UNIVERSITY OF KENT

Institute of Mathematics, Statistics and Actuarial Science

Module MA304

DISCRETE MATHEMATICS AND PROBABILITY

107 Exercises in Probability Theory

1. Suppose that the sample space S consists of the positive integers from 1 to 10 inclusive. Let $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$, $C = \{5, 6, 7\}$. List the members of the following sets:

   (a) $\overline{A} \cap B$ (b) $\overline{A} \cup B$ (c) $\overline{\overline{A} \cap \overline{B}}$ (d) $\overline{A \cap (\overline{B \cup C})}$ (e) $\overline{A \cap (B \cup C)}$

2. Which of the following relationships are true in general, i.e. for any three events A, B and C?

   (a) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$;
   (b) $A \cup B = (A \cap \overline{B}) \cup B$;
   (c) $\overline{A} \cap B = A \cup B$;
   (d) $(\overline{A \cup B}) \cap C = \overline{A} \cap \overline{B} \cap \overline{C}$;
   (e) $(A \cap B) \cap (\overline{B} \cap C) = 0$.

3. Let A, B and C be three events associated with a random experiment. Express the following verbal statements in set notation:

   (a) at least one of the events occurs;

   (b) exactly one of the events occurs;

   (c) exactly two of the events occur;

   (d) not more than two of the events occur simultaneously.

4. Suppose that the sample space consists of all points (x, y) both of whose co-ordinates are integers and which lie inside or on the boundary of the square bounded by the lines $x = 0$, $y = 0$, $x = 6$ and $y = 6$. List the members of the following sets:

   (a) $A = \{(x, y)|x^2 + y^2 \leq 6\}$;

   (b) $B = \{(x, y)|y \leq x^2\}$;

   (c) $C = \{(x, y)|x \leq y^2\}$;

   (d) $B \cap C$;

   (e) $(B \cup A) \cap \overline{C}$.

5. An installation consists of two boilers and one engine. Let A be the event that the engine is in good condition, B_i the event that the i^{th} boiler $(i = 1, 2)$ is in good condition, and C the event that the installation can operate. If the installation requires that at least one boiler and the engine are in good condition before it can operate, express C and $\overline{C}$ in terms of A, B_1 and B_2.

6. The *difference* $A - B$ between two events A and B is defined as an event which occurs if A occurs but B does not occur; the *symmetric difference* $A \Delta B$ is realised when A occurs or B occurs but not both. Use Venn diagrams to illustrate the events $A - B$ and $A \Delta B$, and obtain formal expressions for them in terms of the union, intersection and complements of A and B.

7. Prove that $(\overline{A \cup B}) \cap C = (\overline{A} \cap C) \cup (\overline{B} \cap C)$ holds if and only if $A \cap C = B \cap C$.

8. An experiment $\mathcal{E}$, with which an event A is associated, is performed repeatedly; denote its n^{th} performance by $\mathcal{E}_n$. Let A_n be the event that A occurs on $\mathcal{E}_n$, and let $B_{n,m}$ be the event that A occurs exactly m times in the first n repetitions of $\mathcal{E}$.

   (a) Express the event $B_{4,2}$ in terms of the A_i.

   (b) Give a simple verbal interpretation of the event $B_m = \bigcup_{n=m}^{\infty} \left\{ \bigcap_{k=n}^{\infty} B_{k,m} \right\}$.

9. Two six-sided dice are thrown and the results recorded. On a suitable diagram representing the sample space, identify the following events:

   (a) at least one result is a six;
   (b) the results total at least nine;
   (c) both results are the same;
   (d) one result is twice the other.

10. If in question 9 the dice are unbiased, find the probabilities of the four events.

11. Two letters are randomly chosen, one after another, from the work *tack*. List the sample space.

12. Suppose you plan to make a survey of families having two children. You want to record the sex of each child, in the order of their births. For example, if the first child is a boy and the second a girl, you record (boy, girl). This is one point in the sample space. List all the sample points.

13. A letter is chosen at random from the word *ground*. Which of the following sets are acceptable as sample spaces for the experiment and which are not?
   (a) $\{g, r, o, u, n, d\}$;
   (b) $\{\text{vowel}, g, r, n, d, \}$;
   (c) $\{r, o, u, n, d\}$;
   (d) $\{\text{vowel, consonant}\}$;
   (e) $\{\text{consonant}, u\}$.

14. (a) For general events A, B and C associated with some experiment, show using a Venn diagram that

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(B \cap C) - \Pr(C \cap A) - \Pr(A \cap B) + \Pr(A \cap B \cap C) .$$

   (b) An experiment consists of selecting one of the integers $1, 2, \ldots, 10$ at random (i.e. all outcomes having equal probability). Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5, 6\}$. Select a subset of outcomes to be an event C and verify the result in (a) for your selection.

[Note. This procedure cannot *prove* that the result in (a) is true, but if the result were not to hold for your example then it could not be true in general.]

15. Not all of the following statements are true. Identify and prove those that are true, and devise examples to demonstrate that the others are false:

(a) If A, B are events such that $A \subset B$, then $\Pr(A) \leq \Pr(B)$;

(b) For any events A, B, $\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$;

(c) For any events A, B, $\Pr(A \cap B) \leq \Pr(A)\Pr(B)$;

(d) For any events A, B, C, $\Pr(A \cup B \cup C) \leq \Pr(A) + \Pr(B) + \Pr(C)$;

(e) For any events A, B, the probability that exactly one of the events occurs is $\Pr(A) + \Pr(B) - 2\Pr(A \cap B)$.

16. A, B and C are events such that

$$\begin{array}{ll} \Pr(A) = .4 & \Pr(B \cap C) = .2 \\ \Pr(B) = .7 & \Pr[C \cap (A \cup B)] = .2 \\ \Pr(C) = .3 & \Pr[B \cap (A \cup C)] = .4 \\ \Pr(A \cup B) = .8 & \end{array}$$

(a) Find the probability that exactly two of A, B and C occur.

(b) Find the probability that none of A, B, C occur.

17. The following information is known about the three events A, B and C:

$$\begin{array}{llll} \Pr(A) = \frac{1}{3}\ , & \Pr(B) = \frac{1}{4}\ , & \Pr(C) = \frac{1}{2}\ , & \Pr(A \cup B \cup C) = 1\ , \\ \Pr(B \cap C) = 0\ , & \Pr(C \cap A) = 0\ . & & \end{array}$$

Show that

(a) $\Pr(A \cap B) = \Pr(A).\Pr(B)$,

(b) $\Pr(A \cup B) = \Pr(C)$,

(c) $\Pr(A \cup C) = \Pr(A) + \Pr(C)$.

Write down two of the events A, B, C which are independent.

18. A cube, all sides of which are painted, is sawn into a thousand small cubes of the same dimensions. The small cubes obtained are carefully mixed. Determine the probability that a small cube selected at random will have two painted sides.

19. There are n tickets in a lottery, of which m are winners. How large is the probability of a win for a person holding k tickets?

20. In certain rural communities of Russia there was at one time the following puzzle. A girl clutches in her hand six blades of grass so that the ends of the blades of grass hang above and below; a playmate ties these blades of grass pairwise above and below separately. If all six blades of grass are tied to form a single ring, then this means that the girl will marry during the current year.

(a) Find the probability that the blades of grass, when tied at random, form a ring.

(b) Solve the same problem for the case of $2n$ blades of grass.

21. Perform the following experiment. Toss 2 coins 50 times, keeping a careful tally of the possible outcomes – two heads, one head and one tail, two tails. Compute the proportion of occurence for each of the three outcomes. Do your results appear to support the reasoning of D'Alembert that the three outcomes are equally likely?

22. Open a telephone directory at any page, and obtain the frequency distribution of the last digit for 25 telephone numbers. Find the average value of the last digit and compare it with 4.5, the theoretical value if all digits are equally likely.

23. In the ancient Indian game of Tong, two players simultaneously show their right hands to each other, exhibiting either one or two or three extended fingers. If each player is equally likely to extend one, two or three fingers, what is the probability that the total number of fingers extended is even? Odd? Greater than 4? Less than 2? Prime?
[Note: Set up a sample space as a first step.]

24. Three men and three women sit in a row of six seats. Find the probability that

(a) the three men sit together;

(b) the men and women sit in alternate seats.

25. How many 5-digit numbers can be formed from the integers 1, 2, 4, 6, 7, 8, if no integer can be used more than once? How many of these numbers will be even? How many odd?

26. In how many ways can 3 letters be mailed in 6 mailboxes, if each letter must be mailed in a different box? If the letters are not necessarily mailed in different boxes, how many ways are there of posting them? If the letters are mailed at random, and not necessarily in different boxes, what is the probability that all the letters are put in the same mailbox?

27. In how many ways can 6 students be seated in a classroom with 30 desks?

28. Cards are to be dealt one after the other from a pack of playing cards. Calculate the probability that

(a) the first two cards dealt will both be diamonds;

(b) the third card will be the first diamond;

(c) the first card will be red and the second card will be black;

(d) of the first two cards dealt, one will be a diamond and the other black;

(e) the last four cards dealt will be red.

29. Two balls are drawn with replacement from an urn containing five white and three black balls. Find the probability that

(i) both balls will be white;

(ii) both balls will be the same colour;

(iii) at least one ball will be white.

30. Repeat the above question this time drawing without replacement.

31. Let A and B be events such that $\Pr(A) = 0.4$, and the probability that neither occurs is 0.3. Find the probability of B if

(i) A and B are mutually exclusive;

(ii) A and B are independent.

32. A bag contains red, blue and green counters of equal size and shape. A counter is taken at random from the bag. The probability that it is red is 1.5 times the probability that it is blue, and the probability that it is blue is twice the probability that it is green. Find the probabilities that the counter is (a) red, (b) blue and (c) green.
A counter is taken at random from the bag, its colour is noted and then it is replaced in the bag. The process continues until at least one of each colour has been seen. Considering the order in which the colours are first seen, find the probabilities that
(d) red is seen before green; (e) the order is green, blue and finally red.

33. Of 100,000 persons living at age 20, statistics show that 47,773 will be alive at 70. What is the probability that a person aged 20 will live to be 70? That he will die before he is 70?

34. The probability that A will die within the next 20 years is 0.025, and that B will die within the next 20 years is 0.030. What is the probability that both A and B will die within the next 20 years? That A will die and B will not die? That neither A nor B will die?

35. A committee of 4 is chosen at random from 5 married couples. What is the probability that the committee will not include a husband and wife?

36. An ordinary bridge deck of 52 cards is thoroughly shuffled. The cards are then dealt face up, one at a time, until an ace appears. What is the probability that the first ace appears (a) at the fifth card, (b) at the k^{th} card, (c) at the k^{th} card or sooner?

37. There are k people in a room. What is the probability that at least two of these people have the same birthday, that is, have their birthdays on the same day and month of the year? What is the smallest value of k such that the probability is $\frac{1}{2}$ or better that at least two of the people have the same birthday?

38. The stock of a warehouse consists of boxes of high, medium and low quality bulbs in the respective proportions $1 : 2 : 2$. The probabilities of bulbs of these three types being unsatisfactory are 0, 0.1 and 0.2 respectively. A box is chosen at random and 2 bulbs in it are tested and found to be satisfactory. What is the probability that it is (a) of high quality; (b) of medium quality; (c) of low quality?

39. A gold urn contains 3 red and 4 white balls and a silver urn contains 5 red and 2 white balls. A die is rolled and, if a six shows, one ball is selected at random from the gold urn. Otherwise a ball is selected at random from the silver urn. Find the probability of selecting a red ball. The ball selected is not replaced and a second ball is selected at random from the same urn. Find the probability that both balls are white.

40. Urn 1 contains x white balls and y red balls. Urn 2 contains z white balls and v red balls. A ball is chosen at random from urn 1 and put into urn 2. Then a ball is chosen at random from urn 2. What is the probability that this ball is white?

41. A bag contains three coins, one of which is coined with two heads, while the other two coins are normal and not biased. A coin is chosen at random from the bag and tossed four times in succession. If heads turn up each time, what is the probability that this is the two-headed coin?

42. Some medical tests turn out either 'positive' or 'negative'. 'Positive' indicates that the person tested has the disease in question; 'negative' indicates that he does not have it. Suppose that such a test for a rare disease sometimes makes mistakes: 1 in 100 of those free of the disease have positive test results, and 2 in 100 of those having the disease have negative test results. The rest are correctly identified. One person in 1000 has the disease. Find the probability that a person with a positive test has the disease.

43. Suppose now that, in question 42, the error rates remain the same but now half those tested have the disease and half do not. Among those tested, what is the probability that a person whose test is positive has the disease? That a person whose test is negative is disease-free? Should these two probabilities add to 1?

Do these results accord more with intuition than those of question 42. Why?

44. Three girls, Alice, Betty and Charlotte, wash the family dishes. Since Alice is the oldest, she does the job 40% of the time. Betty and Charlotte share the other 60% equally. The probability that at least one dish will be broken when Alice is washing them is 0.02; for Betty and Charlotte the probabilities are 0.03 and 0.02. The parents do not know who is washing the dishes, but one night they hear one break. What is the probability that Alice was washing up? Betty? Charlotte?

45. It is found that in manufacturing a certain article, defects of one type occur with probability 0.1 and defects of a second type with probability 0.05. What is the probability that:

(a) an article does not have both kinds of defects?

(b) an article is defective?

(c) an article has only one type of defect, given that it is defective?

46. Three students, A, B, C, have equal claims for an award. They decide that each will toss a coin, and that the man whose coin falls unlike the other two wins. (The 'odd man' wins.) If all three coins fall alike, they toss again.
Describe a sample space for the result of the first toss of the three coins, and assign probabilities to its elements. What is the probability thata A wins on the first toss? That B does? That C does? That there is no winner on the first toss?
Given that there is a winner on the first toss, what is the probability that it is A?

47. A fair coin is tossed three times, and the random variable X is defined as the number of heads found. Describe a sample space for the experiment. Write down the probability function of X, and sketch its probability function and cumulative distribution function.

48. Two ordinary six-sided dice are thrown. Find the probability function for the total score on their top faces and graph it. Do you think the points on the graph lie on any simple curve, or curves? Discuss.

49. A white die and a red die are thrown at the same time and the difference $R - W$ is observed, where R is the number on top of the red die and W is that on top of the white. Find the probability function of this difference and sketch its graph. What values of $R - W$ are most probable? Least probable? Compare this probability function and its graph with those obtained in question 48 above. Comment.

50. (i) A random variable X is defined as the larger of the scores obtained in two throws of an unbiased six-sided die. Show that the probability function is

$$p_X(x) = \frac{2x-1}{36}\ , \quad x = 1, 2, \ldots, 6\ .$$

(ii) A random variable X is defined as the highest score obtained in k throws of an unbiased six-sided die. Find an expression for the probability function of X.

[Hint: find the distribution function first.]

51. Detecting trends. This exercise deals with a topic used in studying economic time series, such as daily stock market averages or weekly production of automobiles. A time series is a set of observations or measurements arranged in the order in which they were made. If there is no trend, these measurements should fluctuate about a mean value, some above and some below. If they continually increase or decrease or follow some cyclical patterns, it may be possible to predict the future behaviour of the series.
If among three successive numerical measurements the middle one is the least or the greatest of the three, it is called a turning point of the sequence. Thus in the sequence

$$3,\ 5,\ 4,\ 7$$

the numbers 5 and 4 are turning points because 5 is the greatest of 3, 5, 4 and 4 is the least of 5, 4, 7. In random fluctuations there are more likely to be many turning points in the successive measurements than there would be if the measurements were in general increasing or decreasing.
Suppose four measurements are made, and without loss of generality suppose them to be 1, 2, 3, 4. Write down all the permutations of these numbers, and for each permutation write down the value of X, the number of turning points. If all permutations are equally likely, what is the probability function of X.

52. Draw graphs of the probability and distribution functions of the binomial (10, .2) distribution.

53. A discrete random variable X has probability function

$$p_X(x) = 2^{-x}, \quad x = 1, 2, \ldots$$

Find

(a) $\Pr(X \text{ is even})$;

(b) $\Pr(X > 5)$;

(c) $\Pr(X \text{ is divisible by } 3)$.

54. Three dice were thrown 648 times and the number of times a "5" or "6" appeared was tabulated as follows:

Number of "5" or "6"s	Observed frequency
0	179
1	298
2	141
3	30
Total	648

Obtain the theoretical probability for each outcome, for perfect dice, multiply by 648, and compare the resulting theoretical frequencies with the observed ones.

55. In a binomial experiment with $n = 3$ show that $\Pr(X = 1 \text{ or } 2) = 3pq$.

56. In a quality control test, a random sample of 100 widgets is selected, of which 90 are found to be within specification. The value of p, the proportion within specification in the batch, is not known. Write down the probability of observing 90 'successes' out of 100 as a function of p, and find the value of p for which this probability is a maximum.

57. Repeat exercise 56, but with x 'within specification' widgets out of n randomly sampled, showing that the value of p for which the probability is maximised is x/n.

58. For a binomial random variable X with $n = 10$, what is the smallest value of p such that $\Pr(X \geq 4) \geq 0.95$. [An approximate value is acceptable, as long as the logic is correct; various books contain relevant tables.]

59. *The fruit machine*. A slot machine has 2 dials, each containing ten pictures. On each dial one picture is "apples", four are "bells" and five are "cherries", and, when spun, the dials stop independently, with each picture equally likely. A play costs 10p; the machine pays out £1 for two apples, 20p for two bells and 10p for two cherries, and nothing otherwise. Find the distribution of net profit for a player who plays once, together with the player's mean net profit.

60. In a lottery, 100 tickets are sold at 25p each. There are 4 cash prizes, worth £10, £3, £2 and £1, respectively. What is the expected net gain for a purchaser of two tickets?

61. Two independent binomial experiments, one of n and the other of m trials, both have probability p of success on each trial. Show that the probability of a total of exactly x successes in the two experiments combined is

$$\binom{n+m}{x} p^x (1-p)^{n+m-x} ,$$

and interpret this result.

62. Show that, for a r.v. $X \sim B(n,p),\ \mathrm{E}(s^X) = \{1 - p + ps\}^n$.
[Note: the function $\mathrm{E}(s^X) = \sum s^x \Pr(X = x)$ for a discrete random variable is known as its probability generating function.]

63. Using the result of exercise 62 in the case $p = \mu/n$, $n \to \infty$, show that the probability generating function of the Poisson distribution with parameter μ is $e^{\mu(s-1)}$. Expand this in powers of s and show that the coefficient of s^x is $\Pr(X = x)$, $x = 0, 1, 2, \ldots$.

64. An insurance company has discovered that only about 0.1 percent of the population is involved in a certain type of accident each year. If its 10,000 policy holders were randomly selected from the population what is the probability that not more than 5 of its clients will be involved in such an accident next year?

65. Suppose that a container contains 10,000 particles. The probability that such a particle escapes from the container equals 0.0004. What is the probability that more than 5 such escapes occur? (You may assume that the various escapes are independent of one another.)

66. Suppose that a book of 585 pages contains 43 typographical errors, randomly distributed throughout the book. What is the probability that 10 pages selected at random will be free of errors?
[Hint: Suppose that X, the number of errors on a randomly selected page, has a Poisson distribution.]

67. The number of particles emitted from a radioactive source during a specified period is a random variable with a Poisson distribution. If the probability of no emissions equals $\frac{1}{3}$, what is the probability that 2 or more emissions occur?

68. Two people toss a fair coin n times each. Find the probability that they will score the same number of heads.

69. What is the chance that a hand at bridge (containing 13 cards) contains no ace? If a bridge player has no ace in three consecutive hands, can he fairly claim to be unlucky?

70. In a sequence of Bernoulli trials with probability p of success, find the probability that a successes will occur before b failures.

[Note: The issue is decided after at most $a + b - 1$ trials. This problem played a role in the classical theory of games in connection with the question of how to divide the pot when the game is interrupted at a moment when one players lacks a points to victory, the other b points.]

71. Bernoulli trials with probability p of success are continued until the r^{th} success occurs; let X be the number of trials required. Show that X has probability function

$$\Pr(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

over an appropriate range. What is this range?

72. A lake contains n fish, where n is unknown. A sample of m fish is caught, each fish caught is marked and returned to the lake. Later, a second sample of k fish is caught; of the k fish in the sample X are marked. Making reasonable assumptions about the way the sampling was carried out, find the probability distribution of X, i.e. the function $\Pr(X = x)$.
[Hint: The second sample can be selected in $\binom{n}{k}$ ways.]

73. (Continuation.) The probability $\Pr(X = x)$ is, in practice, a function of known quantities x, k, m and the unknown n. Viewing it as a function $g(n)$ of n examine the ratio $g(n)/g(n-1)$ to obtain the value of n for which $g(n)$ is a maximum.

74. A continuous random variable X has probability density function

$$f_X(x) = \begin{cases} kx, & 0 \le x < 1\,; \\ k, & 1 \le x < 2\,; \\ 3k - kx, & 2 \le x < 3\,; \\ 0, & \text{elsewhere}\,. \end{cases}$$

(a) Find k.

(b) Find, and sketch, the distribution function of X.

(c) If 6 independent repetitions are performed of the experiment leading to X, find the probability that at least three of the values of X observed exceed 2.5.

75. A continuous probability distribution has density

$$f(x) = \begin{cases} a(b-x)^2, & 0 \le x \le b\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

If the mean of the distribution is unity, find the values of a and b.

76. The continuous random variable X has pdf $f(x) = x/2$, $0 \le x \le 2$. Two independent determinations of X are made. What is the probability that both these determinations will be greater than one? If three independent determinations had been made, what is the probability that exactly two of these are larger than one?

77. Find the mean and variance of the continuous distribution with probability density

$$f(x) = \begin{cases} 12x^2(1-x), & 0 \le x \le 1 \; ; \\ 0, & \text{elsewhere} \; . \end{cases}$$

78. For a certain industrial firm it is found that if a person joins the firm the probability that he is still there at time x later, where x is measured in years, is

$$\tfrac{1}{3}\, e^{-\frac{1}{2}x} \;+\; \tfrac{2}{3}\, e^{-\frac{1}{4}x} \; .$$

Find the probability that a person spends between 2 and 4 years with the firm, and find the mean and variance of the length of time he works there.

79. The diameter X of an electric cable is assumed to be a continuous random variable with pdf $f(x) = 6x(1-x), 0 \le x \le 1$.

(a) Check that the above is a pdf and sketch it.

(b) Obtain an expression for the cdf of X and sketch it.

(c) Determine a number b such that $P(X < b) = 2P(X > b)$.

80. A point is chosen at random on a line of length L. What is the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$?

81. Suppose that X is a random variable for which $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Suppose that Y is uniformly distributed over the interval (a, b). Determine a and b so that $\mathrm{E}(X) = \mathrm{E}(Y)$ and $\mathrm{Var}(X) = \mathrm{Var}(Y)$.

82. If Y has an exponential distribution with pdf

$$f_y(y) = \lambda e^{-\lambda y} \; , \quad y \ge 0 \; , \quad \lambda > 0 \; ,$$

obtain an expression for $\mathrm{E}(e^{sY})$ in terms of s and λ, stating the range of values of s for which your expression is valid.
[Note: The function $\mathrm{E}(e^{sY})$ is known as the moment generating function of Y.]

83. Suppose that X has distribution $N(2, 0.16)$. Using the table of the normal distribution, evaluate the following probabilities.
(a) $\Pr(X \ge 2.3)$ (b) $\Pr(1.8 \le X \le 2.1)$.

84. The diameter (in inches) of an electric cable is normally distributed with mean 0.8 and variance 0.0004. What is the probability that the diameter will exceed 0.81 inch.
Suppose that the cable is considered defective if the diameter differs from its mean by more than 0.25 inches. What is the probability of obtaining a defective cable?

85. Suppose that the life lengths of two electronic devices, say D_1 and D_2, have distributions $N(40, 36)$ and $N(45, 4)$, respectively. If the electronic device is to be used for a 45-hour period, which device is to be preferred? If it is to be used for a 49-hour period, which device is to be preferred?

86. Suppose that X has distribution $N(\mu, \sigma^2)$. Determine c (as a function of μ and σ) such that $\Pr(X \geq c) = 2\Pr(X > c)$.

87. An aircraft fuel is to contain a certain percent (say X) of a particular compound. The specifications call for X to be between 30 and 35. The manufacturer will make a net profit on the fuel (per gallon) which is the following function of X:

$$\begin{array}{rcll} T(X) & = & \text{£0.10 per gallon} & \text{if } 30 < X < 35\,, \\ & = & \text{£0.05 per gallon} & \text{if } 35 \leq X < 40 \text{ or } 25 < X \leq 30\,, \\ & = & -\text{£0.10 per gallon} & \text{otherwise.} \end{array}$$

(a) If X has distribution $N(33, 9)$, evaluate $\mathrm{E}(T)$.

(b) Suppose that the manufacturer wants to increase his expected profit, $\mathrm{E}(T)$, by 50 percent. He intends to do this by increasing his profit (per gallon) on those batches of fuel meeting the specifications, $30 < x < 35$. What must his new net profit be?

88. Show that, if $X \sim N(\mu, \sigma^2)$, the random variable X has moment generating function

$$\mathrm{E}(e^{sX}) = e^{\mu s + \frac{1}{2}\sigma^2 s^2}\ .$$

[Hint: Express $\mathrm{E}(e^{sX})$ as the usual integral. The exponent will be a quadratic expression, and the term 'sx' can be incorporated into it, using the technique of completing the square. Terms involving the intergrand x then integrate simply. (Use the fact that the integral I of a Normal pdf, or a function equivalent to it, is 1.)]

89. Show that, for the Normal distribution, the derivative of the moment generating function with respect to s, evaluated at $s = 0$, is the mean.
[Note: This is in fact true for all distributions.]

90. Suppose that a sample of size n is obtained from a very large collection of bolts, 3 percent of which are defective. Find the probability that at most 5 percent of the chosen bolts are defective

(a) if $n = 6$;
(b) if $n = 60$;
(c) if $n = 600$

91. (a) A complex system is made up of 100 components functioning independently. The probability that any one component will fail during the period of operation equals 0.10. In order for the entire system to function, at least 85 of the components must be working. Evaluate the probability of this.

(b) Suppose that the above system is made up of n components each having a reliability of 0.90. The system will function if at least 80 percent of the components function properly. Determine n so that the system has reliability of 0.95.

92. An airline finds that, on average, 4 percent of the persons who reserve seats for a certain flight do not turn up for the flight. Consequently the airline allows 75 persons to reserve seats on a plane which can accommodate only 73 passengers. Using a suitable approximation, find the probability that there will be a seat available for every passenger who turns up for the flight.

93. If 180 seeds are sown, and each has probability 0.3 of germinating, what is the probability that at least 50 germinate? (Use a suitable approximation, and identify clearly any assumptions you are making.)
If it is required that the probability of at least 50 seeds germinating be at least .95, how many further seeds should be sown?

94. Independent random variables X and Y have distributions $N(4, 11), N(6, 14)$ respectively. Show that the probability that $(X - Y)^2$ has a value between 4 and 16 is approximately $\frac{1}{4}$.

95. The following table gives the joint probability function $p_{X,Y}(x, y)$ of a pair of discrete random variables (X, Y).

$y \backslash x$	1	2	3	4
1	.06	.12	.24	.18
2	.02		.08	.06
3		.08	.16	

(a) Find the marginal distributions of X and Y.
(b) Find the conditional distribution of X, given $Y = 2$; and the conditional distribution of Y, given $X = 3$.

96. Suppose that the following table represents the joint probability distribution of the discrete random variables X and Y. Evaluate all the marginal and conditional distributions.

$y \backslash x$	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

97. For the data in question 96, evaluate

(a) $\Pr(X = Y)$, (c) $\Pr(X > 2Y)$,

(b) $\Pr(X > Y)$, (d) $\Pr(X \geq Y | Y \geq 2)$.

98. Suppose that the joint pdf of the two-dimensional random variable (X, Y) is given by

$$\begin{aligned} f(x,y) &= x^2 + \frac{xy}{3}, \quad 0 < x < 1, \quad 0 < y < 2, \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

Compute the following:

(a) $\Pr(X > \frac{1}{2})$; (b) $\Pr(Y < X)$; (c) $\Pr(Y < \frac{1}{2} | X < \frac{1}{2})$.

99. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained and let Y be the number of queens obtained.

(a) Obtain the joint probability distribution of (X, Y).

(b) Obtain the marginal distribution of (X, Y).

(c) Obtain the conditional distribution of X (given Y) and of Y (given X).

100. X and Y are independent random variables, and each is equally likely to take the three values 1, 2, 3. Draw up a table similar to that in question 96 to show the joint probability function of X and Y. Hence write down the probability function of Z, where $Z = X + Y$. Evaluate the mean and variance of Z and compare them with those of X.

101. If, in question 100, $W = X - Y$, find the mean and variance of W. Comment on the result.

102. The distribution of length of two kinds of wooden parts A and B are approximately normal, with means $\mu_A = 2$ inches and $\mu_B = 4$ inches, and standard deviations $\sigma_A = 0.009$ inch, $\sigma_B = 0.40$ inch. An A part and a B part are randomly assembled and laid end to end to form a length about 6 inches long. If an assembly is to fit, it must be between 5.92 and 6.08 inches long. What percentage of random assemblies fail to fit? (Assume that the distribution of the sum of two independent normally distributed random variables is normal.)

103. Two measurements X and Y are independently drawn from the same distribution with mean μ and variance σ^2, and a weighted sum $S = wX + (1 - w)Y$ is computed.

(a) Find μ_S .

(b) Find σ_S^2 .

(c) Find the value of w that minimizes σ_S^2 .

(d) Find the minimum value of σ_S^2.

[Remark: For any w, S is called an unbiased estimate of μ because $\mu_S = \mu$, and, with $w = \frac{1}{2}$, S is called the minimum variance unbiased estimate of μ.]

104. The distribution of weights (in pounds) of a large group of equipped army recruits is very closely approximated by a normal curve with mean 185.0 pounds and standard deviation 15.0 pounds.

(a) If two recruits are picked at random from this group, what is the probability that both their weights are between 170.0 and 200.0 pounds?

(b) If 81 recruits from the large group are to enplane with an allowance of 190.0 pounds per man, what is the chance that the transport will be overloaded?

105. In a certain large society the standard deviation of the number of children in a family is 1.5. If an anthropologist wants the standard deviation of $\overline{X}$, his estimate of the mean number of children per family to be 0.01, how many families should be in his random sample?

106. Independent random variables $X_1, \ldots, X_n$ all have expectation μ, and $V(X_k) = k^2$, $k = 1, \ldots, n$. A random variable S is defined as

$$S = a_1X_1 + \cdots + a_nX_n$$

for constants $a_1, \ldots, a_n$. Find the expectation and variance of S.
What condition must the $\{a_k\}$ satisfy for the expectation of S to be μ? Subject to satisfying this condition, how should one choose $a_1, \ldots, a_n$ so as to minimise $V(S)$?

107. From a distribution with variance $\sigma^2 = 1$, two independent random samples are drawn as in the following table:

	Sample size	Sample average
Sample 1	10	$\overline{X}_1$
Sample 2	5	$\overline{X}_2$

(a) To estimate the population mean; one man weights the sample averages in proportion to their sample sizes and claims that the sample variance of such a weighted estimate, $\frac{2}{3}\overline{X}_1 + \frac{1}{3}\overline{X}_2$, is $\frac{1}{15}$. Justify this result.

(b) A second man merely averages the sample averages and uses $\frac{1}{2}(\overline{X}_1 + \overline{X}_2)$. Show that the sampling variance of such an estimate is $\frac{3}{40}$.

(c) Show that both methods are unbiased (have mean equal to the true mean μ).

(d) Explain (a) from the point of view of one sample of size 15.