Birational Transformations of Algebraic Ordinary Differential Equations

F. Winkler[1]

[1] *Johannes Kepler Universität, Linz, Austria, franz.winkler@risc.jku.at*

An algebraic ordinary differential equation (AODE) is a polynomial relation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

between the unknown function y and its derivatives, possibly involving the variable of differentiation x, where F is a differential polynomial in $K[x]\{y\}$ with K being a differential field and the derivation $'$ being $\frac{d}{dx}$.

In the algebro-geometric approach we consider the algebraic hypersurface defined by the polynomial F. From a rational parametrization of this hypersurface, we can decide the rational solvability of the given AODE, and in fact compute the general rational solution. This approach has been developed for autonomous first order AODEs by Feng and Gao [1], and for non-autonomous first order AODEs by Ngô and Winkler [2].

Transforming the ambient space by some group of transformations, we get a classification of AODEs, such that equivalent equations share the property of rational solvability. The action of the group of affine transformations on AODEs has been investigated by Ngô, Sendra, and Winkler in [3]. Here we describe the action of the group of birational transformations.

For example, the first order AODE

$$F(x, y, y') = 25x^2y'^2 - 50xyy' + 25y^2 + 12y^4 - 76xy^3 + 168x^2y^2 - 144x^3y + 32x^4 = 0$$

can be birationally transformed into the AODE $G(y, y') = y'^2 - 4y = 0$. By the inverse transformation we derive from the rational general solution $y = (x+c)^2$ of $G(y, y') = 0$ the rational general solution $y = \frac{x(2(x+c)^2+1)}{(x+c)^2+3}$ of $F(x, y, y') = 0$.

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References

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