Formal Solutions of Singularly-Perturbed Linear Differential Systems

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Given the singularly-perturbed linear differential system

$$\varepsilon^h \frac{dY}{dx} = A(x,\varepsilon)Y = \sum_{k=0}^{\infty} A_k(x)\varepsilon^k Y,$$

where h is an integer and the $A_k(x)$'s are $n \times n$ matrices whose entries lie in the ring of formal power series in x with complex coefficients.

Such systems are traced back to the year 1817 [10, Historical Introduction] and are exhibited in a myriad of problems within diverse disciplines including astronomy, hydrodynamic stability, and quantum physics [1, 5, 8]. Their study encompasses a vast body of literature as well (see, e.g., [2, 6, 8, 10] and references therein). In the case of a regular perturbation, i.e. $h \leq 0$, this system can be reduced to a set of nonhomogeneous unperturbed linear differential systems which can be solved successively. However, singular perturbations cause major complications. The difficulties are exhibited by the division of the domain $|x| \leq x_0$ into a finite number of subdomains in each of which the solutions, which have yet to be constructed, behave quite differently [3, 4]. The interest of this article is to construct an asymptotic representation of solutions in any of the corresponding subdomains.

We present an algorithm to compute the asymptotic representations of solutions of singularly-perturbed linear differential systems, in a neighborhood of a turning point. Our algorithm is based on an analysis by a Newton polygon and is implemented in the computer algebra system Maple [7].

Keywords: *Singularly-perturbed linear differential systems, turning points, Newton polygon, rank reduction, formal solutions, Maple.*

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