DEPTH AND THE RELATIVE TRACE IDEAL

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Abstract. We prove that for a modular representation, the depth of the ring of invariants is the sum of the dimension of the fixed point space of the p-Sylow subgroup and the grade of the relative trace ideal. We also determine which of the Dickson invariants lie in the radical of the relative trace ideal and we describe how to use the Dickson invariants to compute the grade of the relative trace ideal.

1. Preliminaries

Let G be a finite group, p a prime divisor of the order of G, $\mathbf{k}$ a field of characteristic p and V a finitely generated $\mathbf{k}G$-module. The action of G on V induces an action on the dual, V^*, which extends to an action by algebra automorphisms on the symmetric algebra $A := \mathrm{Sym}(V^*)$. The ring of invariants, $A^G := \{f \in A \mid g(f) = f, \ \forall g \in G\}$, is a finitely generated subring of A. The *depth* of A^G is the length of a maximal regular sequence in A^G. Since V is a modular representation of G, the ring A^G may fail to be Cohen-Macaulay. In other words the depth might be less than the Krull dimension. This difference is measured by the *Cohen-Macaulay defect*, $\mathrm{def}(A^G) := \dim(A^G) - \mathrm{depth}(A^G)$. By the Auslander-Buchsbaum formula, $\mathrm{def}(A^G)$ is equal to the projective dimension of A^G as a module over a homogeneous system of parameters. Thus $\mathrm{def}(A^G)$ is a measure of the structural complexity of A^G. Despite significant recent progress (see, for example, [5], [11], [12], [13], [14], [15], [16], [18]), much of which refers in one way or another to the ground breaking work of Ellingsrud and Skjelbred [7], computing the depth of A^G remains, in general, quite difficult.

For a subgroup $H \leq G$ the *relative trace* (also called the relative transfer) is the map of A^G-modules defined by

$$\begin{array}{rccc} t^G_H: & A^H & \longrightarrow & A^G \\ & f & \longmapsto & \displaystyle\sum_{g \in G/H} g(f). \end{array}$$

The image of t^G_H is an ideal in the ring A^G. Choose a p-Sylow subgroup $P \leq G$ and define the *relative trace ideal* by $A^G_{<P} := \sum_{Q<P} t^G_Q(A^Q)$. The relative trace ideal is the largest ideal of A^G which can be constructed using relative trace maps from proper subgroups of P. Ideals of this form were studied in [8] where it was shown that the radical of the ideal, $\sqrt{A^G_{<P}}$, is always a prime ideal of height $\mathrm{codim}_{\mathbf{k}}(V^P)$. Here V^P denotes the subspace of P-fixed points in V. If $\mathbf{k}$

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is algebraically closed, the ideal $A^G_{<P}$ has a very natural geometric interpretation: its variety, $\mathcal{V}$, in the orbit space V/G, consists of all those G-orbits such that a suitable stabilizer contains P or, in other words,

$$\begin{aligned}\mathcal{V} &= \left\{v^G \in V/G \mid p \text{ does not divide } \left|v^G\right|\right\} \\ &= \left\{v^G \in V/G \mid v \in V^P\right\}\end{aligned}$$

(see [8, Proposition 12.5]). In particular the ideal $\sqrt{A^P_{<P}} \leq A^P$ is the ideal corresponding to the linear subvariety $V^P \leq V/P$, the space of P-fixed points in the space of P-orbits.

It is well known that $\dim(V^P)$ is a lower bound for the depth of A^G (see, for example, [5, Theorem 2]). In Section 2 we prove that the 'missing part' of the depth is provided by regular elements in the relative trace ideal, $A^G_{<P}$. As a consequence, computing the depth of A^G reduces to computing the grade of $A^G_{<P}$.

In Section 3 we describe how the Dickson invariants, the generators of $A^{\mathrm{GL}_n(\mathbf{F}_q)}$, can be used, when $\mathbf{k} = \mathbf{F}_q$, to compute the grade of the relative trace ideal.

We recommend [6] and [4] as basic references for the required commutative algebra and [1] and [17] as basic references for the invariant theory of finite groups.

2. Depth and Localization

Suppose R is a Noetherian ring, M is a finite R-module, and I is an ideal in R such that $IM \neq M$. Then the common length of all maximal M-sequences in I is called the *grade* of I on M and will be denoted by $\mathrm{grade}(I, M)$. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathrm{grade}(\mathfrak{m}, M)$ is the depth of M and if R is graded with augmentation ideal R_+, then $\mathrm{grade}(R_+, M)$ is the the depth of M.

Theorem 2.1. (G. Kemper, [12, Theorem 1.5]) *Let $U \leq V$ be a $\mathbf{k}G$-submodule, such that the kernel of the action of G on U has index coprime to p and let M be a non-zero graded module over* $\mathrm{End}_{\mathbf{k}G}(A)$*, which is finitely generated over A^G. Furthermore let $\mathcal{I}_{A^G}(U) = (U^\perp)A \cap A^G$, then*

$$\mathrm{depth}(M) = \mathrm{grade}(\mathcal{I}_{A^G}(U), M) + \dim_{\mathbf{k}}(U).$$

The hypotheses of this theorem are satisfied when $G = P$, $M = A^P$ and $U = V^P$. In this case $\mathcal{I}_{A^G}(U) = \sqrt{A^P_{<P}}$. Therefore, using the fact that an ideal and its radical have the same grade, we have the following corollary.

Corollary 2.2.

$$\mathrm{depth}\ A^P = \mathrm{grade}\left(\sqrt{A^P_{<P}}, A^P\right) + \dim\ V^P = \mathrm{grade}\left(A^P_{<P}, A^P\right) + \dim\ V^P.$$

The primary purpose of this section is to extend this corollary to a statement about A^G.

For a prime ideal $\mathcal{P} \in \mathrm{Spec}(A)$, let

$$G_{\mathcal{P}} := \{g \in G \mid g(f) - f \in \mathcal{P},\ \forall f \in A\}$$

and $\mathfrak{p} := A^G \cap \mathcal{P}$. Define the Cohen-Macaulay defect of a Noetherian ring R by

$$\mathrm{def}(R) := \dim(R) - \mathrm{depth}(R).$$

For the local ring $A_{\mathfrak{p}}^G$, the dimension is the height of the maximal ideal $\mathfrak{p}$. Thus

$$\operatorname{def}(A_{\mathfrak{p}}^G) = \operatorname{ht}(\mathfrak{p}) - \operatorname{depth}(A_{\mathfrak{p}}^G).$$

Since $\mathcal{P}$ lies over $\mathfrak{p}$ these ideals have the same height. Furthermore, A is Cohen-Macaulay so

$$\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathcal{P}) = \dim(A) - \dim(A/\mathcal{P})$$

and

$$\operatorname{def}(A_{\mathfrak{p}}^G) = \dim(A) - \dim(A/\mathcal{P}) - \operatorname{depth}(A_{\mathfrak{p}}^G). \tag{2.2.1}$$

Proposition 2.3. *Suppose that $\mathcal{P} \in \operatorname{Spec}(A)$ and $[G : G_{\mathcal{P}}]$ is relatively prime to p. Then*
(i) $\operatorname{depth}(A^G) = \operatorname{depth}(A^{G_{\mathcal{P}}})$,
(ii) $\operatorname{def}(A_{\mathfrak{p}}^G) = \operatorname{def}(A^{G_{\mathcal{P}}}) = \operatorname{def}(A^G)$, *and*
(iii) $\operatorname{depth}(A^G) = \dim(A/\mathcal{P}) + \operatorname{depth}(A_{\mathfrak{p}}^G)$.

Proof. Since $[G : G_{\mathcal{P}}]$ is relatively prime to p, $\operatorname{depth}(A^G) \geq \operatorname{depth}(A^{G_{\mathcal{P}}})$ (see, for example, [10, Proposition 14]). On the other hand, from [13, Korollar 5.14] or [14, Theorem A (a)], $\operatorname{depth}(A^G) \leq \operatorname{depth}(A^{G_{\mathcal{P}}})$ and (i) follows. The first equality in (ii) is a clear consequence of either [13, Satz 5.12] or Proposition 2.17 and Theorem 1.9 of [14]. The second equality follows from (i) and the fact that A^G and $A^{G_{\mathcal{P}}}$ have the same dimension.

Using (ii) gives

$$\operatorname{depth}(A^G) = \dim(A^G) - \operatorname{def}(A_{\mathfrak{p}}^G).$$

Substituting Equation 2.2.1 into this gives

$$\operatorname{depth}(A^G) = \dim(A^G) - (\dim(A) - \dim(A/\mathcal{P}) - \operatorname{depth}(A_{\mathfrak{p}}^G)).$$

However, $\dim(A^G) = \dim(A)$ and (iii) follows. □

Theorem 2.4. *Suppose that $\mathcal{P} \in \operatorname{Spec}(A)$ and $[G : G_{\mathcal{P}}]$ is relatively prime to p. Then*
(i) $\operatorname{depth}(A_{\mathfrak{p}}^G) = \operatorname{grade}(\mathfrak{p}, A^G)$ *and*
(ii) $\operatorname{depth}(A^G) = \dim(A/\mathcal{P}) + \operatorname{grade}(\mathfrak{p}, A^G)$.

Proof. Using [4, Proposition 1.2.10(a)]

$$\operatorname{grade}(\mathfrak{p}, A^G) = \inf\{\operatorname{depth}(A_{\mathfrak{q}}^G) \mid \mathfrak{p} \subseteq \mathfrak{q},\ \mathfrak{q} \text{ prime}\}.$$

Therefore

$$\operatorname{grade}(\mathfrak{p}, A^G) \leq \operatorname{depth}(A_{\mathfrak{p}}^G) \tag{2.4.1}$$

and for some prime $\mathfrak{q}$ with $\mathfrak{p} \subseteq \mathfrak{q}$, $\operatorname{grade}(\mathfrak{p}, A^G) = \operatorname{depth}(A_{\mathfrak{q}}^G)$. Since A is an integral extension of A^G, there is a prime $\mathcal{Q}$ lying over $\mathfrak{q}$ with $\mathcal{P} \subseteq \mathcal{Q}$. Therefore $G_{\mathcal{P}} \subseteq G_{\mathcal{Q}}$ and we may apply Proposition 2.3 to the ideal $\mathcal{Q}$. Thus $\operatorname{depth}(A^G) = \operatorname{depth}(A^{G_{\mathcal{Q}}})$ and $\operatorname{def}(A_{\mathfrak{q}}^G) = \operatorname{def}(A^G) = \operatorname{def}(A_{\mathfrak{p}}^G)$. However, since $\mathfrak{p} \subseteq \mathfrak{q}$, $\operatorname{ht}(\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p})$ and

$$\begin{aligned}
\operatorname{grade}(\mathfrak{p}, A^G) = \operatorname{depth}(A_{\mathfrak{q}}^G) &= \operatorname{ht}(\mathfrak{q}) - \operatorname{def}(A_{\mathfrak{q}}^G) = \operatorname{ht}(\mathfrak{q}) - \operatorname{def}(A_{\mathfrak{p}}^G) \\
&\geq \operatorname{ht}(\mathfrak{p}) - \operatorname{def}(A_{\mathfrak{p}}^G) = \operatorname{depth}(A_{\mathfrak{p}}^G).
\end{aligned}$$

Comparing this with Inequality 2.4.1, we have $\mathrm{grade}(\mathfrak{p}, A^G) = \mathrm{depth}(A^G_{\mathfrak{p}})$. Substituting into part (iii) of Proposition 2.3 gives

$$\mathrm{depth}(A^G) = \dim(A/\mathcal{P}) + \mathrm{grade}(\mathfrak{p}, A^G)$$

as required. $\square$

Suppose that $\mathcal{P}$ is the prime ideal in A associated to the linear subspace V^P. Therefore $\dim(A/\mathcal{P}) = \dim(V^P)$. Clearly $P \leq G_{\mathcal{P}}$. Thus $[G : G_{\mathcal{P}}]$ is relatively prime to p. From [8, Corollary 12.3], the prime $\mathcal{P}$ lies over $\sqrt{A^G_{<P}}$. Therefore we may apply Theorem 2.4 with $\mathfrak{p} = \sqrt{A^G_{<P}}$. Since an ideal and its radical have the same grade (see, for example, [4, Proposition 1.2.10(b)]) we have the following.

Corollary 2.5. $\mathrm{depth}(A^G) = \dim(V^P) + \mathrm{grade}(A^G_{<P}, A^G)$.

3. Using the Dickson Invariants

In this section we assume that $\mathbf{k} = \mathbf{F}_q$, the field with $q = p^r$ elements. Let $\overline{\mathbf{F}_q}$ denote the algebraic closure and define $\overline{V} := \overline{\mathbf{F}_q} \otimes V$. Thus elements of A represent distinct functions on $\overline{V}$. Consider the usual representation of $\mathrm{GL}_n(\mathbf{F}_q)$. Choose a basis, $\{x_1, x_2, \ldots, x_n\}$, for V^* and let $V_i^* = \mathrm{Span}_{\mathbf{F}_q}\{x_1, \ldots, x_i\}$. Define $F_i(t) = \prod_{v \in V_i^*}(t - v)$. The ring $A^{\mathrm{GL}_n(\mathbf{F}_q)}$ is generated by the Dickson invariants, $d_{i,n}$, which are defined by $F_n(t) = \sum_{i=0}^n d_{i,n} t^{q^{n-i}}$. Note that $d_{0,n} = 1$. We adopt the convention that $d_{i,n} = 0$ if $i > n$.

As a consequence of Bourguiba & Zarati's proof of the Landweber-Stong Conjecture ([3, Theorem 1.1] for $r = 1$ and [2] or [9] for $r > 1$) we can use the Dickson invariants as a test sequence to measure $\mathrm{depth}(A^G)$. In other words, the depth of A^G is the largest ℓ such that $d_{1,n}, \ldots, d_{\ell,n}$ is A^G-regular. In this section we show that the Dickson invariants can be used as a test sequence to measure $\mathrm{grade}(A^G_{<P}, A^G)$.

Define $h_i = F_{i-1}(x_i)$. It follows from [19, Proposition 1.3b] that

$$d_{i,n} = d_{i,n-1}^q - d_{i-1,n-1} h_n^{q-1}. \tag{3.0.1}$$

Using this inductively gives

$$d_{1,n} = -(h_1^{q^{n-1}(q-1)} + h_2^{q^{n-2}(q-1)} + \cdots + h_n^{q-1}) \tag{3.0.2}$$

and

$$d_{n,n} = (-1)^n (h_1 h_2 \ldots h_n)^{q-1}. \tag{3.0.3}$$

Lemma 3.1. *If $i > 0$ then $d_{i,n} \in (h_1, h_2, \ldots h_{n-i+1})A$.*

Proof. The proof is by a double induction first on i and then on $n - i$.

For $i = 1$ the result follows from Equation 3.0.2. We now assume that $i > 1$. For $n - i = 0$ the result follows from Equation 3.0.3. We now assume that $n - i > 0$.

Using the secondary induction hypothesis $d_{i,n-1} \in (h_1, h_2, \ldots h_{n-i})A$. By the primary induction hypothesis $d_{i-1,n-1} \in (h_1, h_2, \ldots h_{n-i+1})A$. Therefore it follows from Equation 3.0.1 that $d_{i,n} \in (h_1, h_2, \ldots h_{n-i+1})A$. $\square$

Suppose that $G \leq \mathrm{GL}_n(\mathbf{F}_q)$ and $\dim(V^P) = k$. Choose a basis for V so that $\overline{V}^P$ is the variety cut out by $x_1, \ldots, x_{n-k}$. Since $h_1, \ldots, h_{n-k}$ are polynomials in $x_1, \ldots, x_{n-k}$, we conclude that $h_1, \ldots, h_{n-k}$ are zero on $\overline{V}^P$. Therefore it follows from Lemma 3.1 that, if $i > k$ then $d_{i,n}$ is zero on $\overline{V}^P$.

Corollary 3.2. *If* $i > \dim(V^P)$ *then* $d_{i,n} \in \sqrt{A^G_{<P}}$.

Theorem 3.3. *Suppose that* $\mathrm{depth}(A^G) = \ell$ *and* $\dim(V^P) = k$.
(i) If $i \leq k$ *then* $d_{i,n} \notin A^G_{<P} \subseteq \sqrt{A^G_{<P}}$, *and*
(ii) $d_{k+1,n}, \ldots, d_{\ell+1,n}$ *is not* A^G*–regular.*

Proof. Since the Dickson invariants form a homogeneous system of parameters for A^G and $\dim\left(\sqrt{A^G_{<P}}\right) = \dim(V^P) = k$, the ideal $\sqrt{A^G_{<P}}$ can contain at most $n-k$ Dickson invariants. Part (i) then follows from Corollary 3.2. From Corollary 2.5, $\mathrm{grade}(\sqrt{A^G_{<P}}, A^G) = \ell - k$. Since $d_{k+1,n}, \ldots, d_{\ell+1,n}$ is a sequence of $\ell - k + 1$ elements from $\sqrt{A^G_{<P}}$, the sequence can not be A^G–regular. □

Corollary 3.4. *Suppose* $\dim(V^P) = k$. *Then*
(i) $d_{i,n} \in \sqrt{A^G_{<P}}$ *if and only if* $i > k$ *and*
(ii) $\mathrm{grade}(A^G_{<P}, A^G)$ *is the largest* m *such that* $d_{k+1,n}, \ldots, d_{k+m,n}$ *is* A^G*-regular.*

Proof. The first statement follows from Corollary 3.2 and Theorem 3.3(i).

Using Corollary 2.5 and Bourguiba & Zarati/Landweber-Stong, $\mathrm{grade}(A^G_{<P}, A^G)$ is the largest m such that $d_{1,n}, \ldots, d_{k+m,n}$ is A^G-regular. Although in general a permutation of a regular sequence need not be a regular sequence, a regular sequence of homogeneous elements in A^G is still regular after a permutation (See, for example, [4, Proposition 1.1.6]. The proposition is stated for Noetherian local rings but the proof is valid in the current context.). Therefore, if $d_{1,n}, \ldots, d_{k+m,n}$ is A^G-regular, then $d_{k+1,n}, \ldots, d_{k+m,n}$ is A^G-regular. □

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