Algebra + Geometry $\Longrightarrow$ Differential Equation Solving

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Abstract

Consider an autonomous ODE of the form $F(y, y') = 0$, where F is a bivariate polynomial. We can think of F as defining a plane algebraic curve. If this curve admits a rational parametrization, then we can determine whether the ODE has a rational general solution. Feng and Gao have described an algorithm for this problem, based on degree bounds for such parametrizations by Sendra and Winkler.

Here we extend this investigation to the case of non-autonomous algebraic ODEs of the form $F(x, y, y') = 0$. The tri-variate polynomial F defines an algebraic surface, which we assume to admit a rational parametrization. Based on such a parametrization and on knowledge about a degree bound for rational solutions, we can determine the existence of a rational general solution, and, in the positive case, also compute one. This method depends crucially on the determination of rational invariant algebraic curves. We also relate rational general solutions of algebraic ODEs to rational first integrals.

Outline

The problem

An algebraic ordinary differential equation (AODE) is given by

$$F(x, y, y', \ldots, y^{(n)}) = 0 \ ,$$

where F is a differential polynomial in $K[x]\{y\}$ with K being a differential field and the derivation $'$ being $\frac{d}{dx}$.
Such an AODE is autonomous iff F does not depend on x; i.e., $F \in K\{y\}$.

The radical differential ideal $\{F\}$ can be decomposed

$$\{F\} \ = \ \underbrace{(\{F\} : S)}_{general \ \ component} \ \cap \ \underbrace{\{F, S\}}_{singular \ \ component} \ ,$$

where S is the separant of F (derivative of F w.r.t. $y^{(n)}$).
If F is irreducible, $\{F\} : S$ is a prime differential ideal; its generic zero is called a general solution of the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$.

J.F. Ritt, *Differential Algebra* (1950)
E. Hubert, The general solution of an ODE, *Proc. ISSAC 1996*

Problem: Rational general solution of AODE of order 1

given: an AODE $F(x, y, y') = 0$, F irreducible in $\overline{\mathbb{Q}}[x, y, y']$

decide: does this AODE have a rational general solution

find: if so, find it

Example: $F \equiv y'^2 + 3y' - 2y - 3x = 0$.
general solution: $y = \frac{1}{2}((x + c)^2 + 3c)$, where c is an arbitrary constant.
The separant of F is $S = 2y' + 3$. So the singular solution of F is $y = -\frac{3}{2}x - \frac{9}{8}$.

Rational parametrizations

An irreducible algebraic curve $\mathcal{C}$ (in the affine plane over $\mathbb{C}$) is defined as the zero locus of an irreducible polynomial $f(x, y)$; i.e.,

$$\mathcal{C} \;=\; \{(a, b) | f(a, b) = 0\} \; .$$

A rational parametrization of $\mathcal{C}$ is a pair of rational functions

$$\mathcal{P}(t) = \big(x(t), y(t)\big)$$

satisfying

$$f(x(t), y(t)) = 0 \; .$$

A curve having a rational parametrization is called a unirational curve.

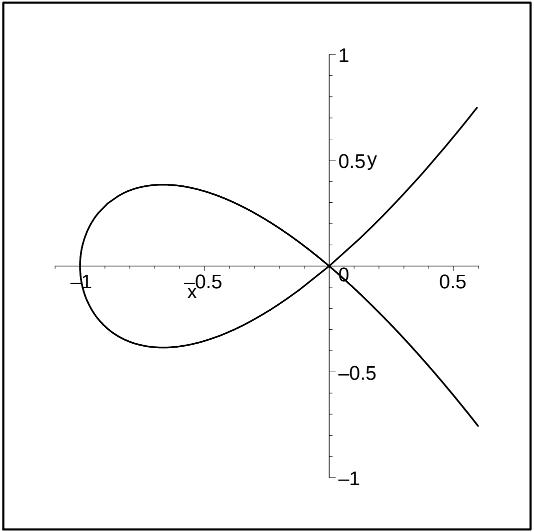
The singular cubic

$$y^2 - x^3 - x^2 = 0$$

has the rational, in fact polynomial, parametrization

$$x(t) = t^2 - 1, \quad y(t) = t^3 - t \ .$$

So this is a unirational curve.


1
0.5y
0
–1
–0.5
x
0.5
–0.5
–1

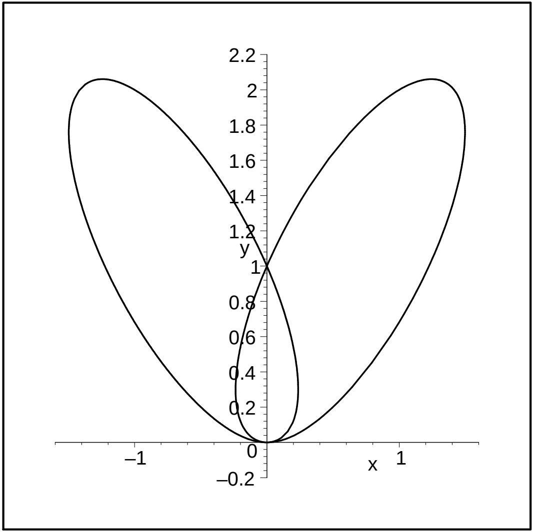
The tacnode curve defined by

$$2x^4 - 3x^2y + y^4 - 2y^3 + y^2 = 0$$

has the parametrization

$$x(t) = \frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9},$$

$$y(t) = \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9}.$$

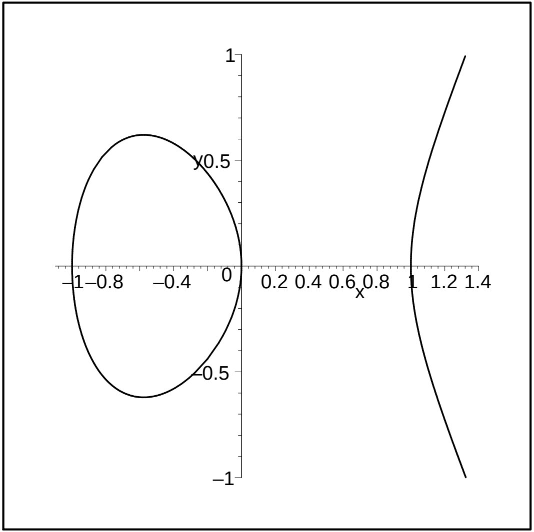


2.2
2
1.8
1.6
1.4
1.2
y
1
0.8
0.6
0.4
0.2
0
–0.2
–1
x
1

The non-singular (elliptic) cubic

$$y^2 - x^3 + x = 0$$

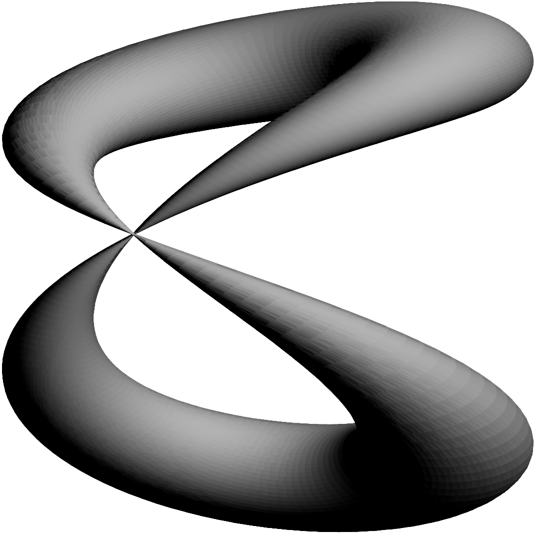
does **not** have a rational parametrization.
It is **not** unirational.



1
y0.5
0
–1–0.8
–0.4
0.2 0.4 0.6 0.8 1 1.2 1.4
x
–0.5
–1

- a parametrization of a curve (or a surface, or an algebraic variety) is a generic point or generic zero of the curve; i.e. a polynomial vanishes on the whole curve if and only if it vanishes on this generic point
- so only irreducible curves (varieties) can have a rational parametrization
- the curves having a rational parametrization are exactly the curves of genus 0

These definitions carry over to hypersurfaces in higher dimensions. For instance, the canal surface around Viviani's temple (intersection of sphere and cylinder) has a rational parametrization.



Proper parametrizations

A parametrization $\mathcal{P}(t)$ of a curve $\mathcal{C}$ is proper iff it is a birational isomorphism between the line and the curve $\mathcal{C}$ (analogously for surface parametrizations $\mathcal{P}(s,t)$); i.e. $\mathcal{P}$ has a rational inverse.
A curve with a proper parametrization is a rational curve.

- every unirational curve is rational (Lüroth)
- every unirational surface is rational (Castelnuovo)
- in dimension ≥ 4 unirationality is not equivalent to rationality

- parametrizations, indeed proper parametrizations, of curves and surfaces can be determined
- from a proper (curve) parametrization $\mathcal{P}(t)$ we get all the other parametrizations by substituting rational functions $R(t)$ for t:
$$\mathcal{P}(R(t))$$
- we know strict bounds for the degree of a proper curve parametrization in terms of the degree of the defining polynomial, and vice versa:
$$\deg(\mathcal{P}(t)) = \max\{\deg_x(f), \deg_y(f)\}$$
$$\deg(x(t)) = \deg_y(f), \quad \deg(y(t)) = \deg_x(f)$$

For details on parametrizations of algebraic curves we refer to

J.R. Sendra, F. Winkler, S. Pérez-Díaz,
Rational Algebraic Curves – A Computer Algebra Approach,
Springer-Verlag Heidelberg (2008)

Autonomous case $F(y, y') = 0$

Rational algebraic curves

First we concentrate on algebraic and geometric questions:

- The algebraic curve $\mathcal{C} : F(s,t) = 0$ is a rational curve iff there exist $(s(x), t(x))$ in $\mathbb{K}(x)^2$ (a rational parametrization) s.t.

$$F(s(x), t(x)) = 0.$$

  If a rational parametrization exists, then we can compute one. But rational parametrizations are not unique.

- Given a rational parametric curve $(s(x), t(x))$, there is a unique irreducible polynomial $F(s,t)$ such that

$$F(s(x), t(x)) = 0.$$

- One can also compute a proper rational parametrization $(s(x), t(x))$ of $F(s, t) = 0$; i.e. an invertible rational mapping and its inverse is also rational.
- If $(s(x), t(x))$ is a proper rational parametrization of $F(s, t) = 0$ and $(\overline{s}(x), \overline{t}(x))$ is another rational parametrization of $F(s, t) = 0$, then there exists a rational function $T(x)$ such that

$$(\overline{s}(x), \overline{t}(x)) = (s(T(x)), t(T(x))).$$

  So proper parametrizations are most general parametrizations.

- A rational solution of $F(y, y') = 0$ corresponds to a proper rational parametrization of the algebraic curve $F(y, z) = 0$.
- Conversely, from a proper rational parametrization $(f(x), g(x))$ of the curve $F(y, z) = 0$ we get a rational solution of $F(y, y') = 0$ if and only if there is a linear rational function $T(x)$ such that $f(T(x))' = g(T(x))$.
  If $T(x)$ exists, then a rational solution of $F(y, y') = 0$ is: $y = f(T(x))$.
  The rational general solution of $F(y, y') = 0$ is (for an arbitrary constant C): $y = f(T(x + C))$
- Feng and Gao described a complete algorithm along these lines.
  R. Feng, X-S. Gao, "Rational general solutions of algebraic ordinary differential equations", Proc. ISSAC2004. ACM Press, New York, 155-162, 2004.
  R. Feng, X-S. Gao, "A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs", J. Symb. Comp., 41, 739-762, 2006.

Remark:

- A degree bound for proper parametrizations of a rational algebraic curve is given in
  J.R. Sendra, F. Winkler, "Tracing index of rational curve parametrizations", Comp.Aided Geom.Design, 18:771–795, 2001.
  This degree bound also provides a degree bound for rational general solutions of a differential equation $F(y, y') = 0$.
- During the parametrization process we need to find regular points on the curve $\mathcal{C} : F(y, z) = 0$. The quality of these points, i.e., the necessary degree of the algebraic field extension, determines the quality of the coefficients in the parametrization and ultimately in the rational general solution of the DE $F(y, y') = 0$. For optimal field extensions see
  J.R. Sendra, F. Winkler, Parametrization of algebraic curves over optimal field extensions, J. Symb. Comp., 23, 191–207, 1997.

The general (non-autonomous) case $F(x, y, y') = 0$

- When we consider the autonomous algebraic differential equation $F(y, y') = 0$, it is necessary that $F(y, z) = 0$ is a rational curve. Otherwise, the differential equation $F(y, y') = 0$ has no non-trivial rational solution.
- It is now natural to assume that the solution surface $F(x, y, z) = 0$ is a rational algebraic surface, i.e. rationally parametrized by

  $$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

  Then $\mathcal{P}(s, t)$ creates a rational solution of $F(x, y, y') = 0$ if and only if we can find two rational functions $s(x)$ and $t(x)$ which solve the following associated system:

  $$s' = \frac{f_1(s, t)}{g(s, t)}, \quad t' = \frac{f_2(s, t)}{g(s, t)}, \tag{1}$$

where $f_1(s,t), f_2(s,t), g(s,t)$ are rational functions in s,t and defined by

$$\begin{aligned} f_1(s,t) =& \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t)\cdot\frac{\partial \chi_1(s,t)}{\partial t}, \\ f_2(s,t) =& \chi_3(s,t)\cdot\frac{\partial \chi_1(s,t)}{\partial s} - \frac{\partial \chi_2(s,t)}{\partial s}, \\ g(s,t) =& \frac{\partial \chi_1(s,t)}{\partial s}\cdot\frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t}\cdot\frac{\partial \chi_2(s,t)}{\partial s}. \end{aligned} \tag{2}$$

The system (1) is called the associated system of $F(x,y,y')=0$ with respect to $\mathcal{P}(s,t)$.

The construction of the associated system and the following theorem can be found in
L.X.C. Ngô, F. Winkler, "Rational general solutions of first order non-autonomous parametrizable ODEs", J. Symb. Comp., 45(12), 1426–1441, 2010.

Properties of the associated system:

The associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}$ has the form

$$s' = \frac{N_1(s,t)}{M_1(s,t)}, \quad t' = \frac{N_2(s,t)}{M_2(s,t)} \tag{3}$$

The corresponding polynomial system of (3) is

$$s' = N_1 M_2, \quad t' = N_2 M_1. \tag{4}$$

Theorem

There is a one-to-one correspondence between rational general solutions of the algebraic differential equation $F(x, y, y') = 0$, which is parametrized by $\mathcal{P}(s, t)$, and rational general solutions of its associated system with respect to $\mathcal{P}(s, t)$.

The associated system is

- autonomous
- of order 1
- of degree 1 in the derivatives

Solving the associated system

Lemma

Every non-trivial rational solution of the associated system (3) *corresponds to a rational algebraic curve* $G(s,t) = 0$ *satisfying*

$$G_s \cdot N_1 M_2 + G_t \cdot N_2 M_1 \in \langle G \rangle \ . \tag{5}$$

Definition

A rational algebraic curve $G(s,t) = 0$ satisfying (5) is called a rational invariant algebraic curve of the system (3).

In case the system (3), (4) has no dicritical singularities, there is an upper bound for irreducible invariant algebraic curves.
M.M. Carnicer, "The Poincaré problem in the nondicritical case", Annals of Mathematics, 140(2):289–294, 1994.

Reparametrization:

Theorem

Let $G(s,t)=0$ be a rational invariant algebraic curve of the associated system (3) *such that $G \nmid M_1$ and $G \nmid M_2$. Let $(s(x),t(x))$ be a proper rational parametrization of $G(s,t)=0$. W.l.o.g. assume $s'(x) \neq 0$.*

Then $(s(x),t(x))$ creates a rational solution of the associated system if and only if there is a linear rational function $T(x)$ such that

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T),t(T))}{M_1(s(T),t(T))}. \qquad (6)$$

In this case, $(s(T(x)),t(T(x)))$ is a rational solution of the associated system.

L.X.C. Ngô, F. Winkler, "Rational general solutions of planar rational systems of autonomous ODEs", to appear in J. Symb. Comp.

Rational general solutions

Invariant algebraic curves come in families depending on parameters. Such families give rise to rational general solutions.

Theorem

Let $\mathcal{R}(x) = (s(x), t(x))$ *be a non-trivial rational solution of the system* (3). *Let* $H(s, t)$ *be the monic defining polynomial of the curve parametrized by* $\mathcal{R}(x)$.
Then $\mathcal{R}(x)$ *is a rational general solution of the system* (3)
if and only if
the coefficients of $H(s, t)$ *contain a transcendental constant.*

associated system (3):

$$s' = \frac{N_1(s,t)}{M_1(s,t)}, \quad t' = \frac{N_2(s,t)}{M_2(s,t)}$$

A first integral is a non-constant bivariate function $W(s,t)$ satisfying

$$\frac{N_1}{M_1} W_s + \frac{N_2}{M_2} W_t = 0 \; .$$

Theorem

The associated system (3) *has a rational general solution if and only if it has a rational first integral* $\frac{U}{V} \in \mathbb{K}(s,t)$ *with* $\gcd(U,V) = 1$ *and any irreducible factor of* $U - cV$ *determines a rational solution curve for a transcendental constant* c *over* $\mathbb{K}$.

Algorithm RATSOLVE
Input: a parametrizable ODE $F(x, y, y') = 0$;
Output: a rational general solution of $F(x, y, y') = 0$, if there is one.

1. Compute a proper rational parametrization $\mathcal{P}(s, t)$ of $F(x, y, z) = 0$.
2. Compute the associated system w.r.t $\mathcal{P}(s, t)$;
3. Compute the set $\mathcal{I}$ of irreducible invariant algebraic curves of the associated system;
4. If $\mathcal{I}$ contains an irreducible invariant algebraic curve $G(s, t) = 0$ with a transcendental coefficient, then check whether $G(s, t) = 0$ is a rational curve.
5. If $G(s, t)$ is a rational curve, then parametrize this curve to find a rational general solution $(s(x), t(x))$ of the system;
6. Compute $c = \chi_1(s(x), t(x)) - x$;
7. Return $y = \chi_2(s(x - c), t(x - c))$.

Example:

Consider the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ has the parametrization

$$\mathcal{P}(s,t) = \left(\frac{t}{s} + \frac{2s+t^2}{s^2}, -\frac{1}{s} - \frac{2s+t^2}{s^2}, \frac{t}{s} \right).$$

This is a proper parametrization and its associated system is

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases}$$

irreducible invariant algebraic curves of the system:

$$G(s,t) = s, \quad G(s,t) = t^2 + 2s, \quad G(s,t) = s^2 + ct^2 + 2cs$$

The first algebraic curve $s = 0$ can be parametrized by $\mathcal{Q}(x) = (0, x)$. Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is

$$T' = T^2.$$

Hence $T(x) = -\frac{1}{x}$. Therefore, the rational solution corresponding to $G(s, t) = s$ is

$$s(x) = 0, \quad t(x) = \frac{1}{x}.$$

However, this solution does not belong to the domain of $\mathcal{P}(s, t)$. Therefore, it is not corresponding to any solution of $F(x, y, y') = 0$ parametrized by $\mathcal{P}(s, t)$.

The second algebraic curve $t^2 + 2s = 0$ can be parametrized by $\mathcal{Q}(x) = \left(-\frac{x^2}{2}, x\right)$. Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is

$$T' = \frac{1}{2}T^2.$$

Hence $T(x) = -\frac{2}{x}$. Therefore, the rational solution corresponding to $G(s,t) = t^2 + 2s$ is

$$s(x) = -\frac{2}{x^2}, \quad t(x) = -\frac{2}{x}.$$

The parametrization $\mathcal{P}(s,t)$ maps this solution to the solution $y(x) = \frac{1}{2}x^2$ of $F(x, y, y') = 0$.

The third algebraic curve $s^2 + ct^2 + 2cs = 0$ can be parametrized by

$$\mathcal{Q}(x) = \left(-\frac{2c}{1+cx^2}, -\frac{2cx}{1+cx^2} \right).$$

Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T' = 1$. Hence $T(x) = x$. So the rational solution in this case is

$$s(x) = -\frac{2c}{1+cx^2}, \quad t(x) = -\frac{2cx}{1+cx^2}.$$

Since $G(s,t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system. Therefore, the rational general solution of $F(x,y,y') = 0$ is

$$y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}(x^2 + 2cx + c^2 + 3c).$$

Generalization to higher order

this is work in progress with L.X.Chau Ngô and Yanli Huang

we give only an example

Example: Consider the differential equation

$$F(x, y, y', y'') = 3xy'' - 3yy'' + 2y'^2 - 6y' = 0.$$

The solution hypersurface $F(x, y, z, w) = 3xw - 3yw + 2z^2 - 6z = 0$ of this differential equation has a proper parametrization

$$\mathcal{P}(s_1, s_2, s_3) = (s_1 + s_2 - s_3^2,\ s_1^2 s_3 + s_2 - s_3^2,\ 3s_1 s_3,\ 6s_3)\ .$$

Therefore, the associated system of the original differential equation with respect to $\mathcal{P}$ is

$$s_1' = 1, \quad s_2' = \frac{2s_3^2}{s_1}, \quad s_3' = \frac{s_3}{s_1}$$

an invariant algebraic space curve for this system is given by the intersection of the 2 surfaces

$$H_1 = x_2 - c_1^2 s_1^2 + c_2, \quad H_2 = s_3 + c_1 s_1 \ .$$

This curve has the proper parametrization

$$(s_1(x), s_2(x), s_3(x)) = (\frac{1}{x}, \frac{c_1^2}{x^2} - c_2, -\frac{c_1}{x}) \ .$$

We check whether we can find a transformation T into a solution of the associated system:

$$T'(x) = \frac{1}{s_1'(T(x))} = -T^2(x) \ .$$

This leads to $T(x) = \frac{1}{x}$. So

$$\begin{aligned} (\hat{s}_1(x), \hat{s}_2(x), \hat{s}_3(x)) &= (s(T(x)), s(T(x)), s(T(x))) \\ &= (x,\ c_1^2 x^2 - c_2,\ -c_1 x) \end{aligned}$$

is a rational solution of the associated system.

Actually the implicit desciption of this rational solution is decribed by the Gröbner basis

$$\mathbb{G} = \{s_3 + c_1 s_1, -s_3^2 + s_2 + c_2\}$$

containing 2 independent transcendental constants in the coefficients.
So $(\hat{s}_1(x), \hat{s}_2(x), \hat{s}_3(x))$ is a rational **general** solution.
We transform it into a rational general solution of the original equation by requiring that $\mathcal{P}(\hat{s}_1, \hat{s}_2, \hat{s}_3)$ should be x:

$$\begin{aligned} y(x) &= \hat{s}_1(x+c_2)^2\hat{s}_3(x+c_2) + \hat{s}_2(x+c_2) - \hat{s}_3(x+c_2)^2 \\ &= -c_1(x+c_2)^3 - c_2 \end{aligned}$$

Classification of AODEs

joint work with L.X.C. Ngô and J.R. Sendra

- consider a group of transformations leaving the associated system of an AODE invariant; equivalence classes contain AODEs of equal complexity in terms of determining rational solutions
- we study some well-known classes of equations and relate them to this algebro-geometric approach
- it turns out that being autonomous is not an intrinsic property of an AODE; certain classes contain both autonomous and non-autonomous AODEs

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Thank you for your attention!