \cap P$ is the "p - focal subgroup"

$$P^* := < [q, g] \mid q^g \ , q \ \in P, \ g \in G >$$

and we have the following isomorphism:

$$G/G'(p) \ \cong \ P/P^*.$$

For any element $g \in G$ we let $S_p(g)$ denote a Sylow p - subgroup of the centralizer $C_G(g)$. For an abelian group A we let A_p denote its Sylow p - subgroup.

Lemma 1.2: *(1) Let $g \in G$ and put $C_g := C_G(g)$ as well as $S_g := (C_g)' \cap Z(C_g)$ and $\overline{C}_g := C_g/S_g$; moreover let $p \in \pi(C_g)$. Then $(S_g)_p \ \leq \ H^2(\overline{C}_g, \mathbf{C}^*)_p \ \leq \ H^2(\overline{S_p(g)}, \mathbf{C}^*)$. In particular, if $(S_g)_p \neq 1$ then $S_p(g)$ is nonabelian and $|(S_g)_p|p^2$ divides $|S_p(g)|$.*

(2) The property W(p) (S(p)) is equivalent to the following:
For each divisor p of G there is an element (a p - element) $g \in G$ such that the p - part g_p is contained in $S_p(g)\backslash S_p(g)'$.
Moreover G satisfies S(p) whenever
(a) The exponent $exp(P')$ is strictly smaller than $exp(P)$, or
(b) the centre of P is not contained in P', or
(c) the Sylow p - subgroups are metacyclic.

(3) The property C(p) is equivalent to any of the following:
(a) For each divisor p of G there is an element $g \in G$ such that the p - focal subgroup $S_p(g)^*$ of the centralizer $C_G(g)$ is a proper subgroup of $S_p(g)$.
(b) There exists $g \in G$ such that p divides the order $o(g)$ of g as well as the index $|C_G(g) : C_G(g)'|$.

(4) We have the following implications: S(p) $\rightarrow$ W(p) $\rightarrow$ C(p).

(5) If $g \in G$ is a p' - element such that $C_G(g)$ satisfies S(p) then G satisfies W(p).

Proof: (1) is contained in [H, Chap.5§23]. To prove (2) we consider the transfer map $V_{C_G(g) \rightarrow S_p(g)}$ from $C_G(g)$ to $S_p(g)$. Being central, g_p is mapped to $g_p^{|C_G(g):S_p(g)|}$. $(S_p(g))'$. By the hypothesis g_p is not in the kernel of the transfer map, hence not contained in $(C_G(g))'$. Now part (a) and (b) follow easily. To prove $2(c)$ we may assume that $< x > \triangleleft < x, y >=: P$. As $P/ < x >$ is cyclic and thus has trivial Schur multiplier; the claim follows. (3) a) and b) are alternative formulations of the definition; (4) and (5) are immediate. ♣

Remarks 1) It is part (1) of Lemma 1.2 that led us to speak of the strong and weak Non - Schur property.

2) In general the property W(p) does not imply S(p). Consider the simple Tits group $Ti \cong {}^2F_4(2)'$ of order $2^{11}3^35^213$. Ti has only one class of 3 - elements, say with representative x. As $C_{Ti}(x) \cong 3^{1+2} : \mathbf{Z}_4$, Ti does not satisfy S(3), but in view of $C_{Ti}(12A) =< 12A >$ for a suitable element $12A$ of order 12 the property W(3) still holds.

Proposition 1.3: *Every minimal counterexample to W(p) (S(p)) is a non - abelian simple finite group.*

Proof: Suppose G is a minimal counterexample with proper normal subgroup $N \triangleleft G$. If p still divides the order of $\overline{G} := G/N$ then there is an element (p - element) $g \in G$ with $\overline{g} \in \overline{G}$ such that $(\overline{g})_p \in S_p(\overline{g})\backslash \underline{S_p(\overline{g})'}$ but $g_p \in S_p(g)'$ ($g_p \in S_p(g_p)'$). Clearly for any $x \in G$ the canonical image $\overline{S_p(x)}$ is contained in $S_p(\overline{x}) \leq C_{\overline{G}}(\overline{x})$. Hence in case of W(p) we get $(\overline{g})_p \in \overline{S_p(g)}' \leq S_p(\overline{g})'$. In case of S(p) we conclude $\overline{g} = \overline{g}_p \in \overline{S_p(g_p)}' \leq S_p(\overline{g})'$, both in contradiction to the hypothesis.
So we may assume that p does not divide G/N. Since $S_p(x)$ is already contained in N for each $x \in N$, there is, by induction, a (p) - element $g \in N$ with $g_p \in S_p(g)\backslash S_p(g)'$. ♣

2. The alternating and sporadic simple groups

Proposition 2.1: *The alternating groups A_n for $3 \leq n$ satisfy S(p) for each prime $p \in \pi(A_n)$.*

Proof: For $n \leq 7$ we easily see that A_n has abelian or metacyclic Sylow p - subgroups for each $p \in \pi(A_n)$; so we are done by 1.2. Henceforth we assume that $G = A_n$ with $8 \leq n$ and $p \in \pi(G)$. If p is odd we put $g = (1, ..., p)$ and observe that $C_G(g) = < g > \times H$ with $H \cong A_{n-p}$. For $p = 2$ we take $g = (1,2,3,4)(5,6)$ and observe that $C_G(g) = < g > \times H$ with $H \cong S_{n-6}$. ♣

Proposition 2.2: *Suppose G is one of the 26 sporadic simple groups and that $p \in \pi(G)$. Then G satisfies W(p); moreover G satisfies S(p) except for $(G,p) \in \{(Ru,3),(J_4,3),(Th,5)\}$. In these exceptions G has Sylow p - subgroups isomorphic to p^{1+2} of exponent p and only one class of elements of p - power order.*

Proof: In order to verify the result we use the information given in [A]. Moreover, in order to simplify the check we use the following criteria:
(1) If $P \in Syl_p(G)$ is abelian or metacyclic then S(p) holds (1.2)
(2) If x is a p - element of G such that $C_G(x)/ < x >$ has cyclic Sylow p - subgroups then G satisfies S(p).
(3) If $P \in Syl_p(G)$ is non - abelian of order p^3 then G satisfies S(p) unless P is extraspecial of exponent p and G has only one class of elements order p. (This is an easy consequence of (1) and (2).)
(4) If x is a p - element of order at least p^2 in G and if $|C_G(x)|_p = | < x > | \cdot p^2$, then G satisfies S(p).
Using the criteria (1) - (4) it is now a straightforward task to verify S(p) for all the sporadics with the exceptions stated above.
Finally we note that in these exceptions we find elements x such that $C_G(x) = < x >$ and $o(x) \in \{15, 66, 36\}$, respectively. So W(p) still holds in these cases. ♣

3. Simple groups of classical Lie type

In this and the next section we look at the infinite series of simple Lie - Chevalley groups G which occur either as commutator subgroups or central factor groups of suitable finite groups of Lie type $\mathbf{G}_F$, i.e. the fixed point groups of a Frobenius endomorphism of a connected reductive algebraic group $\mathbf{G}$ defined over the finite field $\mathbf{F}_q$ with q elements and characteristic $r > 0$. In this context local questions naturally divide up into two cases:
a) the prime p divides q, hence $p = r$ is the "natural characteristic" or
b) p does not divide q and thus $p \neq r$ is a "cross characteristic".

The following lemma contains classical results which are very useful in the present context:

Lemma 3.1: *Let G be a finite Lie - Chevalley group defined over a field of characteristic r. Then the following holds:*
(1) G contains "regular unipotent elements" and for any such element, say x, the order $o(x)$ is a power of r and the centralizer $C_G(x)$ is abelian.
(2) Let W denote the Weyl group with respect to a maximal torus of the ambient algebraic group G. If $p \in \pi(G) \backslash \pi(W)$ is a prime different from r, then the Sylow p - subgroups of G are abelian.

Proof: Both statements hold for the finite groups of Lie type by [SS] III, 1.19, 1.16 and II, 5.19. Since $\pi(Z(G))$ is contained in $\pi(W)$, the statements hold as well for the simple groups. ♣

Obviously by Lemma 3.1 we can assume that p does not divide q. We start with the Chevalley groups of classical type:

Proposition 3.2: *Let G be one of the following finite groups:*

a) for Dynkin type A_{n-1}, untwisted: the general linear group $Gl_n(q)$, the special linear group $Sl_n(q)$ or the simple group $PSl_n(q)$ $^{(*)}$;

b) for Dynkin type A_{n-1}, twisted: the general unitary group $U_n(q)$, the special unitary group $SU_n(q)$ or the simple group $PSU_n(q)$ $^{(*)}$;

c) for Dynkin type B_n: the special orthogonal group $SO_{2n+1}(q)$, or $P\Omega_{2n+1}(q)$ $^{(*)}$;

d) for Dynkin type C_n the symplectic group $Sp_{2n}(q)$, or $PSp_{2n}(q) = Sp_{2n}(q)/Z$ $^{(*)}$, where Z is the centre of $Sp_{2n}(q)$.

e) for Dynkin type D_n untwisted: the special orthogonal group $SO_{2n}(q)$, or $P\Omega_{2n}(q) = (SO_{2n}(q))'/Z$ $^{(*)}$, where Z is the centre of the commutator group $(SO_{2n}(q))'$.

f) for Dynkin type D_n twisted: the special orthogonal group $SO^-_{2n}(q)$, or $P\Omega^-_{2n}(q) = (SO^-_{2n}(q))'/Z$ $^{(*)}$, where Z is the centre of the commutator group $(SO^-_{2n}(q))'$.

Then the following holds:

Let p be a prime number dividing the order of G but not dividing q. Then there exists an element $t \in G$ of order p or, in case $^{(*)}$ and $p = 2$ or 3, an element of order 4, 8 or 9 respectively , such that t is not contained in the commutator subgroup of the centralizer $C_G(t)$.
In particular S(p) holds for G.

Proof: Each of these groups occurs as a section of a suitable finite group of Lie type, i.e. a group $H := \mathbf{G}_F$ of F - fixed points of a connected reductive algebraic group $\mathbf{G}$, defined over $\mathbf{F}_q$ with Frobenius endomorphism $F : \mathbf{G} \longrightarrow \mathbf{G}$. More precisely let $\mathbf{G}$ be one of the following algebraic groups:
in case a) and b) $\mathbf{G} = Gl_n(\overline{\mathbf{F}}_q)$, in case c) $\mathbf{G} = SO_{2n+1}(\overline{\mathbf{F}}_q)$ in case d) $\mathbf{G} = Sp_{2n}(\overline{\mathbf{F}}_q)$ and in case e) and f) $\mathbf{G} = SO_{2n}(\overline{\mathbf{F}}_q)$.

In each of these cases an F - stable maximal torus $\mathbf{T}$ can be described as the set $\mathbf{T} = (\overline{\mathbf{F}}_q^{\,*})^{\mathbf{n}}$ of functions $f : \mathbf{n} := \{1, 2, ..., n\} \longrightarrow \overline{\mathbf{F}}_q^{\,*}$. e.g.

$$f = t = \begin{pmatrix} t_1 & 0... & 0... & 0... \\ 0... & t_2 & ... & \\ 0... & ... & ... & \\ 0... & ... & ... & t_n \end{pmatrix} \in \mathbf{T} \leq Gl_n(\overline{\mathbf{F}}_q)$$

or

$$f = t = \begin{pmatrix} t_1 & 0... & 0... & 0... & & & & \\ 0... & t_2 & ... & ... & & 0 & & \\ 0... & ... & ... & ... & & & & \\ 0... & ... & ... & t_n & & & & \\ & & & & t_n^{-1} & 0... & 0... & 0... \\ & & & & 0... & t_{n-1}^{-1} & ... & ... \\ & 0 & & & 0... & ... & ... & ... \\ & & & & 0... & ... & ... & t_1^{-1} \end{pmatrix} \in \mathbf{T} \leq Sp_{2n}(\overline{\mathbf{F}}_q)$$

All Weyl groups $W = N_G(\mathbf{T})/\mathbf{T}$ of classical groups are subgroups of the wreath product $\mathbf{Z}_2 \wr \Sigma_{\mathbf{n}}$, which is the Weyl group in the cases c) and d). So W is acting naturally on $\mathbf{T}$ (viewed as a set of functions) by ${}^{(z,\sigma)}f(i) = (f(\sigma^{-1}(i)))^{z(i)}$, where $z(i) \in \{1, -1\}$. In case a) and b) we have $W = \Sigma_{\mathbf{n}}$, whereas in e) and f), W is the subgroup of index 2 of $\mathbf{Z}_2 \wr \Sigma_{\mathbf{n}}$ consisting of elements with an even number of 1's in the base group $\mathbf{Z}_2^n$. The action of F on $\mathbf{T}$ can be described as follows: in case a),c),d) and e): $F(f)(i) = f(i)^q \ \forall\, i \in \mathbf{n}$; in case b): $F(f)(i) = f(i)^{-q} \ \forall\, i \in \mathbf{n}$; in case f) $F(f)(i) = f(i)^q \ \forall\, i \in \mathbf{n} \backslash \{n\}$ and $F(f)(n) = f(n)^{-q}$.

Recall that the $\mathbf{G}_F$ - conjugacy classes of F - stable tori are classified by the F - conjugacy classes of W, where $w, w\prime$ are F - conjugate if and only if $w\prime = \tilde{w}^{-1} w F(\tilde{w})$ for suitable $\tilde{w} \in W$. In particular for any element $w \in W$ there is a "finite torus" $\mathbf{T}_{w \cdot F} \leq \mathbf{G}_F$, which can be described as the subgroup $\{f \in \mathbf{T} \mid {}^{w}F(f) = f\ \}$ of $w \cdot F$ - fixed points in $\mathbf{T}$.

We start with a) and b) and let $G_n^\epsilon(q)$ be $Gl_n(q)$ if $\epsilon = 1$ and $U_n(q)$ if $\epsilon = -1$. Similarly let $S_n^\epsilon(q) = (G_n^\epsilon(q))' = Sl_n(q)$ if $\epsilon = 1$ and $SU_n(q)$ is $\epsilon = -1$ and $P_n^\epsilon(q) = S_n^\epsilon(q)/Z(S_n^\epsilon(q))$. Then $|S_n^\epsilon(q)| = q^{n(n-1)/2} \prod_{j=2}^n \epsilon^j ((\epsilon q)^j - 1)$. Let p be a prime divisor of this number that does not divide q and let $e = ord(\epsilon q)\ mod(p)$ denote the multiplicative order of ϵq in $(\mathbf{Z}/p\mathbf{Z})^*$.
Take $\alpha \in \overline{\mathbf{F}}_q^*$ of order p, then e is the size of the $< ()^{\epsilon q} >$ - orbit containing α. If $e > 1$ we define $f_\alpha \in \mathbf{T}$ by $f_\alpha(j) = \alpha^{(\epsilon q)^{j-1}}$ for $1 \leq j \leq e$ and $f_\alpha(k) = 1$ for $e < k \leq n$. Then $f_\alpha \in \mathbf{T}_{(12...e)F} \leq G_n^\epsilon(q)$. Notice that the determinant $det(f_\alpha) = \alpha^{((\epsilon q)^e - 1)/(\epsilon q - 1))} = 1$. As the first e eigenvalues of f_α are distinct and different from 1 , one can easily see that $C_{\mathbf{G}}(f_\alpha) \cong \mathbf{T} \cdot Gl_{n-e}(\overline{\mathbf{F}}_q)$ and $C_{S_n^\epsilon(q)}(f_\alpha) \leq \mathbf{T}_{(12...e)F} \cdot G_{n-e}^\epsilon(q)$ with $f_\alpha \notin S_{n-e}^\epsilon(q)$, which is the commutator subgroup of the centralizer. Moreover if $\tilde{z} = (z, z, ..., z) \in Z(S_n^\epsilon(q))$ such that f_α is conjugate to $f_\alpha \cdot \tilde{z}$ in $G_n^\epsilon(q)$, then $z \cdot \alpha = \alpha^{(\epsilon q)^j}$ for some j, hence $1 = z^{\epsilon q - 1} = \alpha^{((\epsilon q)^j - 1) \cdot (\epsilon q - 1)}$. Since p does not divide $\epsilon q - 1$ we conclude $\alpha^{((\epsilon q)^j - 1)} = 1$ and $z = 1$. So $C_{S_n^\epsilon(q)}(f_\alpha) = C_{S_n^\epsilon(q)}(f_\alpha\ mod\ Z(S_n^\epsilon(q)))$ and we are left with the case $e = 1$.
If $p \neq 2$ we define $f_\alpha := (\alpha, \alpha^{-1}, 1, 1, ..., 1) \in \mathbf{T}_F$. If $p = 2$ we choose $\alpha \in \overline{\mathbf{F}}_q^*$ of order 4 and observe that $f_\alpha := (\alpha, \alpha^{-1}, 1, 1, ..., 1)$ is contained in $\mathbf{T}_F$ if $\epsilon q \equiv 1(\ mod\ 4)$ and in $\mathbf{T}_{(12)F}$ if $\epsilon q \equiv -1(\ mod\ 4)$. It is easy to see that $C_{S_n^\epsilon(q)}(f_\alpha) = C_{S_n^\epsilon(q)}(f_\alpha\ mod\ Z(S_n^\epsilon(q)))$ as long as $3 < n$ or $n = 3$ and $p \neq 3$. If $n = 2$ then any centralizer of a non central semisimple element $t \in S_2^\epsilon(q)$ is abelian. So we can assume $n = p = 3$. If 3^2 does not divide $\epsilon q - 1$ then $Syl_3(P_3^\epsilon(q))$ is abelian.

Otherwise we replace α by an element β of order 9 and take $f_\beta = (\beta, \beta^{-1}, 1)$. Now $C_{S_3^\epsilon(q)}(f_\beta) = C_{S_3^\epsilon(q)}(f_\beta \ mod \ Z(S_3^\epsilon(q)))$. This settles the cases a) and b).

Now let $\mathbf{G}$ be $Sp_{2n}(\overline{\mathbf{F}}_q)$ or $SO_{2n+1}(\overline{\mathbf{F}}_q)$ and notice that for $q = 2^m$ all algebraic groups of type C_n and B_n are isomorphic as abstract groups and the finite groups are simple with exceptions for small q. Put $s = p$ if p is odd and $s = 4$ otherwise; note that s divides $|Sp_{2n}(\mathbf{F}_q)| = |SO_{2n+1}(\mathbf{F}_q)| = q^{n^2} \prod_{j=1}^n (q^{2j} - 1)$ and again let $e = ord(\epsilon q) \ mod(s)$. If $n \le e$ and $s \ne 4$ then $s > n$ hence s does not divide the order of the Weyl group. So by 2.1 we can assume $e < n$ or $s = 4, \ n = 2 = e \ \big(q = -1 \ (mod(4))\big)$. Again we take $\alpha \in \overline{\mathbf{F}}_q^*$ of order s. If $\alpha^{-1} \notin \alpha^{<q>}$ we define $f_\alpha(j) = \alpha^{(q)^{j-1}}$ for $1 \le j \le e$ and $f_\alpha(j) = 1$ else. If $\alpha^{-1} \in \alpha^{<q>}$ we define $f_\alpha := (\alpha, \alpha^q, ..., \alpha^{q^{k-1}}, 1, 1, ..1) \in T_{(1...k)^- F} \cap T_{(1...k)^-(n)^- F}$ with $k = e/2$. Our choice guarantees that the first e, respectively k entries are pairwise different and not inverse and neither 1 nor -1; in particular f_α has order s. If $G_n(q)$ is one of $Sp_{2n}(q), SO_{2n+1}(q)$ or $SO_{2n}^\epsilon(q)$, then f_α is contained in an appropriate finite torus of $G_n(q)$. Moreover it is easy to see that $C_{\mathbf{Z_2} \wr \Sigma_\mathbf{n}}(f_\alpha) = \mathbf{Z_2} \wr \Sigma_{\mathbf{n} \setminus \mathbf{e}}$ or $\mathbf{Z_2} \wr \Sigma_{\mathbf{n} \setminus \mathbf{k}}$ and we see: $C_{G_n(q)}(f_\alpha) = \mathbf{T}_{wF} \cdot G_{n-e}(q) \ (\mathbf{T}_{wF} \cdot G_{n-k}(q))$ where $\mathbf{T}_{wF}$ is a suitable finite maximal torus. Clearly $f_\alpha \notin C_{G_n(q)}(f_\alpha)'$. This settles the classical groups $Sp_{2n}(q), SO_{2n+1}(q), SO_{2n}^\epsilon(q)$ and finishes the proof for $q = 2^m$.
Henceforth we can assume q to be odd. Let $G_n(q) = Sp_{2n}(q)$ and $\overline{G}_n(q) = Sp_{2n}(q)/Z$ where $Z =< (-1, -1, ..., -1) >$ is the centre of $G_n(q)$. We have $e < n$ or $e = n = 2$ and $s = 4$ with $q = -1 \ mod(4)$. The element f_α contains the eigenvalue 1 but not -1. From this we can easily deduce: $C_{\overline{G}_n(q)}(\overline{f}_\alpha) = \overline{C}_{G_n(q)}(f_\alpha)$ and we are done since $f_\alpha Z \notin (C_{\overline{G}_n(q)}(\overline{f}_\alpha))'$.

Now let $G_n(q) = SO_{2n+1}(q)$ or $SO_{2n}(q)$. We can assume that $3 \le n$ by well known isomorphisms of simple Chevalley groups for small n (e.g. [As],[A]). If $s \ne 4$ then clearly $f_\alpha \in G_n(q)'$. So suppose $3 \le n$ and $s = 4$. We choose $\beta \in \overline{\mathbf{F}}_q^*$ of order 8. If $q = \epsilon \ mod(8)$ with $\epsilon \in \{1, -1\}$ then $(\beta, 1, 1, ...) \in T_{(1)^\epsilon F}$ hence $f_\beta := (\beta^2, 1, 1, ...,) \in G_n(q)'$. If $q = 3 \cdot \epsilon \ mod(8)$ then $(\beta, \beta^q, 1, 1, ..., 1) \in T_{(12)(3)^{\pm\epsilon} F}$ and $(1, 1, \beta^2, 1, ..., 1) \in T_{(12)(3)^{-\epsilon} F}$ hence $f_\beta := (\beta, \beta^q, \beta^2, 1, ..., 1) \in T_{(12)(3)^{-\epsilon} F} \cap G_n(q)'$. For these f_β all eigenvalues different from 1 are pairwise different and not inverse so we can argue as above.

We are left with the case $G_n(q)' = SO_{2n}(q)'$ where we have to factor out $Z :=< z := (-1, -1, ..., -1) >$. If $g.c.d(s, 2) = 1$ the f_α's we constructed always contain eigenvalue 1 but not -1, hence $C_{G_n(q)'}(f_\alpha \ mod \ Z) = C_{G_n(q)'}(f_\alpha)$ and we are done. Suppose now that $s = 4$ or 8 and f_β has eigenvalue 1 $\big($not $-1\big)$, then the same arguments hold. So we can assume $s = 8$ and $f_\beta = (\beta, \beta^q, \beta^2)$. In this case ${}^g f_\beta = f_\beta \cdot z$ implies $\beta^2 = -\beta^2$, which is impossible since q is odd now. This finishes the proof. ♣

4. The exceptional Chevalley groups

Throughout this section let $\mathbf{G}$ be a simple, simply connected algebraic group over an algebraically closed field K of characteristic $r > 0$ of exceptional type. Again F denotes a Frobenius endomorphism of $\mathbf{G}$ with finite fixed point group $G = \mathbf{G_F}$. We choose an F - stable torus T of $\mathbf{G}$ with character group $X := Hom_K(T, K^*)$. So G is one of the following groups: ${}^2B_2(2^{2n-1})$, ${}^2G_2(3^{2n-1})$, $G_2(q)$, ${}^2F_4(2^{2n-1})$, $F_4(q)$, $E_6(q)_{sc}$, ${}^2E_6(q^2)_{sc}$, $E_7(q)_{sc}$, $E_8(q)$, where q is a power of r. Recall that the groups $E_6(q)_{sc}$, ${}^2E_6(q^2)_{sc}$, $E_7(q)_{sc}$ have centres of order $(q-1,3)$, $(q+1,3)$ or $(q-1,2)$, respectively, and that the corresponding factor groups are simple; moreover the groups ${}^2B_2(2)$, ${}^2G_2(2)$, $G_2(2)$ and ${}^2F_4(2)$ have a proper commutator subgroup. In all the remaining cases G is simple.
The Dynkin diagrams and fundamental roots for all cases are as given below:

B_2 : $\circ_{\alpha_1} {=}{>}{=} \circ_{\alpha_2}$

G_2 : $\circ_{\alpha_1} {\equiv}{>}{\equiv} \circ_{\alpha_2}$

F_4 : $\circ_{\alpha_1} — \circ_{\alpha_2} {=}{>}{=} \circ_{\alpha_3} — \circ_{\alpha_4}$

E_6 :
$$\begin{array}{ccccccccc} \circ_{\alpha_1} & — & \circ_{\alpha_2} & — & \circ_{\alpha_3} & — & \circ_{\alpha_5} & — & \circ_{\alpha_6} \\ & & & & | & & & & \\ & & & & \circ_{\alpha_4} & & & & \end{array}$$

E_7 :
$$\begin{array}{ccccccccccc} \circ_{\alpha_1} & — & \circ_{\alpha_2} & — & \circ_{\alpha_3} & — & \circ_{\alpha_4} & — & \circ_{\alpha_5} & — & \circ_{\alpha_7} \\ & & & & & & | & & & & \\ & & & & & & \circ_{\alpha_5} & & & & \end{array}$$

E_8 :
$$\begin{array}{ccccccccccccc} \circ_{\alpha_1} & — & \circ_{\alpha_2} & — & \circ_{\alpha_3} & — & \circ_{\alpha_4} & — & \circ_{\alpha_5} & — & \circ_{\alpha_7} & — & \circ_{\alpha_8} \\ & & & & & & & & | & & & & \\ & & & & & & & & \circ_{\alpha_6} & & & & \end{array}$$

If F is standard then G is a nontwisted group and the comorphism F^* of F acts on X by multiplication with q. In the case $G = {}^2E_6(q^2)$ the Frobenius F is not standard but we can chose the torus T such that F^* acts on X by multiplication with $-q$. So F acts trivially on the Weyl group W of $\mathbf{G}$ which is generated by reflections $s(\alpha)$ where α runs through the set of fundamental roots Π.
A semisimple element of G is called regular if the centralizer is a torus. Then it is well known that the G - conjugacy classes of centralizers of regular semisimple elements of G are parametrized by the conjugacy classes of W. If w is an element of W then the order of the centralizer of a regular semisimple element of G parametrized by the W-class $[w]$ of w is given by the characteristic polynomial t_w of $(q \cdot w - 1)$ acting on the real vector space with basis Π. By c_w we denote the number of regular semisimple classes corresponding to the G - class of centralizers parametrized by $[w]$.

Proposition 4.1: *G satisfies W(p) for each $p \in \pi(W)$.*

Proof: By Lemma 3.1 we may assume that $p \in \pi(W)$ and $p \neq r$. So the case G $= {}^2B_2(2^{2n+1})$ is done. If $G = {}^2G_2(3^{2n+1})$ the Sylow-2-subgroups of G are abelian [HB,Ch XI. §13] and so Lemma 1.2 applies. In the remaining cases we will show that there exists a regular semisimple element s in G, i.e. the centralizer $C_G(s)$ is a finite torus of order divisible by p; for this we proceed case by case.

$G = G_2(q)$ (see [Ch] [En]):
The order of W is $2^2 \cdot 3$ and so $p \in \{2,3\}$. We choose the element $x = h(z, z^q, z^{-q-1})$ in [Ch] and [En] as a representative of a semisimple regular class in G. The centralizer of x in G, denoted by $Z(x)$, has order q^2-1 and the number of classes with centralizer $Z(x)$ (up to conjugacy) denoted by $N(x)$ is $(1/4)\cdot(q-1)^2$ resp. $(1/4)\cdot q\cdot(q-2)$ for $r \neq 2$ resp. $r = 2$. $|Z(x)|$ is divisible by 2 and 3. Just in the case when $q = 2$ there does not exist a regular class of order divisible by 3. But then $G = G_2(2)$ and G is isomorphic to $PSU_3(3)\cdot 2$ which has been handled in chapter 3.

$G = {}^3D_4(q^3)$ (see [DM]):
The order of W is $2^6 \cdot 3$ and so $p \in \{2,3\}$.
In [DM] we take the element s_8. Then $|Z(s_8)| = (q^3-1)\cdot(q+1)$ and $N(s_8) =$ $(1/4)\cdot(q^4-2q)$ resp. $N(s_8) = (1/4)\cdot(q^4-2q+1)$ for q even resp. q odd. In any case p divides $|Z(s_8)|$.

$G = F_4(q)$ (see [Shi1] and [Sho]):
The order of W is $2^7 \cdot 3^2$ and so $p \in \{2,3\}$.
For $p = 3$ we choose the element h_{52} in [Shi1] as a representative of a semisimple regular class in G. $Z(h_{52})$ has order $(q+1)\cdot(q^3-1)$ and $N(h_{52}) = (1/12)\cdot q^2\cdot(q^2-1)$. Here q is even and obviously $p = 3$ divides $|Z(h_{52})|$
If $p = 2$ we choose the semisimple regular representative h_{88} in [Sho]. Then the order of $Z(h_{88})$ is q^4-1 and $N(h_{88}) = (1/16)\cdot(q-1)^3\cdot(q+1)$. Again p divides $|Z(h_{88})|$.

$G = {}^2F_4(q^2)$, $q^2 = 2^{2n+1}$ (see [Shi2]):
The order of W is $2^7 \cdot 3^2$ and so $p = 3$ since q is even.
We choose the representative of a semisimple regular class t_6 in [Shi2]. Then p divides $|Z(t_6)| = q^4-1$ and $N(t_6) = (1/4)\cdot q^2\cdot(q^2-2)$.

$G = E_6(q)_{sc}$ (see [FJ1]):
The order of W is $2^7 \cdot 3^4 \cdot 5$ and so $p \in \{2,3,5\}$.
Let $w = s(\alpha_2)\cdot s(\alpha_1+\alpha_2+\alpha_3)\cdot s(\alpha_1)\cdot s(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)\cdot s(\alpha_3+\alpha_4)$ (This is $w20$ for $J = \emptyset$ in [FJ1]). Then $t_w = (q-1)\cdot(q^3+1)\cdot(q^2+1)$ and $c_w = (1/12)\cdot q^3\cdot(q+1)\cdot(q-1)^2$.

$G = {}^2E_6(q)_{sc}$ (see [FJ1]):
As in the nontwisted case $p \in \{2,3,5\}$ but we have to replace q by $-q$, i.e $t_w = (q+1)\cdot(q^3-1)\cdot(q^2+1)$ and $c_w = (1/12)\cdot q^3\cdot(q-1)\cdot(q+1)^2$.

$G = E_7(q)_{sc}$ (see [FJ1]):
The order of W is $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ and so $p \in \{2,3,5,7\}$.
Let $w = s(\alpha_1)\cdot s(\alpha_3)\cdot s(\alpha_5)\cdot s(\alpha_2)\cdot s(\alpha_4)\cdot s(\alpha_7)$ ($w37$ for $J = \emptyset$ in [FJ2]). Then $t_w = (q^6-1)\cdot(q+1)$ and $c_w = (1/24)\cdot q^2\cdot(q+1)\cdot(q^2+q+2)\cdot(q-1)^2$ resp. $c_w =$ $(1/24)\cdot q\cdot(q+1)\cdot(q^3+q^2+2q-1)\cdot(q-1)^2$ for q even resp. q odd. In this case t_w is divisible by $p \in \{2,3,7\}$.
For $p = 5$ we chose $w = s(\alpha_1)\cdot s(\alpha_3)\cdot s(\alpha_2)\cdot s(\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6)\cdot s(\alpha_4)\cdot$ $s(2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+3\alpha_6+4\alpha_7+2\alpha_8)$. Then $t_w = (q^4-1)\cdot(q^3+1)$ and $c_w = (1/48)\cdot q^4\cdot(q+1)\cdot(q-1)^2$ resp. $c_w = (1/48)\cdot q\cdot(q^2-q+1)\cdot(q-1)^2\cdot(q+1)^2$ for q even resp. q odd.

$G = E_8(q)$ (see [FJ2]):
The order of W is $2^{14}\cdot 3^5\cdot 5^2\cdot 7$ and so $p \in \{2,3,5,7\}$. Let $w = s(\alpha_6)\cdot s(\alpha_4)\cdot s(\alpha_1)\cdot s(\alpha_5+\alpha_6)\cdot s(\alpha_2)\cdot s(\alpha 2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7)\cdot s(\alpha_2+2\alpha_3+3\alpha_4+4\alpha_5+2\alpha_6+3\alpha_7+2\alpha_8)$ (w_{73} for $J=\emptyset$ in [FJ2]). Then $t_w = (q^3-1)\cdot(q^3+1)\cdot(q^2+1)$ and $c_w = (1/72)\cdot q^4\cdot(q-1)^2\cdot(q+1)$ and p divides t_w.

In all cases above (except $G_2(2)$) the number of classes with the given conjugacy class of centralizer is different from zero. This implies the existence of regular classes. Since p divides the corresponding centralizer order the proof is complete. ♣

Proposition 4.2: *For each exceptional finite simple group G of Lie type and each $p \in \pi(G)$ the property $W(p)$ holds.*

Proof: If G is an exceptional simply connected finite group of Lie type then as mentioned before G is simple except when G is ${}^2B(2)$, $G_2(2)$, ${}^2G_2(3)$, ${}^2F_4(2)$, $E_6(q)_{sc}$, ${}^2E_6(q^2)_{sc}$ or $E_7(q)_{sc}$.
We have the isomorphisms ${}^2B(2) \cong 5:4$, $G_2(2) \cong PSU_3(3)\cdot 2$, ${}^2G_2(3) \cong PSl_2(8)\cdot 3$ and ${}^2F_4(2) \cong Ti\cdot 2$, where Ti is the finite simple Tits group. Thus the groups ${}^2B_2(2)$, $G_2(2)$ and ${}^2G_2(3)$ have already been considered in the previous sections. In the case of the Tits group one can find a selfcentralizing element whose order is divisible by p for each prime divisor p of the group order (i.e $p \in \{2,3,5,13\}$) (see [A]).
We are left with $G = E_6(q)_{sc}$, $G = {}^2E_6(q^2)_{sc}$ and $G = E_7(q)_{sc}$. These groups have have a centre Z of order $(q-1,3)$, $(q+1,3)$ and $(q-1,2)$ respectively, and the quotient G/Z is simple.

Let $p \neq r$ be a prime. Then as in the proof of Proposition 4.1 one can find a regular semisimple element g of G such that the order of g is divisible by p. The centralizer $C = C_G(g)$ is a finite torus, thus abelian. We consider g as a product of its p - and p' - parts g_p, g'_p. Let A be the Sylow p - subgroup of C. If $g_p = h^p$ for $h \in A$ then the centralizer of $h\cdot g'_p$ C in G is C as well. So we can choose g such that g_p is not contained in A^p. Put $\tilde{C} = C_G(g \ mod \ Z)$. If p does not divide $|Z|$ then A is an abelian Sylow p - subgroup of $\tilde{C}$. By Lemma 1.2 and Proposition 4.1 the result follows.

Henceforth we may assume that $p = |Z|$; so $p = 2$ for $G = E_7(q)_{sc}$ and $p = 3$ for $E_6(q)_{sc}$ and ${}^2E_6(q^2)_{sc}$. As in the proof of 4.1 we choose $w = s(\alpha_2)\cdot s(\alpha_1+\alpha_2+\alpha_3)\cdot s(\alpha_1)\cdot s(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5)\cdot s(\alpha_3+\alpha_4)$ for $E_6(q)_{sc}$, ${}^2E_6(q^2)_{sc}$ and $w = s(\alpha_1)\cdot s(\alpha_3)\cdot s(\alpha_5)\cdot s(\alpha_2)\cdot s(\alpha_4)\cdot s(\alpha_7)$ for $E_7(q)_{sc}$. Then C is conjugate by an element $h \in \mathbf{G}$ to $T_w := \{s \in T | wF(s) = s\}$ the twisted torus parametrized by the element w. We let $\tilde{T}_w$ be the group ${}^h\tilde{C}$.
If $x \in \tilde{T}_w$ then x has to be contained in the normalizer of T_w thus $\tilde{T}_w/T_w \cong C_{C_W(w)}({}^hg)$. $C_W(w)$ is just $< w >$ (see [FJ1]). So it follows that $\tilde{T}_w$ is contained in $< \dot{w} > T_w$. Let ϵ be 1 in the nontwisted and -1 in the twisted case. Using the fact that $w(t^{\epsilon q}) = t$ holds for an element t in T_w an easy calculation shows that $(\tilde{T}_w)'$ is contained in $T_w^{q-\epsilon}$ Thus g_p is not contained in $\tilde{C}'$ and the proof is complete. ♣

Finally note that the claim of the Theorem now follows by the Classification of the finite simple groups together with the results above.

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