Finite Fields, Root Systems and Orbit Numbers of Chevalley groups

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Abstract

We describe combinatorial techniques to determine the numbers of semisimple conjugacy classes and adjoint orbits with fixed class of centralizers for simply connected finite groups of Lie type.

1 Introduction

For any finite group H let $Irr_K(H)$ denote the set of irreducible characters of H, defined over the field K. If G is a finite group of Lie type one has an explicit parametrization of the characters in $Irr_{\overline{\mathbb{Q}}_\ell}(G^*)$, due to fundamental work of G. Lusztig ([20]). Here $\overline{\mathbb{Q}}_\ell$, is the algebraic closure of the ℓ - adic rationals and ℓ is a prime different from the defining characteristic of G.
If (G, G^*) is a pair of dual groups such that the center of the ambient algebraic group $\mathbf{G}^*$ is connected (e.g. $(SL_n(q), PGL_n(q))$), then this parametrization can be described as follows:

$$Irr_{\overline{\mathbb{Q}}_\ell}(G^*) = \biguplus_{[s]} \mathcal{E}(G_s, [1]), \qquad (*)$$

where $\mathcal{E}(G_s, [1])$ denotes the set of irreducible 'unipotent characters' of the centralizer G_s and $[s]$ runs through the different semisimple conjugacy classes of G. Whenever two centralizers G_{s_1} and G_{s_2} of semisimple elements are conjugate in G, the sets $\mathcal{E}(G_{s_i}, [1])$ are conjugate as well. Hence it is important to describe those semisimple classes $[s]$ of G that have G - conjugate centralizers G_s.

We will view G as fixed - point group $\mathbf{G}_F$ of a Frobenius endomorphism F on a connected reductive algebraic group $\mathbf{G}$, defined over $\mathbb{F}_q$. Notice that the Lie algebra $Lie\mathbf{G}$ is defined over $\mathbb{F}_q$ as well. Its corresponding Frobenius endomorphism will also be denoted by F. The restriction of the adjoint action of $\mathbf{G}$ on $Lie\mathbf{G}$ yields an action of G on $LieG := (Lie\mathbf{G})_F$. Hence we can also look at semisimple G - orbits δ^G in $LieG$. We are interested in those semisimple conjugacy classes x^G of G or G - orbits on $LieG$, which have G - conjugate 'connected algebraic stabilizers' $\mathbf{G}_x^0$. Following Carter [3], such classes are said to be of the same **genus**. The number of classes of given genus will be called a **genus number**.

Notice that in the situation of $(*)$, centralizers of semisimple elements are connected reductive subgroups of $\mathbf{G}$ (see [4]). This implies that for semisimple elements $s_1, s_2 \in G$ the centers of centralizers satisfy $Z(G_{s_i}) = Z(\mathbf{G}_{s_i})_F$, hence the G_{s_i} are conjugate in G if and only if $[s_1]$ and $[s_2]$ are of the same genus.

We are especially interested in 'generic genus numbers', i.e. formulae depending on q, which give these numbers uniformly for a whole series of groups with the same Dynkin type (e.g. [24],[23] for groups of type F_4, [11], [12] for $E_n, n = 6, 7, 8$ and [8] for general linear and unitary groups).

Since stabilizers of semisimple elements are subgroups of $\mathbf{G}$ of maximal Lie rank, they can be described by 'closed subsystems' of the root system Φ of $\mathbf{G}$. This yields combinatorial interpretations of genus numbers in terms of lattices of 'stable subsystems' of Φ. In this article we will give a survey on these connections and show, how elementary combinatorics, including polynomials over finite fields, can be used to obtain the Möbius functions of stable subsystems and eventually compute genus numbers.

2 Semisimple classes and root systems

Combinatorial methods to describe semisimple genera and genus numbers in the group case have been applied by different authors (e.g. see [24], [23] and [5]). Our presentation here will also cover the case of Lie algebras.

The group $\mathbf{G}$ contains, up to G conjugacy, a unique F - stable maximal torus $\mathbf{T} = F(\mathbf{T})$, which is quasisplit, i.e. contained in an F - stable Borel subgroup $\mathbf{B}$. Let $X := Hom(\mathbf{T}, \overline{\mathbb{F}_q}^*) \cong \mathbb{Z}^n$ with $n = \dim \mathbf{T}$ be the group of rational characters of $\mathbf{T}$, $\Phi \subseteq X$ the root system of $\mathbf{G}$ with respect to $\mathbf{T}$, Φ^+ the positive system with respect to $\mathbf{B}$ and $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ the Weyl group of $\mathbf{G}$. Then $F = \rho \circ F_0 = F_0 \circ \rho$ with a rational automorphism ρ of $\mathbf{G}$ of finite order such that $F_0(t) = t^q$ for each $t \in \mathbf{T}$ and $F(w) = \rho w \rho^{-1} \in Aut(\mathbf{T})$ for any $w \in W$. Let $\tilde{W}$ be the group $\langle W, \rho \rangle \leq Aut(\mathbf{T})$. Then $\tilde{W}$ acts on X by $(ax)(t) := x(a^{-1}(t))$, i.e. the comorphism a^* (as defined in [25]) of $a \in Aut(\mathbf{T})$ coincides with the action of a^{-1} on X. Moreover $F(\mathbf{X}_\alpha) = \mathbf{X}_{\rho(\alpha)}$ for all root subgoups $\mathbf{X}_\alpha \leq \mathbf{G}$.

Definition 2.1 *A subset $x \subset \Phi$ is called* **closed** *if $\langle x \rangle_{\mathbb{Z}} \cap \Phi = x$.*
Let $a \in Aut(X)$, then we define $\mathcal{S}_a$ to be the (inclusion ordered) poset of all closed subsystems x of Φ such that $a(x) = x$. We also will write $\mathcal{S}_{\Phi,a}$ and $\mathcal{S}_\Phi$ for $\mathcal{S}_a$ and $\mathcal{S}_{id}$ respectively.
For any $\Phi_1 \in \mathcal{S}_\Phi$ let $\mathbf{G}_{\Phi_1}$ be the (connected reductive) subgroup $\langle \mathbf{T}, \mathbf{X}_\alpha | \alpha \in \Phi_1 \rangle$ of $\mathbf{G}$, generated by $\mathbf{T}$ and the root subgroups $\mathbf{X}_\alpha$ with $\alpha \in \Phi_1$.

Any semisimple element $s \in G$ is contained in some F - stable maximal torus $\tilde{\mathbf{T}}$ of $\mathbf{G}$ (see [3]). Similarly any semisimple element $\delta \in Lie G$ is contained in $Lie\tilde{\mathbf{T}} \leq Lie\mathbf{G}$ with some $\tilde{\mathbf{T}}$ as above (see [Bo] 11.8, pg. 150). Moreover since all maximal tori in a connected linear algebraic group are conjugate, there is $g \in \mathbf{G}$ with $t = {}^g s \in \tilde{\mathbf{T}} = {}^g\mathbf{T}$ or $\delta = {}^g\gamma \in Lie\tilde{\mathbf{T}} = {}^g(Lie\mathbf{T})$. It is well known that for $x = s$ or γ the 'connected stabilizers' (= the 1 - components of the stabilizers) are reductive groups of the form:

$$\mathbf{G}_x^o = \langle \mathbf{T}, \mathbf{X}_\alpha \mid \alpha \in \Phi_x \rangle$$

where $\Phi_s := \{\alpha \in \Phi \mid \alpha(s) = 1\}$, $\Phi_\gamma := \{\alpha \in \Phi \mid d\alpha(\gamma) = 0\}$ are the root systems of the stabilizer (notice that for any root $\alpha \in \Phi$ the differential $d\alpha$ is an element in the dual space $(Lie\mathbf{T})^*$).
Clearly Φ_x is a closed subsystem of Φ. Moreover for $h = {}^g k \in {}^g\mathbf{G}_{\Phi_x}$ we have: $F({}^g k) = {}^{g\mathcal{L}_F(g)}F(k)$ with $\mathcal{L}_F(g) := g^{-1}F(g) \mapsto w \in W$. (Recall that $\mathcal{L}_F : \mathbf{G} \to \mathbf{G},\ g \mapsto g^{-1}F(g)$ is surjective by a theorem of Lang [25]). Hence for $y = t$ or δ and $y = {}^g x$:

$$(\mathbf{G}_y^0)_F = ({}^g(\mathbf{G}_{\Phi_x}))_F = {}^g((\mathbf{G}_{\Phi_x})_{\mathcal{L}_F(g)F});$$

in particular $\Phi_x = w\rho(\Phi_x)$ is $w\rho$ - stable. On the other hand, if $\Phi_1 \in \mathcal{S}_\Phi$ with $w\rho(\Phi_1) = \Phi_1$, we can find $h \in \mathbf{G}$ such that $\mathcal{L}_F(h) \mapsto w$ and $F({}^h\mathbf{X}_\alpha) = {}^{h\mathcal{L}_F(h)}F(\mathbf{X}_\alpha) = {}^h(\mathbf{X}_{w\rho\alpha})$. So ${}^h\mathbf{G}_{\Phi_1}$ is F - stable.

Let $\Phi_1 \in \mathcal{S}_{w\rho}$ for some $w \in W$. Then $\Phi_1^+ := \Phi_1 \cap \Phi^+$ and $w\rho(\Phi_1^+)$ are two 'positive systems' of Φ_1 and hence are conjugate in $W_{\Phi_1} := N_{\mathbf{G}_{\Phi_1}}(\mathbf{T})/\mathbf{T}$. In particular we always find $w_1 \in W$ with $w_1\rho\Phi_1^+ = \Phi_1^+$. Then, by a result in

[2], the set of F - stable elements in the $\mathbf{G}$ - orbit $(\mathbf{G}_{\Phi_1})^{\mathbf{G}}$ splits into $\mathbf{G}_F$ - classes which are parametrized by pairs

$$(\Phi_1, [ww_1\rho]) \text{ with } [ww_1\rho] \in N_W(\Phi_1^+)w_1\rho \backslash N_W(\Phi_1^+).$$

Here $N_W(\Phi_1^+)w_1\rho$ is a coset in $\tilde{W}$ and $N_W(\Phi_1^+)w_1\rho\backslash N_W(\Phi_1^+)$ denotes the set of $N_W(\Phi_1^+)$ - conjugacy classes on this coset. A representative of the G - class attached to $(\Phi_1, [ww_1\rho])$ is given by ${}^g\mathbf{G}_{\Phi_1}$, where $\mathcal{L}_F(g)$ maps to $ww_1 \in W$.

Definition 2.2 *We define $\mathcal{O}(\Phi_1, [ww_1\rho])$ to be the genus number corresponding to the parameter $(\Phi_1, [ww_1\rho])$ and will write $\mathcal{O}_G$ and $\mathcal{O}_{LieG}$ to distinguish the group case and the Lie algebra case.*
For $\mathcal{T} = \mathbf{T}$ or $Lie\mathbf{T}$, $a \in \tilde{W}$ and $\Phi' \in \mathcal{S}_\Phi$ with $a(\Phi') = \Phi'$ we define

$$\mathcal{T}(\Phi', a) := \{\ t \in \mathcal{T} \mid a \circ F_0(t) = t,\ \Phi_t = \Phi'\ \};$$

$$\mathcal{Z}(\Phi', a) := \{\ t \in \mathcal{T} \mid a \circ F_0(t) = t,\ \Phi_t \supseteq \Phi'\ \}.$$

Notice that in the group case $\mathcal{Z}(\Phi', a) = Z((\mathbf{G}_{\Phi'})_{\dot{a}\circ F_0}) = Z(\mathbf{G}_{\Phi'})_{a\circ F_0}$ is the center of $(\mathbf{G}_{\Phi'})_{\dot{a}\circ F_0}$ where $\dot{a}$ is a rational automorphism whose restriction to $\mathbf{T}$ coincides with a. It is well known that the cardinality $g_{\mathbf{T}}(\Phi', a) := |\mathcal{Z}(\Phi', a)|$ is given by

$$cp((ww_1\rho)^*|_{(X/\langle\Phi'\rangle)\otimes\mathbb{R}})(q) \cdot |\ \mathcal{O}_{p'}(Tor(X/\langle\Phi'\rangle))_{q(ww_1\rho)^*}\ |.$$

Here $cp((ww_1\rho)^*|_{(X/\langle\Phi'\rangle)\otimes\mathbb{R}})(q)$ denotes the characteristic of the induced operator of $ww_1\rho$ on $X/\langle\Phi'\rangle) \otimes \mathbb{R}$, evaluated at q and $\mathcal{O}_{p'}(Tor(X/\langle\Phi'\rangle))_{q(ww_1\rho)^*}$ denotes the maximal subgroup of order prime to p in the group of $q(ww_1\rho)^*$ - fixed points of the torsion group $Tor(X/\langle\Phi'\rangle)$ (see [5]). In the Lie algebra case we have

$$\mathcal{Z}(\Phi', a) = (\langle d\alpha \mid \alpha \in \Phi'\rangle^\perp)_{\dot{a}F} \subseteq Lie\mathbf{T}.$$

Here the endomorphism $\dot{a}F$ is the Frobenius endomorphism of an $\mathbb{F}_q$ - structure on the $\overline{\mathbb{F}}_q$ - vector space $d(\Phi')^\perp \leq Lie\mathbf{T}$. For such an $\overline{\mathbb{F}}_q$ - space V with $\mathbb{F}_q$ - structure one always has $|V_F| = q^{dimV}$. So in the Lie algebra case we get:

$$g_{Lie\mathbf{T}}(\Phi', a) = |\mathcal{Z}(\Phi', a)| = q^{n - dim_{\overline{\mathbb{F}}_q}\langle d\alpha\ \mid\ \alpha\in\Phi'\rangle_{\overline{\mathbb{F}}_q}}.$$

where $n = \dim Lie\mathbf{T}$ is the Lie rank of $\mathbf{G}$.

Now we make the following technical assumptions for the rest of the paper:
In case of conjugacy classes we assume that the center of the dual group $\mathbf{G}^*$ (in the sense of [3]) is connected.

In case of orbits in the Lie algebra we assume that $p = char\ \mathbb{F}_q$ is not a torsion prime (in the sense of [26]).

For example GL_n is self - dual and PGL_n is the dual of SL_n, hence GL_n and SL_n satisfy this assumption in case of conjugacy classes. Torsion primes are all primes dividing either $|X^*/\langle\Phi^*\rangle|$ or $|Tor(\langle\Phi^*\rangle/\langle\Phi_1^*\rangle)|$ for some $\Phi_1 \in \mathcal{S}_\Phi$. Here $\Phi^* \subseteq X^*$ denote root system and character group of $\mathbf{G}^*$ and $^* : \Phi \to \Phi^*$ the duality bijection of root systems (see [4]).

In particular GL_n and SL_n do not have any torsion primes. For quasi simple groups of Dynkin type other than type A the torsion primes are ≤ 5 (see [26]).

As a consequence of these assumptions all centralizers of semisimple elements in $\mathbf{G}$ and $Lie\mathbf{G}$ are connected reductive algebraic subgroups of the form $\mathbf{G}_{\Phi_1}$ for $\Phi_1 \in \mathcal{S}_\Phi$ (see [26]).

We fix ${}^g\mathbf{G}_{\Phi_1}$ such that $\mathcal{L}_F(g) \mapsto ww_1 \in W$ and $w_1\rho(\Phi_1^+) = \Phi_1^+$. Then we get $N_{\mathbf{G}}({}^g\mathbf{G}_{\Phi_1}) = {}^g(\mathbf{G}_{\Phi_1} \cdot N_{N_{\mathbf{G}}(\mathbf{T})}(\Phi_1))$ and hence

$$N_{\mathbf{G}_F}({}^g\mathbf{G}_{\Phi_1})/({}^g\mathbf{G}_{\Phi_1})_F \cong {}^g(N_W(\Phi_1)/W_{\Phi_1})_F \cong N_W(\Phi_1^+)_{ww_1\rho}.$$

Since $\{y \in {}^g\mathcal{T} \mid F(y) = y, \mathbf{G}_y = {}^g\mathbf{G}_{\Phi_1}\} = {}^g\{x \in \mathcal{T} \mid ww_1\rho F_0(x) = x, \mathbf{G}_x = \mathbf{G}_{\Phi_1}\}$ we get:

$$|\mathcal{O}(\Phi_1, [ww_1\rho])| = \frac{1}{|\ N_W(\Phi_1^+)_{ww_1\rho}\ |} \cdot |\mathcal{T}(\Phi_1, ww_1\rho)|.$$

From the definitions we get:

$$\mathcal{Z}(\Phi_1, ww_1\rho) = \biguplus_{\Phi' \in \mathcal{S}_{ww_1\rho},\ \Phi_1 \subseteq \Phi'} \mathcal{T}(\Phi', ww_1\rho)$$

and therefore by Möbius inversion

$$f_{\mathcal{T}}(\Phi_1, ww_1\rho) := |\mathcal{T}(\Phi_1, ww_1\rho)| = \sum_{\Phi' \in \mathcal{S}_{ww_1\rho}} \mu_{ww_1\rho}(\Phi_1, \Phi') g_{\mathcal{T}}(\Phi', ww_1\rho). \quad (1)$$

Here $\mu_{ww_1\rho}$ is the Möbius function of the inclusion ordered poset $\mathcal{S}_{ww_1\rho}$ of $ww_1\rho$ - stable closed subsystems in $\mathcal{S}_\Phi$.

3 Genus numbers and Coxeter Arrangements

There are close connections between the posets $\mathcal{S}_\Phi$ and Coxeter arrangements of hyperplanes as studied e.g. in [21]. In special characteristics the results there give valuable information on semisimple genus numbers. In case of regular semisimple classes and orbits (i.e. those with abelian stabilizers) these connections are discussed in [FJ1] as well as in [L2].

Let us consider the poset, $\mathcal{H} := \mathcal{H}_\Phi := \{ \cap_{\alpha\in s}\alpha^\perp | \ s \subseteq \Phi \}$ of intersections of reflection hyperplanes in the n - dimensional complex space $V = X \otimes \mathbf{C}$, ordered by reversed inclusion. The action of the Weyl group W on the complement $M := V - \cup_{\alpha\in\Phi}(\langle\alpha\rangle_{\mathbf{C}})^\perp$ induces an action on cohomology $H^*(M, \mathbf{C})$.

It has been shown in [21] that:

$$\mathcal{P}_\Phi(w,x) := \sum_{i=0}^{\infty} trace(w, H^i(M,\mathbf{C}))\cdot x^i = \sum_{y\in\mathcal{H}_w} \mu_{\mathcal{H}_w}(0,y)\ (-x)^{n-dim\ y}.$$

The polynomial $\mathcal{P}_\Phi(w,x)$ is called the Poincaré polynomial of $\mathcal{H}_w := \{y \in \mathcal{H}_\Phi \mid w(y) = y \ \}$. Similarly we define

$$\mathcal{P}_\Phi(a,x) := \sum_{y\in\mathcal{H}_a} \mu_{\mathcal{H}_a}(0,y)\ (-x)^{n-dim\ y}$$

for $a \in \tilde{W}$ and Möbius function $\mu_{\mathcal{H}_a}$ of $\mathcal{H}_a$.
The map $\theta : \mathcal{S}_\Phi \longrightarrow \mathcal{H}_\Phi,\ \Phi_1 \mapsto (\langle\Phi_1\rangle_{\mathbf{C}})^\perp$ is an isotone epimorphism of $\tilde{W}$ - posets. The map $\check{\theta} : \mathcal{H}_\Phi \longrightarrow \mathcal{S}_\Phi,\ x \mapsto x^\perp \cap \Phi$ is isotone and satisfies $\theta\circ\check{\theta} = id$ as well as $\Phi_1 \leq \check{\theta}\circ\theta(\Phi_1) =: \overline{\Phi_1}$. Hence $\check{\theta}\circ\theta$ is a 'closure operator' in the sense of combinatorics and $\theta_| : \check{\theta}\circ\theta(\mathcal{S}_a) \to \mathcal{H}_a$ is an isomorphism of posets. Now the lemma of Rota ([22]) tells us:

$$\sum_{s'\in\mathcal{S}_a,\theta(s')=x} \mu_{\mathcal{S}_a}(s,s') = \begin{cases} \mu_{\mathcal{H}_a}(\theta(s),x) & \text{if } s = \check{\theta}\circ\theta(s) \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Lemma 3.1 *If the characteristic p of $\mathbb{F}_q$ does neither divide any $|\langle\Phi_1\rangle/\langle\Phi_2\rangle|$ for $\Phi_2 \leq \Phi_1 \in \mathcal{S}$ nor $|Tor(X/\langle\Phi\rangle)|$, then*

$$dim_{\overline{\mathbb{F}_q}}\ \langle d\alpha \mid \alpha \in \Phi_1\rangle\ = rank\langle\Phi_1\rangle.$$

These primes are called **very good** *for* $\mathbf{G}$.

Proof: Let $\mathcal{B} := \{x_1, ..., x_n\}$ be a $\mathbb{Z}$ - basis of X. Then the differential d maps each $x := \sum_i z_i x_i$ onto $\sum_i z_i (\mathrm{mod}\, p) \otimes_{\mathbb{Z}} x_i \in \overline{\mathbb{F}}_q \otimes X \cong (Lie\mathbf{T})^*$. Since p neither divides $|Tor(\langle\Phi\rangle/\langle\Phi_1\rangle)|$ nor $|Tor(X/\langle\Phi\rangle)|$ it does not divide $|Tor(X/\langle\Phi_1\rangle)|$ either. Hence from this we immediately get the claim. $\circ$

If Φ is indecomposable, very good primes are never torsion primes. Now we conclude:

Theorem 3.2 *Let R be the least common multiple of all primes which are not very good and let i be the order of $ww_1\rho$.*
In case $\mathcal{T} = \mathbf{T}$ suppose $\gcd(q^i - 1, R) = 1$. In case $\mathcal{T} = Lie\mathbf{T}$ suppose that $p =$ char $\mathbb{F}_q$ is a very good prime.
Then the cardinality $f_{\mathcal{T}}(\Phi_1, ww_1\rho)$ is zero if $\Phi_1 \neq \overline{\Phi}_1$. If $\Phi_1 = \overline{\Phi}_1$ then:

$$f_{\mathbf{T}}(\Phi_1, ww_1\rho) = \sum_{y \in (\mathcal{H}_\Phi)_{ww_1\rho}} \mu_{(\mathcal{H}_\Phi)_{ww_1\rho}}(\theta(\Phi_1), y)\ cp|_y((ww_1\rho)^*)(q);$$

$$f_{Lie\mathbf{T}}(\Phi_1, ww_1\rho) = \sum_{y \in (\mathcal{H}_\Phi)_{ww_1\rho}} \mu_{(\mathcal{H}_\Phi)_{ww_1\rho}}(\theta(\Phi_1), y)\ q^{dim\ y}.$$

In particular we get for $\Phi_1 = \emptyset$:

$$f_{Lie\mathbf{T}}(\emptyset, w\rho) = x^n \cdot \mathcal{P}_\Phi(w\rho, -1/x)|_{x=q}.$$

Proof: In case $\mathcal{T} = \mathbf{T}$ the condition $\gcd(q^i - 1, R) = 1$ implies that $\gcd(q^i - 1, |\ \overline{\langle\Phi'\rangle}/\langle\Phi'\rangle\ |) = 1$, hence $|\mathcal{O}_{p'}(Tor(X/\langle\Phi'\rangle))_{q(ww_1\rho)^*}| = 1$. The characteristic polynomials appearing in (1) only depend on $\theta(\Phi')$. Now the statement follows from (2). For $\mathcal{T} = Lie\ \mathbf{T}$ the proof is similar. Since p is a very good prime we have $dim_{\mathbf{C}}\ y = n -\ rank\ (\Phi') = n - dim\langle\Phi'\rangle_{\overline{\mathbb{F}_q}}$ for $y = \theta(\Phi')$ and again the result follows from (2). $\circ$

Remark 3.3 (i) *If one considers the rank function $rk : \mathcal{H} \to \mathbb{N},\ h \mapsto \mathrm{codim}\ h$, then*

$$f_{Lie\mathbf{T}}(\Phi_1, a) = \sum_{y \in (\mathcal{H}_\Phi)_a} \mu_{(\mathcal{H}_\Phi)_a}(\theta(\Phi_1), y)\ q^{dim\ y}$$

is the **characteristic polynomial** *of the interval $[\theta(\Phi_1), \mathbf{1}]_a \subseteq \mathcal{H}_a$ as defined in combinatorics.*
(ii) *If char $\mathbb{F}_q$ is very good, then, as a result of 3.2, the root systems Φ_1 of stabilizers of semisimple elements in $LieG$ satisfy $\Phi_1 = \overline{\Phi_1}$, hence the corresponding algebraic stabilizers $\mathbf{G}_{\Phi_1}$ are Levi subgroups of $\mathbf{G}$.*

In ([OS2]), Orlik and Solomon proved the identity

$$\mathcal{P}_\Phi(id, x) = \prod_{i=1}^{rank\ \Phi} (1 + (d_i - 1) \cdot x)$$

where the d_i's are the 'fundamental degrees' of the Coxeter group W_Φ (see [16]). Using this we obtain the following closed formulae for the number of regular elements in 'split tori' and their Lie algebras:

Corollary 3.4 *Let F be a split Frobenius endomorphism, i.e. $F^* = q \cdot Id|_X$. In case $\mathcal{T} = \mathbf{T}$ let* $\gcd(q-1, R) = 1$. *In case $\mathcal{T} = Lie\mathbf{T}$ let $p =$ char $\mathbb{F}_q$ be a very good prime. Then*

$$f_T([\emptyset],[id]) = x^n \cdot \mathcal{P}_\Phi(id, -1/x)|_{x=q-1} = (q-1)^{n-rank\ \Phi} \prod_{i=1}^{rank\ \Phi} (q - d_i);$$

$$f_{Lie\mathbf{T}}([\emptyset],[id]) = x^n \cdot \mathcal{P}_\Phi(id, -1/x)|_{x=q} = q^{n-rank\ \Phi} \prod_{i=1}^{rank\ \Phi} (q - d_i + 1).$$

The polynomials $\mathcal{P}_\Phi(w,x)$ have been determined by Lehrer and Solomon [LS] for $\Phi = A_n$ and by Lehrer [L1] for $\Phi = B_n, C_n$. In [FJ1] one can find a unified computation, valid for all classical root systems including D_n. For exceptional groups the Poincaré polynomials have been computed in [FJ3] using the computer.

4 Möbius functions of stable subsystems

In general the hypotheses of 3.2 are not satisfied. To evaluate (1), one has to determine the Möbius functions μ_a of $(\mathcal{S}_\Phi)_a$ for $a \in \tilde{W}$. This can be done, using the connection between irreducible polynomials over $\mathbb{F}_q$ and stable partition lattices, given by decomposition patterns of characteristic polynomials of matrices in $Mat_n(\mathbb{F}_q)$. This section will give a brief outline of these arguments.

For a group H acting on the finite set M and $h \in H$, let M_h denote the set of h - fixed points in M. For a ring R let $R^{\underline{n}}$ denote the set of functions from $\underline{n} := \{1, 2, \dots n\}$ to R and let $\Pi(\underline{n})$ denote the set of all partitions of $\underline{n}$. Then $\Sigma_{\underline{n}}$ acts naturally on $\Pi(\underline{n})$ and $R^{\underline{n}}$ and for each $w \in \Sigma_{\underline{n}}$ we call an element of $\Pi(\underline{n})_w$ a w **- stable partition**. We consider $\Pi(\underline{n})$ as a lattice with "refinement" ordering on $\Pi(\underline{n})$:
In this ordering $\lambda \leq \lambda'$, if each block $m \in \lambda$ is contained in a block $m' \in \lambda'$; moreover $\lambda \wedge \lambda' := \{m \cap m' \mid m \in \lambda, m' \in \lambda'\}$. Notice that any subposet $\Pi(\underline{n})_w$ has a unique minimal element $0_{\underline{n}} := \{\{x\} \mid x \in \underline{n}\}$ and a unique maximal one $1_{\underline{n}} := \{\underline{n}\}$.
Now let us recall the usual description of indecomposable classical root systems as subsets of $\mathbb{Z}^{\underline{n}}$:

Definition 4.1 *If $\{e_i \mid i \in \underline{n}\}$ denotes the canonical basis of $\mathbb{Z}^{\underline{n}}$, then:*

$$
\begin{array}{lll}
\Phi_{D_n} & := & \{ \pm(e_i \pm e_j) \mid i \ < j \in \underline{n} \ \} \\
\Phi_{B_n} & := & \{ \pm(e_i \pm e_j) \ , \pm e_i \mid i \ < j \in \underline{n} \ \} \\
\Phi_{C_n} & := & \{ \pm(e_i \pm e_j) \ , \pm 2e_i \mid i \ < j \in \underline{n} \ \} \\
\Phi^c_{A_{n-1}} & := & \{ \ v \in \Phi_{D_n} \mid [v, \sum_{i \in \underline{n}} c_i \cdot e_i] = 0 \ \}
\end{array}
$$

where $[,]$ *denotes the bilinear form on* $\mathbb{Z}^{\underline{n}}$ *given by* $[e_i, e_j] := \delta_{ij}$ *and* $c = (c_1, c_2, ..., c_n)$ *is a vector in* $\mathbb{Z}^{\underline{n}}$ *with* $c_k = \pm 1$ *for all* k.

For the rest of this section Φ is always one of these root systems. The corresponding Weyl groups are subgroups of $W_{B_n} = W_{C_n} \cong (\mathbb{Z}/2\mathbb{Z})^{\underline{n}} \cdot \Sigma_{\underline{n}}$ and act naturally on $\underline{n}$ and $\Pi(\underline{n})$.

The connection with partition posets is the following:
If Φ is one of the root systems above and $\Phi_1 \in \mathcal{S}_\Phi$ a closed subsystem, then one can define an equivalence relation on $\underline{n}$ by setting $i \ \cong_{\Phi_1} \ j$ if and only if $i = j$ or there is a root $\alpha \in \Phi_1$ involving the basis vectors e_i and e_j with non-vanishing coefficient. If Φ is not of type B then the map

$$\Lambda : \mathcal{S}_\Phi \longrightarrow \Pi(\underline{n}), \quad \Phi_1 \mapsto \{ \ [i]_{\cong_{\Phi_1}} \mid i \in \underline{n} \ \}$$

establishes a Galois connection of W_Φ - posets:

For $m \subseteq \underline{n}$ and $\Phi_1 \in \mathcal{S}_\Phi$ we define: $(\Phi_1)_m := \Phi_1 \cap \langle e_i \mid i \in m \rangle_{\mathbb{Z}}$. Since $e_i \in \langle \Phi_1 \rangle_{\mathbb{Z}}$ if and only if Φ is of type B_n one can easliy see that

$$\Phi_{1,\lambda} := \Phi \cap \sum_{m \in \lambda} \langle (\Phi_1)_m \rangle_{\mathbb{Z}} \ = \biguplus_{m \in \lambda} (\Phi_1)_m$$

and

$$\Lambda(\Phi_{1,\lambda}) = \lambda \wedge \Lambda(\Phi_1)$$

for any $\lambda \in \Pi(\underline{n})$ and $\Phi \neq B_n$.

On the other hand we have for all $\lambda < 1_{\Pi(\underline{n})}$:

$$\biguplus_{m \in \lambda} (\Phi_{B_n})_m < \Phi_{B_n,\lambda} = \Phi_{B_n} \cap \sum_{m \in \lambda} \langle (\Phi_{B_n})_m \rangle_{\mathbb{Z}} = \Phi_{B_n}.$$

Hence we get

Lemma 4.2 *Suppose* Φ *is of type* A, C *or* D. *Then for any* $w \in W_\Phi$ *and any interval* $[\ \Phi_1, \Phi_2\ ]_w \subseteq (\mathcal{S}_\Phi)_w$, *the restriction*

$$\Lambda_| : [\ \Phi_1, \Phi_2\ ]_w \longrightarrow [\ \Lambda(\Phi_1), \Lambda(\Phi_2)\ ]_w \subseteq \Pi(\underline{n})_w, \quad \Phi' \mapsto \Lambda(\Phi')$$

together with the map

$$\Phi_{2,(\ )} : [\ \Lambda(\Phi_1), \Lambda(\Phi_2)\ ]_w \to [\ \Phi_1, \Phi_2\ ]_w, \ \lambda \mapsto \Phi_{2,\lambda}$$

form a closure operator $\theta := \Phi_{2,(\ )}\circ\Lambda_|$ *with* $\theta([\ \Phi_1, \Phi_2\ ]_w) \cong [\ \Lambda(\Phi_1), \Lambda(\Phi_2)\ ]_w$.
If $\mathcal{S}^*_{\Phi_{B_n},w}$ *denotes the dual of* $\mathcal{S}_{\Phi_{B_n},w}$ *(i.e. with reversed inclusion order), then the map*

$$d : \mathcal{S}^*_{\Phi_{B_n},w} \longrightarrow \mathcal{S}_{\Phi_{D_n},w},\ \Phi_1 \mapsto \Phi_1 \cap \Phi_{D_n}$$

together with the embedding $\Phi_{D_n} \hookrightarrow \Phi_{B_n}$ *is a Galois connection (in the sense of [22]).*

Proof: It is clear that the assignment $\lambda \mapsto \Phi_{2,\lambda}$ is a right inverse of $\Lambda_|$. Moreover $\Phi' \leq \Phi_{2,\Lambda(\Phi')}$ for all $\Phi' \in [\Phi_1, \Phi_2]_w$. The rest is clear. $\circ$

If Φ is indecomposable of type A then Λ is an isomorphism of posets. Using 4.2 the determination of Möbius functions of stable subsystems of classical root systems can be reduced to the stable partition poset.
These have been widely investigated in combinatorics. In particular Hanlon [15] computed the values $\mu^n_w(0,1)$ for $\Pi(\underline{n})_w$ with cohomological methods. Here we give an independent derivation of the values $\mu^n_w(\lambda', \lambda)$. The starting point is the following polynomial identity for the characteristic polynomial of a stable partition lattice:

Theorem 4.3 *Let* $w \in \Sigma_{\underline{n}}$ *be a product of* n_i *cyles of length* i*, then:*

$$\sum_{\lambda\in\Pi(\underline{n})_w} \mu^n_w(0,\lambda)\cdot q^{|\lambda|} = \prod_{i=1}^{n}(i^{n_i}\cdot n_i!)\cdot\binom{\Psi_i(q)}{n_i}.$$

($\Psi_i(q) = 1/i\cdot\sum_{d|i}\mu(d)q^{i/d} = |\{f \in \mathbb{F}_q[X] \mid f \text{ monic irreducible of degree } i\}|$*.)*

Proof: We view the general linear group $GL_n(q)$ as a finite group of Lie type $\mathbf{G}_F$ with $\mathbf{G} = GL_n(\overline{\mathbb{F}}_q)$ and $F : GL_n(\overline{\mathbb{F}}_q) \longrightarrow GL_n(\overline{\mathbb{F}}_q),\ (g_{ij}) \mapsto ({g_{ij}}^q)$. Then $LieG = (Mat_n(\mathbb{F}_q), [\ ,\ ])$ where $[\ ,\ ]$ is the commutator of matrices. The adjoint action of G on $LieG$ is given by conjugation of matrices.
Two semisimple elements α and $\beta \in LieG$ are conjugate under G if and only if they have the same characteristic polynomial $cp(\alpha) = cp(\beta)$. Moreover α is regular if and only if $cp(\alpha)$ is multiplicity free. If $LieG_{ss,r}/G$ denotes the set of regular, semisimple G - orbits in $LieG$ and $\mathcal{P}(n)$ the set of number partitions j^{c_j} with $\sum_j c_j\cdot j = n$, then we get a map:

$$m : LieGL_n(q)_{ss,r}\backslash GL_n(q) \to \mathcal{P}(n),\ \alpha^{GL_n(q)} \mapsto i^{c_i},$$

where c_i denotes the number of all $\mathbb{F}_q$ - irreducible divisors of $cp(\alpha)$ with degree i. Its fibres satisfy: $|m^{-1}((i^{c_i}))| = \prod_{i=1,\dots,n}\binom{\Psi_i(q)}{c_i}$. Recall from section 2 that semisimple genera of $LieGL_n(q)$ are parametrized by pairs $(\Phi_1, [w])$.

In particular a regular semisimple element α has genus $(\emptyset, [w])$, where $[w]$ is a conjugacy class of $W \cong \Sigma_n$ of cycle type $m(\alpha^{GL_n(q)})$. Hence we get

$$|\mathcal{O}_{LieG}(\emptyset, [w])| = |m^{-1}((i^{c_i}))| = \prod_{i=1,\ldots,n} \binom{\Psi_i(q)}{c_i}.$$

Now the statement of the theorem follows from 3.2 since $\mathcal{H}$ and $\Pi(\underline{n})$ are isomorphic $\Sigma_{\underline{n}}$ - posets and all primes are very good for $GL_n(q)$. Notice that for any $\Phi' \in \mathcal{S}_\Phi$ we have $\dim \langle \Phi' \rangle_{\overline{\mathbb{F}}_q} = \sum_{m \in \Lambda(\Phi')} |m| - 1 = dim \mathbf{T} - |\Lambda(\Phi')|$, $w_1 \rho = id$ here and $|(\Sigma_{\underline{n}})_w| = \prod_{i=1,\ldots,n} i^{n_i} \cdot (n_i)!$. $\circ$

A permutation $w \in \Sigma_{\underline{n}}$ is called **homogeneous**, if it is a product of cycles of the same length. A comparison of coefficients yields:

Corollary 4.4 *Let $w \in \Sigma_{\underline{n}}$ with order $ord(w)$ and $k := n/ord(w)$.*
a) (Hanlon[15])

$$\mu_w^n(0,1) = \begin{cases} (-1)^{k-1}(k-1)! \cdot \mu(ord(w)) \cdot ord(w)^{k-1} & \text{if } w \text{ is homogeneous} \\ 0 & \text{otherwise} \end{cases}.$$

b) Suppose that $\lambda \in \Pi(\underline{n})_w$ consists of a single w - orbit. Then:

$$\mu_w^n(0,\lambda) = \begin{cases} (-1)^{k-1}(k-1)! \cdot \mu(s) \cdot s^{k-1} & \text{if } w \text{ is homogeneous} \\ 0 & \text{otherwise.} \end{cases}$$

Here $s = ord(w)/|\lambda|$.

Proof: For b) we consider an $m \in \lambda$. Then the assignment $\lambda' \mapsto \{m' \in \lambda' \mid m' \subseteq m\}$ is an isomorphism of intervals $[0_{\underline{n}}, \lambda]_w$ and $[0_m, 1_m]_{w^{|\lambda|}} \subseteq \Pi(m)_{w^{|\lambda|}}$. $\circ$

Let $w \in \Sigma_{\underline{n}}$ and $\lambda' \in \Pi(\underline{n})_w$. Then for every $\lambda'' \in [\lambda', 1_{\underline{n}}]_{\underline{n}}^w$ we can define an equivalence relation on the set of sets λ' by setting $m_1 \cong_{\lambda''} m_2$ if and only if $m_1 \cup m_2 \subseteq m$ for some $m \in \lambda''$. If $\overline{w}$ denotes the image of w in $\Sigma_{\lambda'}$, we get an isomorphism of posets:

$$\theta : [\ \lambda'\ ,\ 1_{\underline{n}}\ ]_{\underline{n}}^w \longrightarrow \Pi(\lambda')_{\overline{w}},\ \lambda'' \mapsto \{\ [m]_{\cong_{\lambda''}} \mid m \in \lambda'\ \}$$

and therefore

$$\mu_w^n(\lambda', \lambda) = \mu_{\overline{w}}^{|\lambda'|}(0, \theta(\lambda)).$$

It is straightforward to reduce the computation of the values $\mu_{\overline{w}}^{|\lambda'|}(0, \theta(\lambda))$ to the case where $\theta(\lambda)$ consitsts of only one $\overline{w}^{|\lambda'|}$ - orbit. This completely describes the Möbius functions of stable partition posets.

Here we give the values for $\mu_w(0,1) := \mu_\Phi^w(0,1)$ which are of independent combinatorial interest and refer to [9] for more details.

Theorem 4.5 *Let $w \in W_\Phi$ be conjugate to a product $\prod_{i=1}^n w_i$, where each w_i is a product of n_i^+ positive and n_i^- negative i - cycles. Then :*
a) $\mu_w(0,1) = 0$ unless w is homogeneous.
b) If w is homogeneous, i.e. $w = w_i$ for an i with $i \cdot (n_i^+ + n_i^-) = n$, then one of the following holds:

1) $\Phi = A_{n-1}$ and $\mu_w(0,1) = \mu_w^n(0,1) := (-1)^{k-1}(k-1)!\mu(i)i^{k-1}$, where $k := n/i$;

$$2)\ \Phi = D_n \text{ and } \mu_w(0,1) = \begin{cases} \mu_w^n(0,1) & \text{if } n_i^- \neq 0 \text{ and } i \text{ is even or} \\ & n_i^+ \cdot n_i^- \neq 0 \text{ and } i \text{ is odd;} \\ (1-2^{n_i^+-1}) \cdot \mu_w^n(0,1) & \text{if } n_i^- = 0 \text{ and } i \text{ odd;} \\ (1-2^{n_i^--1}) \cdot \mu_w^n(0,1) & \text{if } n_i^+ = 0 \text{ and } i \text{ odd;} \\ (1-2^{n_i^+}) \cdot \mu_w^n(0,1) & \text{if } n_i^- = 0 \text{ and } i \text{ even ;} \end{cases}$$

$$3)\ \Phi = C_n \text{ and } \mu_w(0,1) = \begin{cases} 0 & \text{if } n_i^- \neq 0 \text{ and } i \text{ even or} \\ & n_i^+ \cdot n_i^- \neq 0 \text{ and } i \text{ odd ;} \\ -2^{n_i^+-1} \cdot \mu_w^n(0,1) & \text{if } n_i^- = 0 \text{ and } i \text{ odd;} \\ -2^{n_i^--1} \cdot \mu_w^n(0,1) & \text{if } n_i^+ = 0 \text{ and } i \text{ odd;} \\ -2^{n_i^+} \cdot \mu_w^n(0,1) & \text{if } n_i^- = 0 \text{ and } i \text{ even;} \end{cases}$$

$$4)\ \Phi = B_n \text{ and } \mu_w(0,1) = \begin{cases} (-1)^n \cdot n! & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: (Sketch) If Φ is of type A then the claim is clear since in this case $\Lambda : \mathcal{S}_{\Phi,w} \longrightarrow \Pi(\underline{n})_w$ is an isomorphism. Suppose Φ is of type D. Then 4.2 and Rota's theorem on Galois - connections yield: $\mu_w(0, \Phi_{D_n}) = (1 - A_w) \cdot \mu_w^n(0,1)$, where A_w is the number of w - stable subsystems of type A_{n-1} in Φ_{D_n}. In case $\Phi = \Phi_{C_n}$ the number $1 - A_w$ has to be replaced by $-A_w$ because $\Phi_{C_n,\{\{1\},\dots,\{n\}\}} \cong C_1^n \neq \emptyset$. By the results on $\mu_w^n(0,1)$ it suffices to consider A_w only for those w that are homogeneous. This gives the results above. Suppose now that Φ is of type Φ_{B_n}. Then we get from 4.2

$$0 = \sum_{\Phi'' \in \mathcal{S}_{D_n,w}, \Phi'' = \Phi_{B_n}} \mu_w(0, \Phi'') = \sum_{\Phi' \in \mathcal{S}_{B_n,w}, \Phi' \cap \Phi_{D_n} = \emptyset} \mu_w(\Phi', \Phi_{B_n}).$$

Hence $\mu_w(0, \Phi_{B_n}) =$

$$-\sum_{j \in \underline{n},\ w(j)=j} \mu_w(\Phi_{B_1(\{j\})}, \Phi_{B_n}) = -\sum_{j \in \underline{n},\ w(j)=j} \mu_{w|_{\Phi_{B_{n-1}(\underline{n}\setminus\{j\})}}}(0, \Phi_{B_{n-1}(\underline{n}\setminus\{j\})}).$$

Here $\Phi_{B_{|m|}(m)}$ is the root system of type $B_{|m|}$ ' on the cyphers $m \subseteq \underline{n}$'. Now the statement follows easily by induction. $\circ$

Using the methods sketched in this section, all genus numbers of regular semisimple conjugacy classes for simply connected groups of classical type have been computed in [10]. In general these formulae can be rather complicated (see also [13]). However the generic formulae for the total numbers of regular semisimple classes often have a simple shape. It has been proved by Lehrer [18] that in case of semisimple simply connected groups, these numbers are always odd. Using methods similar to the one sketched here, the following result has been obtained in [14]:

Theorem 4.6 *Let $S_n^\epsilon(q)$ denote $SL_n(q)$ and $SU_n(q^2)$ for $\epsilon = 1$ and -1 respectively. Then the number of regular semisimple conjugacy classes in G is given by:*
$\frac{q^{n+1}-q^n+(-1)^{n+1}\cdot\epsilon^{\left[\frac{n}{2}\right]}\cdot(q-\epsilon^n)}{q^2-\epsilon}$ *if n is odd or q is even and*
$\frac{q^{n+1}-q^n-\epsilon^{\frac{n}{2}}\cdot(q-1)}{q^2-\epsilon} - \epsilon^{\frac{n}{2}+1}$ *if n is even and q is odd.*

Finally, let us once more consider Lusztig's parametrization of irreducible characters (section 1 $(*)$). Since $|\mathcal{E}(G_s,[1])|$ is constant (and known) for all semisimple classes of given genus in G, one can use genus numbers to compute the total number of irreducible characters or, equivalently, the generic class number of G^*.

In [11] and [12] all generic semisimple genus numbers $f_{\mathbf{T}}(\Phi_1,[w])$ have been calculated for all simply connected groups of type E_6, E_7 and E_8 (see also [6] for E_6). The computations were done by implementing the formulae in section 2 on a computer. The following example shows the generic class number of the simple groups of type $E_8(q)$:

$$\begin{array}{ll}
q^8+q^7+2q^6+3q^5+\ \,9q^4+14q^3+32q^2+47q+\ \,70 & \text{for } q=2^n\ ;\\
q^8+q^7+2q^6+3q^5+10q^4+16q^3+39q^2+65q+102 & \text{for } q=3^n\ ;\\
q^8+q^7+2q^6+3q^5+10q^4+16q^3+40q^2+67q+111 & \text{for } q=5^n\ ;\\
q^8+q^7+2q^6+3q^5+10q^4+16q^3+40q^2+67q+112 & \text{otherwise.}
\end{array}$$

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