On pointwise conjugacy of distinguished coset representatives in Coxeter groups

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Let (W, S) be a Coxeter system. For a standard parabolic subgroup $W_K,\ K \subseteq S$ let D_K be the set of *distinguished coset representatives*, i.e. representatives of cosets $W_K w$ of minimal Coxeter length. If $L = K^c \subseteq S$ with $c \in W$, then D_K and $D_L = c^{-1} D_K$ are in general not conjugate as sets. However it is shown that if W_K is finite, they are conjugate 'pointwise', i.e. there is a bijection $\theta :\ D_K \to D_L$ such that $\theta(d) = d^{wc}$ for some $w \in W_K$ depending on $d \in D_K$. In particular for each conjugacy class C of W the cardinalities $\#(D_K \cap C)$ and $\#(D_L \cap C)$ are the same. The case of infinite standard parabolic subgroups is also discussed and a corresponding result is proved.

1. INTRODUCTION

Consider the group $G := \Sigma_n$ of all permutations on the set $\mathbf{n} := \{1, 2, 3, \cdots, n\}$ and let $\Lambda := \{m_1, m_2, \cdots, m_k\}$ be a set-partition of $\mathbf{n}$. Define D_Λ to be the subset of G consisting of all permutations σ, which are isotone on all subsets $m_\ell \in \Lambda$, i.e. satisfying $i^\sigma < j^\sigma$ for each pair $i < j$ with $i, j \in m_\ell \in \Lambda$. Now let $\gamma \in G$ and $\Delta := \Lambda^\gamma$ be the conjugate partition. Then $\Delta = \Lambda^\tau$ with $\tau = \delta\gamma \in D_\Lambda$, where δ is a suitable element in $\prod_{m \in \Lambda} \Sigma_m$ and it is easy to see that $D_\Delta = \tau^{-1} D_\Lambda$. However, the sets D_Δ and D_Λ will in general not be conjugate. It is therefore somewhat surprising that they contain the same number of elements of each given cycle type, provided that both Λ and Δ consist of intervals of $\mathbf{n}$. In other words, if all sets m_i and m_i^τ are of the form

$$\{s, s+1, \cdots, s + |m_i| - 1\},$$

then each conjugacy class C of G intersects D_Λ and D_Δ with the same cardinality. This turns out to be a general behaviour of conjugacy classes in Coxeter groups and distinguished representatives for cosets of standard parabolic subgroups. Let (W, S) be a Coxeter system with distinguished set of generators S. Following [2] or [5] chapter 5 (see also section 4), we will view $W = \langle S \rangle$ as a linear group, acting faithfully on a euclidean $\mathbb{R}$ - vector space $(V, (\ ,\ ))$ with basis Π, such that W is generated by the set of reflections $S := \{s_\alpha \mid \alpha \in \Pi\}$ with

$$v^{s_\alpha} := v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \ \forall\, v \in V.$$

The set $\Phi := \{\alpha^w \mid \alpha \in \Pi,\ w \in W\}$ is then called the *root system* of W. For finite W (and hence finite Φ) the root systems and their combinatorial structure are described in [1]. For infinite W we follow the treatment in [2] or [5], where $\Pi := \{\epsilon_s \mid s \in S\}$ is a set of unit vectors and $(\ ,\ )$ is the bilinear form extending $(\epsilon_s, \epsilon_{s'}) := -\cos\frac{\pi}{|ss'|}$ with $|ss'| \in \mathbb{N} \cup \{\infty\}$ being the order of $ss' \in W$. As in the finite case it can be shown that $\Phi = \Phi^+ \uplus -\Phi^+$ with

$$\Phi^+ := \langle \Pi \rangle_{\mathbb{R}^+} \cap \Phi = \{\sum_{\alpha \in \Pi} c_\alpha \alpha \mid 0 \le c_\alpha \in \mathbb{R}\} \cap \Phi$$

and many other results carry over from the situation of finite root systems (see [2] and [5] for more details).

For each subset $X \subseteq \Phi^+$ we denote by $W_X := \langle s_\alpha \mid \alpha \in X \rangle$ the subgroup generated by the reflections with respect to roots in X and set $D_X := \{w \in W \mid \alpha^w \in \Phi^+,\ \forall \alpha \in X\}$. It is known (see [2] 3.1. or [5] 5.7. pg. 116), that

$$D_X = \{w \in W \mid \ell(s_\alpha w) > \ell(w),\ \forall \alpha \in X\},$$

where $\ell(w)$ denotes the Coxeter length of w, i.e. the length k of a reduced expression $w = s_1 s_2 \cdots s_k$ with $s_i \in S$. Moreover D_X is a cross section of cosets $W_X x$ consisting of elements of minimal length in their coset (see Proposition 3.1). If $X \subseteq \Pi$ then W_X is called a *standard parabolic* subgroup. The following is the main result of this paper:

THEOREM 1.1. *Let $K, L \subseteq \Pi$ and $c \in W$ with $L = K^c$ and assume that the group W_K has finite order. Then D_K and D_L are 'pointwise conjugate', i.e. there is a bijection*

$$\theta :\ D_K \to D_L$$

satisfying $\theta(d) = d^{wc}$ for some $w \in W_K$ which depends on d. In particular for each conjugacy class C of W, restriction of θ induces a bijection between $D_K \cap C$ and $D_L \cap C$.

Note that if W is for example an affine Weyl group, then all proper standard parabolic subgroups are finite, so Theorem 1.1 applies. Note further that the symmetric group Σ_n can be viewed as a finite Coxeter group of type A_{n-1} with root system

$$\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j\}.$$

If we choose $\Pi := \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ with $\alpha_i := \epsilon_i - \epsilon_{i+1}$, then

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\}.$$

Each $K \subseteq \Pi$ gives rise to a unique equivalence relation on $\mathbf{n}$ by

$$i \sim j \iff \epsilon_i - \epsilon_j \in \Phi_K := \langle \alpha \mid \alpha \in K \rangle_{\mathbb{Z}} \cap \Phi.$$

The equivalence - classes of this relation form a partition Λ consisting of intervals and it is easy to see that $D_K = D_\Lambda$. On the other hand each partition of $\mathbf{n}$ consisting of intervals is obtained in this way. Thus the claim stated above, that for each conjugacy class C of Σ_n the cardinalities $\#(D_\Lambda \cap C)$ and $\#(D_\Delta \cap C)$ are equal, is a consequence of Theorem 1.1.

For $x \in W$ and $U, V \subseteq W$ let $x^U := \{x^u \mid u \in U\}$ and $V^U := \cup_{v \in V} v^U$. Theorem 1.1 will follow from the following two results:

PROPOSITION 1.1. *Let (W, S) be an arbitrary Coxeter system and $d \in D_K$, $K \subseteq \Pi$, then*

$$d^{W_K} \cap D_K = \{d\}$$

i.e. if $d, c \in D_K$, $x \in W_K$ with $d^x = c$, then $d = c$.

THEOREM 1.2. *If $K \subseteq \Pi$ with W_K finite, then*

$$(D_K D_K^{-1})^{W_K} = D_K^{W_K};$$

in other words, for $d, c \in D_K$ there is $w \in W_K$ such that $(dc^{-1})^w \in D_K$. Moreover $D_K^{W_K} = (D_K c^{-1})^{W_K}$ for any $c \in W$ with $K^c \subseteq \Pi$.

The paper is organized as follows:

In section 2 we will give examples of the investigated situation in the case of finite Coxeter groups, showing the necessity of the hypotheses made in the statements above. In section 3 we will carry out the proofs of Proposition 1.1 and Theorems 1.2 and 1.1. In section 4 we will discuss versions of Theorem 1.1 and Theorem 1.2 for infinite parabolic subgroups W_K. Therefore we will modify some of the arguments of section 3 to work in this setup.

2. EXAMPLES

Note that $L = K^c$ implies $W_L = (W_K)^c$, but $D_L = c^{-1}D_K$ will in general not be conjugate to D_K:

EXAMPLE 2.1. Let W be of type A_4 with Dynkin Diagram

$$\circ_{\alpha_1} \text{---} \circ_{\alpha_2} \text{---} \circ_{\alpha_3} \text{---} \circ_{\alpha_4},$$

$$K := \{\alpha_1, \alpha_2\} \leftrightarrow \Lambda := 123|4|5, \text{ and}$$

$$L := \{\alpha_2, \alpha_3\} = K^c \leftrightarrow \Delta := 1|234|5$$

with $c := s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1} = (12345)$ and $C := (123)^W$, the class of three - cycles. Then

$$D_K \cap C := \{(345), (354), (235), (234)\} \text{ and}$$

$$D_L \cap C := \{(152), (345), (132), (145)\}.$$

Note that $D_L \cap C$ does not contain a pair of inverses, so $D_K \cap C$ and $D_L \cap C$ are not conjugate.

The next example shows that the statement of Theorem 1.1 is false if applied to non-standard parabolic subgroups.

EXAMPLE 2.2. Let W be as above, $\beta := \epsilon_3 - \epsilon_5 = \alpha_3 + \alpha_4$,

$$K := \{\alpha_1, \alpha_2, \alpha_3\} \leftrightarrow 1234|5,$$

$$L := \{\alpha_1, \alpha_2, \beta\} \leftrightarrow 1235|4$$

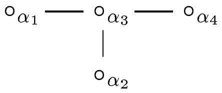
and $W_L := \langle s_\alpha \,|\alpha \in L\rangle = \Sigma_{\{1,2,3,5\}} \le W$. Then $L = K^c$ with $c := s_{\alpha_4} = (45)$ and we get

$$D_K := \{1, (45), (345), (2345), (12345)\},\ D_L := \{1, (34), (45), (234), (1234)\}.$$

In particular $\#(D_K \cap C) = 1 < 2 = \#(D_L \cap C)$, for the class C of transpositions in W.

The following example shows that $\#(D_K \cap C) = \#(D_L \cap C)$ can fail, if K and L are conjugate by a symmetry of the Dynkin diagram, but not under W:

EXAMPLE 2.3. Let W be of type D_4 with Dynkin diagram



Let $K := \{\alpha_1, \alpha_4\}$, $L := \{\alpha_2, \alpha_4\}$. Note that $L = K^\tau$, where τ is an involution in $\mathrm{Aut}(W)$ swapping s_{α_1} and s_{α_2} and fixing $s_{\alpha_3}, s_{\alpha_4}$. Calculations with MAGMA ([6]) show that exactly two of the thirteen conjugacy classes of W satisfy $\#(D_K \cap C) \neq \#(D_L \cap C)$:
($[i_1, i_2, i_3, \cdots]$ will denote the element $s_{\alpha_{i_1}} s_{\alpha_{i_2}} s_{\alpha_{i_3}} \cdots$.)
for $C_1 := w_1^W$ with $w_1 := [3,1,4,3]$ we have $D_K \cap C_1 = \{w_1, w_2, w_3\}$ with $w_2 := [2,3,1,2,3,4,3,1,2,3]$ and $w_3 := [2,3,1,4,3,2]$, whereas $D_L \cap C_1 = \{w_1\}$. Applying τ we get the analogous result for $C_2 := (w_1^\tau)^W = (C_1)^\tau$, i.e. we have $D_K \cap C_2 = (D_L \cap C_1)^\tau = \{w_1^\tau\}$ and $D_L \cap C_2 = \{w_1, w_2, w_3\}^\tau$.

An immediate analogue to Theorem 1.1 for distinguished double coset representatives $D_{KJ} := D_K \cap D_J^{-1}$ fails to be true:

EXAMPLE 2.4. Let $W = A_4 \cong \Sigma_5$, $K := \{\alpha_1\}$, $J := \{\alpha_2\} =: L$, $S := \{\alpha_3\}$. Then $(K^c, J^c) = (L, S)$ for $c := (1234)$. Let $C := (123)^W$ and $C' := (12345)^W$,then
$D_{KJ} \cap C = \{(243), (123), (253), (345), (354), (125), (124)\}$,
$D_{LS} \cap C = \{(235), (134), (154), (135), (234), (152), (132), (354)\}$,
$D_{KJ} \cap C' = \{(12435), (12345), (12453), (12354), (12543), (14253), (14325), (12534)\}$
$D_{LS} \cap C' = \{(15234), (12354), (13524), (13542), (13452), (12345), (15432)\}$.
For the remaining classes C we have $\#(D_{KJ} \cap C) = \#(D_{LS} \cap C)$.

3. THE PROOFS

We use standard notation for Coxeter groups, as for example in [5] and [1]. Unless otherwise specified (W, S) denotes an arbitrary Coxeter system with possibly infinite group W. Note that we are working with cosets $W_K x$ rather than $x W_K$. For $w \in W$ we define $N(w) := \Phi^+ \cap (\Phi^-)^{w^{-1}}$, i.e. the set of all positive roots that are changed into negative ones by w; it is known that $\ell(w) = \#N(w)$.

The following result is well known mainly for parabolic subgroups and finite W. Due to a lack of reference for the general case it will be proved here for arbitrary reflection subgroups of a general Coxeter group W:

PROPOSITION 3.1. *Let $X \subseteq \Phi^+$; then $W = W_X D_X$ and D_X is a cross section of W_X/W such that for each $d \in D_X$:*

$$\ell(d) = \min\{\ell(x) \mid x \in W_X d\}.$$

Proof. The group W_X stabilizes the set $\langle X\rangle_{\mathbb{R}} \cap \Phi$. It is known that W_X is again a Coxeter group with respect to a canonical set of generators Y (see [3], [4]). Moreover the proof in [3] shows that the (positive) root system of W_X is a subset Ψ of $\langle X\rangle_{\mathbb{R}} \cap \Phi^{(+)}$. Define $W^X := \{x \in W \mid \ell(wx) \geq \ell(x)\ \forall w \in W_X\}$, then clearly $W^X \subseteq D_X$. First we show $W = W_X W^X$: let $y \in W \backslash W_X W^X$ of minimal length. Then $0 < \ell(y)$ and $y \notin W^X$, hence there is $1 \neq w \in W_X$ with $\ell(wy) < \ell(y)$. This implies $wy \in W_X W^X$ hence $y \in W_X W^X$, a contradiction. Let $d = wx \in D_X$ with $w \in W_X, x \in W^X$. If $w \neq 1$, there is $\alpha \in \Psi^+ \subseteq \langle X\rangle_{\mathbb{R}^+} \cap \Phi^+$ with $\alpha^w \in \Phi^- \cap \langle X\rangle$. Hence $\alpha^d = \alpha^{wx} \in \Phi^-$, because $x \in D_X$; but this contradicts the fact that $\alpha^d \in \Phi^+$, due to d lying in D_X. ∎

We need some more definitions:

DEFINITION 3.1. Let $U \subseteq V \subseteq \Phi$ be subsets; we call U *closed in* V, if $\alpha, \beta \in U$ and $\alpha + \beta \in V$ imply $\alpha + \beta \in U$. The set U will be called *bi-closed* in V, if U and $V \backslash U$ are closed in V.

We will need the following result on bi - closed subsets, which is stated for finite root systems as exercise 16 on page 225 in [1]:

PROPOSITION 3.2. *The map $w \mapsto N(w)$ from W to the set of all subsets of positive roots which are bi - closed in Φ^+ is injective. It is bijective if Φ is finite (i.e. W is finite).*

Proof. Assume that $N(w) = N(w')$ and $\alpha \in \Phi^+$. If $\alpha \in N({w'}^{-1})$, then $-\alpha^{{w'}^{-1}} \in N(w') = N(w)$ and $\alpha^{{w'}^{-1}w} \in \Phi^+$. If $\alpha \in \Phi^+ \backslash N({w'}^{-1})$, then $\alpha^{{w'}^{-1}w} \in \Phi^+$, for otherwise $\alpha^{{w'}^{-1}} \in N(w) = N(w')$ and $\alpha = \alpha^{{w'}^{-1}w'} \in \Phi^-$. Hence $N({w'}^{-1}w) = \emptyset$ and ${w'}^{-1}w = 1$. This shows that the mapping $\psi:\ w \mapsto N(w)$ is injective. Now assume that Φ is finite and let $U \subseteq \Phi^+$ be bi - closed in Φ^+, then $T := \Phi^+ \backslash U \uplus (-U)$ is easily seen to be closed in Φ with $T \uplus -T = \Phi$. It follows from [1] Corollaire 1, page 161, that $T := (\Phi^+)^{w^{-1}}$ for some uniquely determined $w \in W$, whence $U = \Phi^+ \cap (\Phi^-)^{w^{-1}} = N(w)$, which defines the inverse map $U \mapsto w$. (Note that in [1] Corollaire 1, page 161 Φ is assumed to be finite.) ∎

Proof of Proposition 1.1: First we show that d stabilizes the set $N(x) \subseteq \Phi_K^+$. So let $\alpha \in N(x)$. Then $\alpha^x \in \Phi_K^-$ and $\alpha^d \in \Phi^+$, so $\alpha^{dx} = \alpha^{xc} \in \Phi^- \cap (\Phi^+)^x$. Hence $\alpha^d \in \Phi^+ \cap (\Phi^-)^{x^{-1}} = N(x)$. Now let $x = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}$ be a reduced expression with $\alpha_i \in K$. Then, as is well known,

$$N(x) = \{\alpha_1, \alpha_2^{s_{\alpha_1}}, \alpha_3^{s_{\alpha_2} s_{\alpha_1}}, \cdots, \alpha_\ell^{s_{\alpha_{\ell-1}} \cdots s_{\alpha_1}}\}.$$

Hence $x \in \langle s_{\alpha_1}, \cdots, s_{\alpha_\ell}\rangle = \langle s_\beta \mid \beta \in N(x)\rangle =: H \leq W_K$. It follows that $y := {}^d x = x^{d^{-1}} \in H^{d^{-1}} = H \leq W_K$.
Now we show that $N(y) = N(x)$: so let $\alpha \in N(y)$; then, as $y \in W_K$, we know that $\alpha \in \Phi_K^+$ with $\alpha^y \in \Phi_K^-$. It follows that $\alpha^{dx} = \alpha^{yd} \in (\Phi_K^-)^d \subseteq \Phi^-$ with $\alpha^d \in \Phi^+$, so $\alpha^d \in N(x)$, hence $\alpha \in N(x)$.
Now let $\alpha \in N(x) \subseteq \Phi_K$; then so is α^d and we get $\alpha^{dx} \in \Phi_K^-$. If $\alpha^{dxd^{-1}}$ were positive, then it would lie in Φ_K^+, because $dxd^{-1} \in W_K$. We would then arrive at the contradiction $\alpha^{dxd^{-1}} \in \Phi_K^+ \cap (\Phi^-)^{d^{-1}} = \Phi_K^+ \cap N(d) = \emptyset$. Hence $\alpha \in \Phi^+ \cap (\Phi^-)^{y^{-1}} = N(y)$. Now Proposition 3.2 yields $x = y$ and therefore $d = c$.

Remark 3. 1. The statement of Proposition 1.1 is false if $K \not\subseteq \Pi$: in Example 2.2 we have $(34)^x = (45) \in D_L^x \cap D_L$ with $x = (35) \in W_L$.

DEFINITION 3.2. Let $K \subseteq \Pi$ and $c \in W$ with $|c| \in \mathbb{N} \cup \{\infty\}$ being the order of c and let $U \subseteq \Phi_K^+$. We define

$$\bar{U} := \Phi_K^+ \backslash U;$$

$$X := X(c) := \Phi_K^+ \cap (\Phi_K^+)^c = \{\alpha \in \Phi_K^+ \mid \alpha^{c^{-1}} \in \Phi_K^+\};$$

$$Y := Y(c) := \Phi_K^+ \cap (\Phi^-)^c = \Phi_K^+ \cap N(c^{-1}) \subseteq \overline{X(c)};$$

$$\mathcal{X}(c) := \cup_{i=1}^{\infty} (X \cap X^c \cap \cdots \cap X^{c^{i-1}} \cap Y^{c^i}) =$$

$$\{\alpha \in X \mid \exists\, 1 \leq s \text{ with } \alpha^{c^{-i}} \in X \text{ for } 0 \leq i < s \text{ and } \alpha^{c^{-s}} \in Y\},$$

$$X^\infty := X(c)^\infty := X \cap X^c \cap \cdots \cap X^{c^{|c|-1}}.$$

Note that $\mathcal{X}(c), Y(c)$ and $X(c)^\infty$ are pairwise disjoint subsets of Φ_K^+.

Remark 3. 2. For $i \geq |c| - 1 < \infty$ we have

$$(X \cap X^c \cap \cdots X^{c^{(i-1)}} \cap Y^{c^i}) = (\Phi_K^+)^{c^i} \cap (\Phi^-)^{c^{i+1}} \cap (\Phi_K^+)^{c^{i-1}} \cap \cdots \cap \Phi_K^+,$$

which is contained in $(\Phi^-)^{c^k} \cap (\Phi_K^+)^{c^k} = \emptyset$, whenever $i + 1 = |c| + k \geq |c|$. Hence

$$\mathcal{X}(c) = \cup_{i=1}^{|c|-2} (X \cap X^c \cap \cdots \cap X^{c^{i-1}} \cap Y^{c^i}) =$$

$$\{\alpha \in X \mid \exists\, 1 \leq s \leq |c| - 2 \text{ with } \alpha^{c^{-i}} \in X \text{ for } 0 \leq i < s \text{ and } \alpha^{c^{-s}} \in Y\},$$

LEMMA 3.1. *Let $d_1, d_2 \in D_K$ with $K \subseteq \Pi$ and $w \in W_K$. Set $c := d_2 d_1^{-1}$ and $A := N(w^{-1})$. Then $d_1 d_2^{-1} \in D_K^w$ if and only if*

$$A = Y(c) \uplus \mathcal{X}(c) \uplus U$$

with $U^c = U \subseteq X(c)^\infty$.

Proof. Clearly $d_1 d_2^{-1} \in D_K^w$ if and only $(\Phi_K^+)^{wd_1} \subseteq (\Phi^+)^{wd_2}$. Since $N(w) = \Phi_K^+ \cap N(w)$, this is equivalent to

(i) $N(w)^{wd_1} \subseteq (\Phi^+)^{wd_2}$ and
(ii) $(\Phi_K^+ \backslash N(w))^{wd_1} \subseteq (\Phi^+)^{wd_2}$.

Note that the following hold:

$$N(w)^w = -N(w^{-1}), (\Phi^+ \backslash \Phi_K)^w = \Phi^+ \backslash \Phi_K,$$

$$N(w)^{wd_2} = (\Phi_K^+ \cap N(w))^{wd_2} \subseteq (\Phi_K^-)^{d_2} \subseteq \Phi^-,$$

$$(\Phi_K^+ \backslash N(w))^{wd_2} \subseteq (\Phi_K^+)^{d_2} \subseteq \Phi^+, \text{ and } (\Phi^+ \backslash \Phi_K^+)^{wd_2} = (\Phi^+ \backslash \Phi_K^+)^{d_2}.$$

We see that (i) is equivalent to

$$-N(w^{-1})^{d_1} \subseteq -N(w^{-1})^{d_2} \uplus -(\Phi^+ \cap (\Phi^- \backslash \Phi_K)^{d_2}).$$

From $\Phi^+ \cap (\Phi_K^-)^{d_2} = \emptyset$ we get $\Phi^+ \cap (\Phi^- \backslash \Phi_K)^{d_2} = N(d_2^{-1})$, so (i) is equivalent to

$$N(w^{-1})^{d_1} \subseteq N(w^{-1})^{d_2} \uplus N(d_2^{-1}).$$

From $(\Phi_K^+ \backslash N(w))^w = (\Phi_K^+ \cap (\Phi_K^+)^{w^{-1}})^w = \Phi_K^+ \backslash N(w^{-1})$ we see that (ii) is equivalent to

$$(\Phi_K^+ \backslash N(w^{-1}))^{d_1} \subseteq (\Phi_K^+ \backslash N(w^{-1}))^{d_2} \uplus \Phi^+ \cap (\Phi^+ \backslash \Phi_K^+)^{d_2}.$$

Hence (i) and (ii) are equivalent to

(i)

$$A \subseteq A^c \uplus (\Phi_K^+ \cap (N(d_2^{-1}))^{d_1^{-1}}) = A^c \uplus Y;$$

(ii)

$$\bar{A} \subseteq (\bar{A})^c \uplus (\bar{Y} \cap \bar{X}),$$

the latter, because $\Phi_K^+ \cap (\Phi^+)^{d_1^{-1}} \cap (\Phi^+ \backslash \Phi_K)^c = \Phi_K^+ \cap (\Phi^+ \backslash \Phi_K^+)^c$. Taking complements in Φ_K^+, (ii) is equivalent to

$$A \supseteq [\Phi_K^+ \backslash (\Phi_K^+ \backslash A)^c] \cap (Y \uplus X) =$$

$$[(\Phi_K^+ \backslash (\Phi_K^+)^c) \cup (\Phi_K^+ \cap A^c)] \cap (Y \uplus X) =$$

$$(\bar{X} \cup A^c) \cap (Y \uplus X) = Y \uplus (A^c \cap X),$$

because $A^c \cap Y \subseteq (\Phi^+)^c \cap (\Phi^-)^c = \emptyset$. Since $A \cap A^c \subseteq X$, we conclude:

$$\text{(i)} + \text{(ii)} \iff Y \uplus (X \cap A^c) \subseteq A \subseteq A^c \uplus Y \iff A = Y \uplus (X \cap A^c).$$

Assume that $A = Y \uplus (X \cap A^c)$; then iteration yields

$$A \cap X = A^c \cap X = (Y \uplus (X \cap A^c))^c \cap X = (Y^c \cap X) \uplus (X \cap X^c \cap A^{c^2}) =$$

$$(Y^{c^2} \cap X^c \cap X) \uplus (Y^c \cap X) \uplus (X \cap X^c \cap X^{c^2} \cap A^{c^3}) = \cdots =$$

$$\mathcal{X}(c) \uplus (A \cap X(c)^\infty).$$

Indeed, $\mathcal{X}(c) = (X \cap Y^c) \uplus (X \cap \mathcal{X}(c)^c)$, so for $\alpha \in (A \cap X) \backslash \mathcal{X}(c)$, $\alpha^{c^{-1}} \in (A \backslash Y) \backslash \mathcal{X}(c) = (A \cap X) \backslash \mathcal{X}(c) \Rightarrow \cdots \Rightarrow \alpha^{c^{-i}} \in A \cap X$, $\forall i \in \mathbb{N}$ and $\alpha \in A \cap X^\infty$. Hence $(A \cap X) \backslash \mathcal{X}(c) = A \cap X^\infty$. Moreover

$$(A \cap X^\infty)^c \cap X = A^c \cap X \cap (X^\infty)^c = A \cap X \cap (X^\infty)^c = A \cap X^\infty \subseteq (A \cap X^\infty)^c.$$

Since $A \cap X^\infty$ is a finite set, we conclude $U := A \cap X^\infty = (A \cap X^\infty)^c = U^c$. On the other hand, if $A = Y \uplus \mathcal{X}(c) \uplus U$ with c - stable subset $U \subseteq X^\infty$, then

$$Y \uplus (X \cap A^c) = Y \uplus (X \cap (Y^c \uplus \mathcal{X}(c)^c \uplus U^c)) = Y \uplus \mathcal{X}(c) \uplus U = A,$$

and (i) + (ii) hold. ▌

Remark 3. 3. For finite W let w_0 denote the longest element in W and w_K the longest element in W_K. Then $d_2 := w_K \cdot w_0$ satisfies $K^{d_2} = (-K)^{w_0} =: L \subseteq \Pi$ and

$$D_K^{w_0} = D_L.$$

Indeed, for $\beta \in L$ we have $\beta = \alpha^{d_2}$ with $\alpha \in K$ and for each $d \in D_K$:

$$\beta^{w_0 d w_0} = \alpha^{w_K d w_0} \in (-K)^{d w_0} \subseteq (\Phi^-)^{w_0} \subseteq \Phi^+.$$

This can also be derived from Lemma 3.1:
Note that $N(w_0 w_K) = \Phi^+ \backslash \Phi_L$ and for $c := w_K w_0 d_1^{-1}$,

$$Y := Y(c) = \{\alpha \in \Phi_K^+ \mid \alpha^{d_1} \in N(w_0 w_K)\} = \Phi_K^+ \cap (\Phi^+ \backslash \Phi_L)^{d_1^{-1}}.$$

Hence $X = X(c) = \Phi_K^+ \cap (\Phi_K^+)^{w_K w_0 d_1^{-1}} = \Phi_K^+ \cap (\Phi_L^+)^{d_1^{-1}} = \bar{Y}$. Each $\beta \in \bar{Y} \cap \bar{\mathcal{X}}$ satisfies $\beta^{c^{-i}} \in X$ for $i = 1, 2, \cdots$, so

$$\Phi_K^+ = Y \uplus \mathcal{X}(c) \uplus X^\infty = N(w_K)$$

and we conclude that $d_1 d_2^{-1} \in D_K^{w_K}$ for each $d_1 \in D_K$. So $D_K \subseteq D_K^{w_K} d_2$, hence $D_L = d_2^{-1} D_K \subseteq D_K^{w_K d_2} = D_K^{w_0}$.

Lemma 3.2. *Let $c := d_2 d_1^{-1}$ with $d_1, d_2 \in D_K$ and $K \subseteq \Pi$. Let $Y := Y(c)$ and $\mathcal{X} := \mathcal{X}(c)$. Then the set $Y \uplus \mathcal{X}$ is bi- closed in Φ_K^+.*

Proof. First we show

$$(Y + X) \cap \Phi_K^+ \subseteq Y. \tag{$*$}$$

So let $\alpha \in Y, \beta \in X, \gamma := \alpha + \beta \in \Phi_K^+$ and let $x := c^{-1}$. Then $\alpha^x = \alpha^{d_1 d_2^{-1}} \in \Phi^-$ and $\alpha^{d_1} = (\alpha^x)^{d_2} \in \Phi^+$ implies that $\alpha^x \in \Phi^- \backslash \Phi_K$, whereas $\beta^x \in \Phi_K^+$. Hence $\gamma^x = \alpha^x + \beta^x \in \Phi^-$ and $\gamma \in Y$. Obviously Y and X are closed in Φ_K^+. To show that $\mathcal{S} := Y \uplus \mathcal{X}$ is closed, let $\alpha, \beta \in \mathcal{S}$ with $\gamma := \alpha + \beta \in \Phi_K^+$. If one of the two summands lies in Y, then by the argument above, $\gamma \in Y \subseteq \mathcal{S}$ and we are done. So we can assume that $\alpha, \beta \in \mathcal{X}$. In particular for some $s \in \mathbb{N}_0$ we have $\alpha^{x^s} \in Y$ with $\alpha^{x^i} \in X$ for all $0 \leq i < s$. By choosing α and β appropriately, we can assume that $\beta^{x^i} \in X$ for all $i < s$ and $\beta^{x^s} \in Y \uplus X$. Using $(*)$ we conclude that $\gamma^{x^s} \in Y$ and $\gamma^{x^i} \in X$ for all $i < s$, whence $\gamma \in \mathcal{X} \subseteq \mathcal{S}$.
To show that $\mathcal{S}$ is bi - closed, let $\alpha, \beta \in \overline{\mathcal{X}} \cap \bar{Y}$ with $\gamma = \alpha + \beta \in \Phi_K^+$. Then $\gamma^x = \alpha^x + \beta^x \in \Phi^+$, so $\gamma \notin Y$. Assume $\gamma \in \mathcal{X}$. Then there is s such that $\gamma^{x^s} \in \Phi^-$ with $\gamma^{x^i} \in \Phi_K^+$ for $0 \leq i < s$. In particular $\gamma^x \in \Phi_K^+$ and $\alpha^x, \beta^x \in \Phi^+$ imply $\alpha^x, \beta^x \in \Phi_K^+$. If $\alpha^x \in Y$, then $\alpha \in Y^c \cap \Phi_K^+ = Y^c \cap X \subseteq \mathcal{X}$; if $\alpha^x \in \mathcal{X}$, then $\alpha \in \mathcal{X}^c \cap \Phi_K^+ \subseteq \mathcal{X}$. Both give a contradiction and the same argument for β shows that $\alpha^x, \beta^x \in \overline{\mathcal{X}} \cap \bar{Y}$. Now we can assume inductively that $\alpha^{x^{s-1}}, \beta^{x^{s-1}} \in \overline{\mathcal{X}} \cap \bar{Y}$ and obtain the final contradiction

$$\gamma^{x^s} = \alpha^{x^s} + \beta^{x^s} \in \Phi^+.$$

∎

Now we can prove the main results of section 1:

Proof of Theorem 1.2: Since W_K is finite, so are the sets $Y \uplus \mathcal{X}$ appearing in Lemma 3.2. Now the first statement follows immediately from Lemmas 3.1, 3.2 and Proposition 3.2. Let $c \in W$ be such that $L := K^c \subseteq \Pi$. Then $c^{-1} \in D_L$ and the first statement, applied to L yields $c^{-1}D_K c = D_L c \subseteq D_L^{W_L}$. Hence $D_K \subseteq D_L^{W_L c^{-1}} =$

$$D_L^{c^{-1}W_K} = (D_K c^{-1})^{W_K} \subseteq D_K^{W_K}.$$

COROLLARY 3.1. *Let $X = K^c \subseteq \Phi^+$ with $K \subseteq \Pi$; then $D_X^{W_X} \subseteq D_K^{W_K c}$ with equality, if $X \subseteq \Pi$.*

Proof. Since $c \in D_K$ and $cD_X = D_K$ we get $D_K c^{-1} \subseteq D_K^{W_K}$, hence $D_X \subseteq D_K^{W_K c}$. From $W_X = W_K^c$ we conclude

$$D_X^{W_X} \subseteq D_K^{W_K c W_X} = D_K^{W_K c}.$$

If $X \subseteq \Pi$, then the same argument gives

$$D_K^{W_K} \subseteq D_X^{W_X c^{-1}} \subseteq D_K^{W_K c c^{-1}} = D_K^{W_K}.$$

▮

Proof of Theorem 1.1: Let $d \in D_K$ and $K^c = L \in \Pi$. Then by Theorem 1.2 there exists $w^{-1} \in W_K$ and $d' \in D_K$ with $d = (d'c^{-1})^{w^{-1}}$, so $d^w c \in D_K$. Hence $d^{wc} \in c^{-1}D_K = D_L$. For any other $w' \in W_K$ with $d^{w'c} \in D_L$ we have

$$d^{w'c} = (d^{wc})^{c^{-1}w^{-1}w'c} = (d^{wc})^x$$

with $x := (w^{-1}w')^c \in W_L$. Hence $d^{w'c} = d^{wc}$ by Proposition 1.1 and $d \mapsto d^{wc}$ is a well defined mapping θ from D_K to D_L. In the same way we can define a mapping $\psi :\ D_L \to D_K,\ d' \mapsto d'^{w'c^{-1}}$ with suitable $w' \in W_L$. Now

$$\psi\theta(d) = \psi(d^{wc}) = d^{wcw'c^{-1}} = d^{w(w'^{c^{-1}})} \in d^{W_K} \cap D_K = \{d\},$$

so $\psi \circ \theta = id_{D_K}$. Similarly $\theta \circ \psi = id_{D_L}$. Clearly θ and ψ preserve intersections with conjugacy classes.

4. INFINITE STANDARD PARABOLIC SUBGROUPS

In this section we discuss analogues of Theorems 1.1 and 1.2 for infinite Coxeter groups and infinite parabolic subgroups W_K. Infinite root systems require some extra care: for example the mapping $w \mapsto N(w)$ from W to the collection of bi - closed subsets of Φ^+ is not surjective, because clearly the infinite set Φ^+ is bi - closed in itself, whereas each $N(w)$ is finite. In other words, there is no 'longest element' in W changing signs on all roots. Therefore we have to use a slightly modified definition of 'closed' and 'bi - closed' sets, which also takes into account the fact that positive roots are linear combinations of the basic roots with coefficients in $\mathbb{R}$ rather than $\mathbb{Z}$.

For any subset $M \subseteq \Phi^+$ let

$$\langle M\rangle_{\mathbb{R}^+} := \{\sum_{m\in M} c_m m \mid 0 \le c_m \in \mathbb{R},\ c_m = 0 \text{ for almost all } m \in M\}.$$

DEFINITION 4.1. Let $U \subseteq \Phi^+$ be a subset; we call U *admissible* if

$$\langle U\rangle_{\mathbb{R}^+} \cap \langle \Phi^+ \backslash U\rangle_{\mathbb{R}^+} = 0.$$

Remark 4. 1.

1. If U is admissible, then it is bi - closed in Φ^+.
2. For each element $w \in W$, the set $N(w)$ is admissible. Indeed let

$$v = \sum_{\alpha\in N(w),\ c_\alpha\ge 0} c_\alpha \alpha = \sum_{\beta\in\Phi^+\backslash N(w),\ b_\beta\ge 0} b_\beta \beta. \text{ Then}$$

$$v^w \in \langle \Phi^-\rangle_{\mathbb{R}^+} \cap \langle \Phi^+\rangle_{\mathbb{R}^+} = 0.$$

PROPOSITION 4.1. *Let (W,S) be an arbitrary Coxeter system with root system Φ. Then the map $w \mapsto N(w)$ is a bijection between W and the set of all finite, admissible subsets of Φ^+.*

Proof. Due to Proposition 3.2 it suffices to prove that every finite admissible set $U \subseteq \Phi^+$ appears as some $N(w)$. We argue by induction on $|U|$, the claim being obvious for $|U| = 0$. So we can assume that there is $\beta = \sum_{\alpha\in\Pi} c_\alpha \alpha \in U$; since $\beta \ne 0$, there is at least one $\alpha \in U \cap \Pi$ with $c_\alpha > 0$. Now let

$$U' := (U\backslash\{\alpha\})^{s_\alpha} = U^{s_\alpha}\backslash\{-\alpha\} = U^{s_\alpha} \cap \Phi^+ \not\ni \alpha.$$

The set U' is admissible: indeed, let

$$v = \sum_{\substack{u \in U \setminus \{\alpha\} \\ c_u \geq 0}} c_u u^{s_\alpha} = \sum_{\substack{y \in (\Phi^+ \setminus U) \cup \{-\alpha\} \\ \delta_y \geq 0}} \delta_y y^{s_\alpha}$$

(notice that $\Phi^+ \setminus U' = ((\Phi^+ \setminus U) \uplus \{-\alpha\})^{s_\alpha})$. Then $v^{s_\alpha} = \sum_{\substack{u \in U \setminus \{\alpha\} \\ c_u \geq 0}} c_u u = \sum_{y \in \Phi^+ \setminus U} \delta_y y - \delta_\alpha \alpha$ with $\delta_\alpha \geq 0$. Hence

$$\sum_{\substack{u \in U \setminus \{\alpha\} \\ c_u \geq 0}} c_u u + \delta_\alpha \alpha = \sum_{\substack{y \in \Phi^+ \setminus U \\ \delta_y \geq 0}} \delta_y y \in \langle U \rangle_{\mathbb{R}^+} \cap \langle \Phi^+ \setminus U \rangle_{\mathbb{R}^+} = 0.$$

Since $U \subseteq \Phi^+$, $\alpha \in \Pi$, c_u, $\delta_\alpha \geq 0$, we conclude that $c_u = \delta_\alpha = 0 = \delta_y$ for all $u \in U$ and all y's and therefore $v = 0$.
By induction on $|U|$ there is $\tilde{w} \in W$ with $\alpha \notin U' = N(\tilde{w})$, which implies $\alpha^{\tilde{w}} \in \Phi^+$ and therefore $\ell(s_\alpha \tilde{w}) > \ell(\tilde{w})$. We conclude that

$$N(s_\alpha \tilde{w}) = \{\alpha\} \uplus N(\tilde{w})^{s_\alpha} = \{\alpha\} \uplus (U \setminus \{\alpha\}) = U.$$

∎

LEMMA 4.1. *Let $Y, \mathcal{X}$ be as in Lemma 3.2. Then the set $Y \uplus \mathcal{X}$ is admissible in Φ_K^+.*

Proof. The proof of Lemma 3.2 can be modified appropriately to cover this case. Using the notation there, we first show that if $v = \mathfrak{y} + \mathfrak{x} = \mathfrak{z}$ with $\mathfrak{y} \in \langle Y \rangle_{\mathbb{R}^+}$, $\mathfrak{x} \in \langle X \rangle_{\mathbb{R}^+}$ and $\mathfrak{z} \in \langle \bar{Y} \rangle_{\mathbb{R}^+}$, then $\mathfrak{y} = 0$: Indeed, let

$$v = \sum_{y \in Y,\ c_y \geq 0} c_y y + \sum_{\mu \in X,\ c_\mu \geq 0} c_\mu \mu = \sum_{\beta \in \Phi_K^+ \setminus Y,\ c_\beta \geq 0} c_\beta \beta.$$

Then for all $c_\beta > 0$, $\beta^x \in \Phi^+$, the y^x lie in $\Phi^- \setminus \Phi_K$ and all μ^x with $c_\mu > 0$ lie in Φ_K^+. Hence for each appearing y we have $y^x = \sum_{\alpha \in \Pi} d_{\alpha,y} \alpha$ with $d_{\alpha,y} \leq 0$ and $d_{\alpha_y,y} < 0$ for some $\alpha_y \in \Pi \setminus K$. Consider

$$\sum_{\mu \in X,\ c_\mu \geq 0} c_\mu \mu^x = \sum_{y \in Y,\ c_y \geq 0} -c_y y^x + \sum_{\beta \in \bar{Y},\ c_\beta \geq 0} c_\beta \beta^x.$$

Since $\mu^x \in \Phi_K^+$ and the β^x are positive roots, a comparison of coefficients at each $\alpha_0 \in \Pi \setminus K$ gives

$$0 = \sum_{y, \alpha_y = \alpha_0} -c_y d_{\alpha_y,y} + \sum_{\beta \in \bar{Y}} c_\beta \cdot \mathrm{coeff}_{\alpha_0}(\beta^x).$$

All summands involved are non- negative, so they must all be zero and in particular $c_y = 0$ for all $y \in Y$.

Hence

$$\langle Y \uplus \mathcal{X}\rangle_{\mathbb{R}^+} \cap \langle \bar{Y} \cap \overline{\mathcal{X}}\rangle_{\mathbb{R}^+} \subseteq \langle \mathcal{X}\rangle_{\mathbb{R}^+} \cap \langle \bar{Y} \cap \overline{\mathcal{X}}\rangle_{\mathbb{R}^+}.$$

So let $0 \neq v = \sum_{\mu\in\mathcal{X},\ c_\mu\geq 0} c_\mu\mu = \sum_{\beta\in\bar{Y}\cap\overline{\mathcal{X}},\ c_\beta\geq 0} c_\beta\beta$ belong to the right hand side; to finish the proof, we will derive a contradiction.

For each $\mu \in \mathcal{X}$ let $s_\mu \in \mathbb{N}$ be such that $\mu^{x^{s_\mu}} \in Y$ and $\mu^{x^k} \in X$ for all $0 \leq k < s_\mu$, and define $s := \min\{s_\mu \mid c_\mu > 0\}$. Then it follows that $v^{x^i} \in \langle X\rangle_{\mathbb{R}^+}$ for all $i \leq s-1$, and that $v^{x^s} = \mathfrak{y} + \mathfrak{x}$ with $0 \neq \mathfrak{y} \in \langle Y\rangle_{\mathbb{R}^+}$ and $\mathfrak{x} \in \langle X\rangle_{\mathbb{R}^+}$. Now we will show that v^{x^s} lies in $\langle \bar{Y}\rangle_{\mathbb{R}^+}$, which then contradicts the statement in the beginning of the proof.

For this purpose we first show by induction that $\beta^{x^i} \in X$ for all β with $c_\beta > 0$ and all $0 \leq i \leq s-1$:

If $i = 0$, then $\beta \in \Phi_K^+,\ \beta^x \in \Phi^+$ and all $\mu^x \in \Phi_K^+$; hence $\beta \in X$.

For the induction step $i-1 \mapsto i$ assume that $\beta, \beta^x, \cdots, \beta^{x^{i-1}} \in X$. Then $\beta^{x^i} \in \Phi_K^+$. If $\beta^{x^{i+1}} \in \Phi^-$, then $\beta^{x^i} \in Y$, $\beta \in \mathcal{X}$ and $c_\beta = 0$. Hence $\beta^{x^{i+1}} \in \Phi^+$, but since $v^{x^i} \in \langle \Phi_K^+\rangle_{\mathbb{R}^+}$, we have $\beta^{x^{i+1}} \in \Phi_K^+$ and $\beta^{x^i} \in X$.

But for all $c_\beta > 0$ we also have $\beta^{x^s} = (\beta^{x^{s-1}})^x \in X^x \subseteq \Phi_K^+$, hence $\beta^{x^s} \in \bar{Y}$, because $\beta \notin \mathcal{X}$. We therefore conclude that $v^{x^s} \in \langle \overline{\bar{Y}}\rangle_{\mathbb{R}^+}$, which finishes the proof. ∎

COROLLARY 4.1. *Let (W, S) be an arbitrary Coxeter group, $K \subseteq \Pi$, d_1, $d_2 \in D_K$ and $x := d_1 d_2^{-1}$. Then $x \in D_K^{W_K}$ if and only if the set $\mathcal{K}(x) := \mathcal{X}(x^{-1}) =$*

$$\{\alpha \in \Phi_K^+ \mid \exists\, 1 < s \in \mathbb{N},\ \alpha^{x^s} \in \Phi^-,\ \alpha^{x^i} \in \Phi_K^+\ \forall 0 \leq i < s\} \subseteq \cup_{1<s\in\mathbb{N}} N(x^s)$$

is finite.

Proof. The set $Y(c)$ appearing in 3.1 is finite, therefore $Y(c) \uplus \mathcal{X}(c)$ is finite (and admissible by Lemma 4.1) if and only if $\mathcal{K}(x)$ is finite. Now the statement follows from Lemma 3.1. ∎

We can now formulate the analogues of Theorems 1.2 and 1.1:

THEOREM 4.1. *Let (W, S) be an arbitrary Coxeter system and let W^{fin} denote the subset of W consisting of the elements of finite order.*

If $K \subseteq \Pi$, then

$$(D_K D_K^{-1})^{W_K} \cap W^{fin} = D_K^{W_K} \cap W^{fin};$$

moreover $D_K^{W_K} \cap W^{fin} = (D_K c^{-1})^{W_K} \cap W^{fin}$ for any $c \in W$ with $K^c \subseteq \Pi$.

Proof. For each $x = d_1 d_2^{-1} \in D_K D_K^{-1}$ of finite order only finitely many different (finite) sets $N(x^s)$ exist. Therefore the set $\mathcal{K}(x)$ appearing in Corollary 4.1 is finite and $x \in D_K^{W_K}$. Since W^{fin} is stable under conjugation, the remaining part of the proof of Theorem 1.2 now carries over to the current situation. ▌

Using the second statement of Theorem 4.1 and the proof of Theorem 1.1 we get:

THEOREM 4.2. *Let (W, S) be an arbitrary Coxeter system. Let $K, L \subseteq \Pi$ and $c \in W$ with $L = K^c$. Then the sets $D_K \cap W^{fin}$ and $D_L \cap W^{fin}$ are pointwise conjugate, i.e. there is a bijection*

$$\theta : \; D_K \cap W^{fin} \to D_L \cap W^{fin}$$

satisfying $\theta(d) = d^{wc}$ for some $w \in W_K$ depending on d. In particular for each conjugacy class C of W, θ induces a bijection

$$D_K \cap C \cap W^{fin} \cong D_L \cap C \cap W^{fin}.$$

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