Symbolic Computation for Rankin-Cohen Differential Algebras

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Rankin-Cohen algebras were defined by Zagier [Z], who reprised the role of differential operators in the theory of modular forms, central in the 19th century but "surprisingly little (...) in more modern investigations". In brief, for $f(\tau)$ and $g(\tau)$ two modular forms of weights k, l respectively, on some group $\Gamma \subset PSL(2,\mathbb{R})$, let D be the differential operator $\dfrac{1}{2\pi\imath}\dfrac{d}{d\tau}$, given the expansion of the modular forms in τ, where $\dfrac{1}{2\pi\imath}\dfrac{d}{d\tau} = q\dfrac{d}{dq}$, $q = e^{2\pi\imath\tau}$ as usual. The n-th Rankin-Cohen bracket (so named by Zagier, after R.A. Rankin who studied the derivations on modular forms and H. Cohen who gave examples) of f and g is the only bilinear differential operator of degree $2n$ that acts on the graded vector space of modular forms on Γ, and is defined as follows (denoting $D^r f$ by $f^{(r)}$ for a form f):

$$[f,g]_n(\tau) = \sum_{r+s=n} (-1)^r \binom{n-k+1}{s}\binom{n-l+1}{r} f^{(r)}(\tau) g^{(s)}(\tau).$$

Zagier pursues the study of the algebraic structure that this operation gives to the ring of modular forms viewed as a differential module, observing that it is "not clear how far we would have to go to get the first relation or how much further to ensure that all subsequent relations obtained would be consequences of ones already found". Instead of determining the relations, he proposes the abstract concept of a Rankin-Cohen differential algebra and gives a "partial structure theorem".

In this work, we propose to use Symbolic Computation to detect minimal sets of relations for the case study of $\Gamma(7)$, the modular group of the Klein curve, the only algebraic curve of genus three with the largest possible group of automorphisms, motivated by the first-named author's Ph.D. Thesis [Farr], which uses techniques that allow us to deal explicitly with certain modular forms.

We apply the theory of Gröbner bases (as in [EGÔP]) to control the weight of the relations, and then perform a search (implemented in *Maple* syntax) for complete, minimal sets ot relations weight-by-weight; in consequence, our results only reach a(ny) finite given weight, but these relations are of interest, given the large number of open problems that concern the Klein curve (more specifically stated below).

Then, in order to further exploit the power of computation, we propose to study Rankin-Cohen differential algebras over finite fields; indeed, when giving their abstract definition in [Z], "We will suppose the ground field K to be of characteristic 0 (in our examples it is usually $\mathbb{Q}$ or $\mathbb{C}$) although it is clear that the theory makes sense in any characterisitc or, for that matter, even if we work over $\mathbb{Z}$ rather than a field." Since our strategy is to reduce cusp forms modulo a prime p, we assume that p does not divide the level [CFW], therefore $p \neq 7$ throughout.

1 Wronskians

The problem of determining the set of Weierstrass points on curves of arithmetic interest, such as the Fermat curves $x^N + y^N + z^N = 0$ and the modular curves $X(N)$, remains unsolved for all but a few values of N.

Klein's curve has been studied by its many different aspects according to the properties that were best accessible through one or the other: as a covering of $\mathbb{P}^1$, [FK1, VII.3], it is a Riemann surface M given by the algebraic equation

$$w^7 = z(z-1)^2.$$

The function z is ramified (of ramification number 7) at the points 0, 1 and ∞, and we set: $P_0 = z^{-1}(0), \quad P_1 = z^{-1}(1), \quad Q = z^{-1}(\infty)$, and consider the following divisors: $(z) = \frac{P_0^7}{Q^7}$, $(dz) = \frac{P_0^6 P_1^6}{Q^8}$, $(w) = \frac{P_0 P_1^2}{Q^3}$. Per this calculation, the differentials

$$\frac{dz}{w^3}, \quad (z-1)\frac{dz}{w^5}, \quad (z-1)\frac{dz}{w^6}$$

have divisors $P_0^3 Q, \quad P_0 P_1^3, \quad P_1 Q^3$, hence give a basis for $\Omega^1(M)$. Using this basis we can find an embedding of M in $\mathbb{P}^2$ [FK1, III.10]. In fact, if we set $w = -XY^{-1}, \quad z - 1 = X^3 Y^{-2}$ we find that the projective equation for the algebraic curve M is the quartic: $X^3 Y + Y^3 Z + Z^3 X = 0$.

We can immediately conclude from the divisors of the differentials that the points P_0, P_1 and Q are Weierstrass points of weight 1. We turn to the Wronskian of to finish the search for the Weierstrass points. We recall that, denoting $W(f_1, \ldots, f_g)$ the Wronskian determinant for a basis $f_1(z), \ldots, f_r(z)$ of the canonical linear system, $|K|$, with associated linear series $\mathscr{L}(K)$, over an algebraic curve X of genus $g \geq 2$, in a local coordinate z, the zeros of $W(f_1, \ldots, f_g)(dz)^{g(g+1)/2}$ are the Weierstrass points for the curve X, the multiplicities of the zeros being their Weierstrass weights [M, VII.4]. Using the function z above as a local coordinate, since we already took into account the points over 0, 1 and ∞ where it ramifies, we compute $W(z) = 3!(z^3 - 8z^2 + 5z + 1)/(z^8(z-1)^5)$ The polynomial

$p(z) = z^3 - 8z^2 + 5z + 1$ has three distinct real roots, each of which corresponds to 7 distinct points on M. Thus M has 24 Weierstrass points, each of weight one.

We now consider a second method for finding the ordinary Weierstrass points, as in [R]. When $X(\Gamma)$ is the modular curve $\Gamma \diagdown \mathscr{H}^*$, for Γ a subgroup of finite index in $SL_2(\mathbb{Z})$ and $\mathscr{H}^*$ the upper half plane with the cusps of Γ adjoined, the set of weight-2 cusp forms for Γ, $S_2(\Gamma)$, is isomorphic to the set of holomorphic 1-forms for the Riemann surface. Thus to build a Wronskian for $\Gamma \diagdown \mathscr{H}^*$ we may use a basis $f_1, f_2, \ldots, f_g$ for $S_2(\Gamma)$, the Wronskian $W(f_1, f_2, \ldots, f_g)$ being a modular form of weight $g(g+1)$ for Γ.

The Klein curve X is isomorphic to the modular curve $X(7)$, with $\Gamma = \Gamma(7)$. Since $\Gamma(7)$ is normal in $SL_2(\mathbb{Z})$, this Wronskian is a modular form for $SL_2(\mathbb{Z})$ itself, with character $\det \rho$, for ρ the natural representation of $SL_2(\mathbb{Z})$ on the space of cusp forms of weight 2 for Γ. The choice of basis only affects the Wronskian by a nonzero complex multiple, while we are only concerned about its zeros; to eliminate the dependence on the choice of basis entirely we may require that the first nonzero coefficient in the Fourier expansion of the Wronskian at the cusp at ∞ be 1. Thus we can talk about the Wronskian for $\Gamma \diagdown \mathscr{H}^*$.

In general, if the ramification index of Γ in $SL_2(\mathbb{Z})$ is r at ∞, we can express the Fourier expansion of $W(z)$ at ∞ as

$$W(z) = \sum_{n \geq n_0} a_n e^{2\pi i n z/r}, \quad a_{n_0} = 1.$$

For the case of $X(7)$, $g = 3$, so $W(z)$ is a cusp form of weight 12 with character for $SL_2(\mathbb{Z})$. The character factors through $SL_2(\mathbb{Z})/\{\pm 1\}\Gamma(7)$, hence is trivial, thus $W(z)$ is a cusp form for $SL_2(\mathbb{Z})$ itself. The only possibility is that $W(z) = \Delta$. Since Δ is never zero on $\mathscr{H}$, we find that the Weierstrass points are the cusps.

The Wronskian for the pluricanonical series, $\mathscr{L}(nK)$, $n \geq 2$ (associated to $|nK|$) gives the higher-order Weierstrass points [FK1, III.5]. In the pluricanonical case, the Wronskian for a modular curve $X(\Gamma)$ is an automorphic form of weight $(2n-1)^2 g(g-1)/2$ [FK2, 3.1].

Using the model for $X(7)$ given by $w^7 = z(z-1)^2$, we have found bases for the pluricanonical series $\mathscr{L}(nK)$ for X. Indeed, we observed that for $2 \leq n \leq 5$, pairwise multiplication of the elements of our previously found basis for $\mathscr{L}(K)$ leads to exactly $\dim \mathscr{L}(nK) = (2n-1)(g-1) - 1$ independent differentials. For example for $n = 2$, pairwise multiplication of the basis elements of $\mathscr{L}(K)$ above led to

$$\{\frac{1}{w^6}, \frac{1}{wz(z-1)}, \frac{1}{zw^3}, \frac{1}{w^2 z(z-1)}, \frac{1}{w^4 z}, \frac{1}{w^5 z}\}.$$

To use these Wronskians in the Rankin-Cohen algebra, we must find their q-expansion: our strategy is to first identify them as automorphic forms constructed from theta constants [FK2, III.2]; then use classical identities to embed (as Klein did) the curve in $\mathbb{P}^2$ [FK2, III.8.4]; and lastly, use an algebraic map to convert $\mathbb{P}^2$-coordinates into the meromorphic functions w, z on the curve as the 7-sheeted cover; retracing our steps, we have written the pluricanonical Wronskians as classical automorphic forms, and can Fourier-expand them. As Zagier notes, a "canonical" Rankin-Cohen algebra can be generated by a form in degree four and a degree-2 differentiation; our Wronskians are of course of higher degree, but he also considers, for comparison, a homogeneous generator F of arbitrary degree, provided it is not a zero-divisor, so our case study is a legitimate example of his theory.

2 Finite Fields

Modular forms in positive characteristic (we are only considering reduction of coefficient modulo a prime p, not Katz' theory which has an algebro-geometric definition and may give rise to non-liftable forms, an unsettled issue) still present challenges, such as the structure of their Hecke algebra [BK]. The Hecke operator makes sense in characteristic p, but others do not exist in characteristic zero, particularly "multiplication by the Hasse invariant"; the "theta operator" ϑ is defined in characteristic zero, in fact it is precisely what we called D following [Z], where it "destroys modularity" [K], but in positive characteristic it raises the weight by $p+1$: this $\vartheta := q\dfrac{d}{dq}$ acts formally on the q expansion of the discriminant Δ and the Eisenstein series E_4, E_6, and these can be chosen as generators of the (graded) ring of modular forms. In the recent monograph [K], the author implements some such operations in computation, using both MAGMA and its open-source counterpart SAGE, primarily with the goal of computing Fourier coefficients.

We propose to use our case-study $\Gamma(7)$ and computation in characteristic $p \neq 7$ (over a finite field or its algebraic closure), not only to study the structure of Rankin-Cohen algebras, but also with the goal of computing "theta cycles": these are specific to positive characteristic, and arise as follows. The multiplication $f \mapsto Af$, where A is the Hasse invariant, in characteristic p raises the weight by $p-1$ and leaves the q-expansion unchanged: the smallest weight in which a form f appears is called its "filtration" $w(f)$. Since $w(\vartheta^p f) = w(\vartheta f)$, one can attach to any mod p modular form f a $(p-1)$-tuple of integers, $\left(w(\vartheta f), w(\vartheta^2 f), \ldots, w(\vartheta^{p-1} f)\right)$, and this is called its theta cycle. These were investigated by J. Tate and classified by N. Jochnowitz in her thesis: they have applications to estimates on the number of local components of Hecke algebras. We study the action of the Rankin-Cohen brackets on theta cycles: this might give us an extra handle on the relations of Rankin-Cohen algebras in characterisitc p.

3 Conclusions

Our underlying theme is that the use of differential operators in the theory of modular forms, especially as regards their dual nature as algebro-geometric or number-theoretic objects, should be revived in the spirit of the nineteeth century and made powerful by means of symbolic computation. We use cusp forms over the Klein curve, obtain a relationship between the algebraic and modular aspects, and computationally obtain explicit identities for the little-known Rankin-Cohen differential (graded) algebras; in positive characteristic, even over finite fields, our case-study potentially aids the quest for the structure of the Hecke algebra. Further motivation for using the Klein curve is a computational study of its differential-Galois aspects (when viewed as an algebraic cover) [SU], which can be related to the algebraic Wronskians, and which we plan to relate to its cusp forms, particularly in positive characteristic since the previous work was carried out over the complex numbers.

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