Computing Resolutions for Linear Differential Systems

Werner M. Seiler[1]

[1] *Institut für Mathematik, Universität Kassel, Germany, seiler@mathematik.uni-kassel.de*

Rings of linear differential operators represent a typical example of *polynomial algebras of solvable type*. This class of rings was first introduced by Kandry-Rody and Weispfenning [4] and various variants of it appear for instance in the work of Bueso et al. [3], Kredel [5], Levandovskyy [6] or the author [8]. It contains weakly non-commutative rings whose elements may be considered as polynomials in variables which do not necessarily commute and which may operate on the coefficients. From a theoretical point of view, these rings may have very different algebraic properties. However, from an algorithmic point of view, to all these rings the classical theory of Gröbner bases can be applied without changes, as one can meaningful work with leading terms and monomials.

In this talk we are mainly concerned with the problem of computing free resolutions over polynomial algebras of solvable type with special emphasis on the case of linear differential operators. In order to motivate this topic, we will first recall the relation between free resolutions over the ring of linear differential operators and compatibility complexes as they appear in the theory of overdetermined systems (see e. g. [10]). The former ones can be algorithmically computed – as discussed below – whereas the latter ones depend on the function space on which the operators act and may or may not exist. In the case that the function space is an injective cogenerator the two notions are dual to each other and each resolution yields immediately an exact compatibility complex (this observation follows essentially from results of Oberst [7]).

The main part of the talk will consists of a discussion under which conditions the theoretical results of [9] and the algorithmic results of [1] and [2] for the commutative polynomial ring remain valid for a polynomial algebra of solvable type.

In [8], the author presented an involutive version of the well-known Schreyer Theorem for Janet and Pommaret bases: given a Janet or Pommaret basis of a polynomial module, it shows how to read off a Janet or Pommaret basis of the first syzygy module from the involutive standard representations of the non-multiplicative prolongations of the generators. In contrast to the classical form of the Schreyer Theorem (which underlies most algorithms for computing resolutions), the involutive version allows to predict without any further computations the shape of the complete resolution obtained by iterating the theorem. In the case of a Pommaret basis, this resolution has even the same "bounding box" as the minimal one, i. e. we can read off the projective dimension and the Castelnuovo-Mumford regularity. As a trivial corollary, one then obtains Hilbert's Syzgy Theorem, i. e. the global dimension of the commutative polynomial ring.

In [1], the approach of [9] is complemented by Algebraic Discrete Morse Theory [11] in order to determine not only the shape but also the differential of the arising resolution. More precisely, it is shown how the results of [12] can be made effective via Pommaret bases (in [2], this is extended to Janet bases). This combination leads to a novel algorithm for the determination of free resolutions with very different features than previous algorithms. In particular, in the case that the notion of a minimal resolution is defined and thus one can speak of Betti numbers, this approach allows for the determination of individual Betti numbers without computing the minimal resolution. As benchmarks with a first prototypical implementation showed, the new algorithm is for many examples by orders of magnitude faster than previous ones.

References

[1] M. Albert, M. Fetzer, E. Sáenz-de-Cabezón and W.M. Seiler, *On the Free Resolution Induced by a Pommaret Basis*, J. Symb. Comp. **68**, pp. 4–26 (2015)

[2] M. Albert, M. Fetzer and W.M. Seiler, *Janet Bases and Resolutions in CoCoALib*, Preprint Universität Kassel (2015)

[3] J.L. Bueso, J. Gómez-Torrecillas and A. Verschoren, *Algorithmic Methods in Non-Commutative Algebra*, Mathematical Modelling: Theory and Applications 17, Kluwer, Dordrecht (2003)

[4] A. Kandry-Rody and V. Weispfenning, *Non-commutative Gröbner Bases in Algebras of Solvable Type*, J. Symb. Comp. **9**, pp. 1–26 (1990)

[5] H. Kredel, *Solvable Polynomial Rings*, Verlag Shaker (1993)

[6] V. Levandovskyy, *Non-commutative Computer Algebra for Polynomial Algebras: Gröbner Bases, Applications and Implementation*, Ph.D. thesis, Kaiserslautern (2005)

[7] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math. **20**, pp. 1–175 (1990)

[8] W.M. Seiler, *A Combinatorial Approach to Involution and δ-Regularity I: Involutive Bases in Polynomial Algebras of Solvable Type*, Appl. Alg. Eng. Comm. Comp. **20**, pp. 207–259 (2009)

[9] W.M. Seiler, *A Combinatorial Approach to Involution and δ-Regularity II: Structure Analysis of Polynomial Modules with Pommaret Bases*, Appl. Alg. Eng. Comm. Comp. **20**, pp. 261–338 (2009)

[10] W.M. Seiler, *Involution—The Formal Theory of Differential Equations and Its Applications in Computer Algebra*, Algorithms and Computations in Mathematics 24, Springer-Verlag, Heidelberg (2010)

[11] Sköldberg, E.: *Morse theory from an algebraic viewpoint*. Trans. Amer. Math. Soc. 358, 115–129 (2006)

[12] Sköldberg, E.: *Resolutions of modules with initially linear syzygies*. Preprint arXiv:1106.1913 (2011)