# Jacobi algebras, in-between Poisson, differential, and Lie algebras

L. Poinsot[1,2]

[1] *LIPN - CNRS (UMR 7030), University Paris 13, Sorbonne Paris Cité, France, laurent.poinsot@lipn.paris13.fr*
[2] *CReA, French Air Force Academy, Base aérienne 701, France*

In the non-differential setting there is a functorial relation between Lie algebras and associative algebras: any algebra becomes a Lie algebra under the commutator bracket, and, conversely, to any Lie algebra is attached a universal associative envelope. In the realm of differential algebras, there are two such adjoint situations. The most obvious is obtained by lifting the above correspondence to differential algebras. At the contrary, the second connection is proper to the differential setting. Any commutative differential algebra admits the *Wronskian* bracket $xy' - x'y$ as a Lie bracket, and to any Lie algebra is provided a universal differential and commutative associative envelope.

A natural question is to know under which conditions a given Lie algebra embeds into its differential envelope. While an answer, for the non-differential setting, is known – the Poincaré-Birkhoff-Witt theorem – there are no yet any such solution in differential algebra. In a first part of this talk, after having briefly recalled the above construction, I will present some classes of Lie algebras for which the canonical map to their differential algebra is one-to-one.

Moreover, differential commutative algebras are merely not just Lie algebras, with help of their Wronskian bracket, but rather Lie-Rinehart algebras [2], the algebraic counterpart of a Lie algebroid.

However the Lie-Rinehart structure on a differential commutative algebra is just the consequence of a more abstract structure, namely that of a Jacobi algebra. A Jacobi algebra [1] is a commutative algebra $A$ together with a Lie bracket $[-,-]$ (called *Jacobi bracket*) satisfying the following version of Leibniz rule:

$$[ab,c] = a[b,c] + b[a,c] - ab[1_A,c],\ a,b,c \in A.$$

A Jacobi bracket provides a derivation and an alternating biderivation. Hence forgetting one or the other of those differential operators provides a differential or a Poisson algebra, and these relations are functorial.

In a second part of the talk I will also present some of the functorial relations between Jacobi, differential, and Lie algebras, such as, e.g., the Jacobi envelope of a Lie algebra. I will also explain that the Lie algebra of global smooth sections of a line bundle $E$ over a smooth manifold $M$ (i.e., a vector bundle over $M$ each fibre of which is one-dimensional) embeds, when $E$ is trivial, into its Jacobi envelope.

# References


[1] J. Grabowski, *Abstract Jacobi and Poisson structures. Quantization and star-products*, Journal of Geometry and Physics **9**, pp. 45-73 (1992).

[2] G. Rinehart, G., *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108**, pp. 195-222 (1963).