One symbolical method for solving differential equations with delayed argument

Natasha Malaschonok

Tambov State University

Consider an equation

$$x^{(n)}(t)+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}x^{(n-j)}(t-t_k)=f(t), \tag{1}$$

with initial conditions $x^{(n-j)}(0)=x_0^{(n-j}, j=1,\ldots,n$. The function $f(t)$ in the right-hand part is in general composite. We may consider for it the same partition points t_k.

All functions of the argument t are supposed to satisfy the conditions for existing of their Laplace transform, and they equal zero for negative t. The points $t_k, t_{k-1}<t_k$, are rational and taken in the set of $t\in\mathbf{T}: 0\leq t\leq T$. Writing t_k as $t_k=\frac{\tau_k}{\sigma_k}$, denote $\sigma=LCM_k(\sigma_k)$, and $t_k=\frac{\tilde{\tau}_k}{\sigma}$.

Preparation for Laplace transform

The unknown function $x(t)$ and $f(t)$ satisfy the properties, put on above, so the equation (1) may be written using the Heaviside function $\eta(t)$ in the following way:

$$x^{(n)}(t)+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}\eta(t-t_k)x^{(n-j)}(t-t_k)=f(t), \tag{2}$$

$f(t)$ is also written by means of Heaviside function.

Laplace transform

It permits to write symbolically the Laplace image of the equation (2):

$$\left(p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}e^{-pt_k}p^{n-j}\right)X(p)= \tag{3}$$

$$\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}e^{-pt_k}+F(p), \tag{4}$$

where $X(p)$ and $F(p)$ are the Laplace images of $x(t)$ and $f(t)$, correspondingly, and $F(p)$ in general is also a sum of exponents with polynomial coefficients.

Solving the algebraic equation

Denote

$$Q(p)=\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}e^{-pt_k}+F(p), \tag{5}$$

$$D(p)=p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}e^{-pt_k}p^{n-j}, \tag{6}$$

then

$$X(p)=\frac{Q(p)}{D(p)}. \tag{7}$$

Expansion of the solution in a series

Denote $e^{-\frac{p}{\sigma}}=z$. Then

$$X(p)=\frac{\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}z^{\tilde{\tau}_k}+F(p)}{p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}z^{\tilde{\tau}_k}p^{n-j}}. \tag{8}$$

Formally we expand (5) in a Taylor serious by z at the point $z=0$. It corresponds to $p:\mathrm{Re}p=+\infty$. Substituting $e^{-\frac{p}{\sigma}}$ instead of z, we obtain the series for $X(p)$ by $e^{-\frac{np}{\sigma}}$, which converges in some neighbourhood of ∞:

$$\sum_{n}A_n e^{-\frac{np}{\sigma}}, \tag{9}$$

where A_n are proper fractions, and can be represented as sums of partial fractions.

Inverse Laplace transform

For the series (6) the Inverse Laplace transform may be written symbolically.

We restrict ourselves to the consideration of one equation, but the method works similarly with systems of equations of such type.