Computing localizations iteratively

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Weyl algebra

- Let K be a field of characteristic zero. (Think: $K = \mathbb{C}$)
- Affine space: $X = K^n$.
- Weyl algebra: an associative algebra

$$D_X = K\langle \boldsymbol{x}, \boldsymbol{\partial} \rangle = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

  where $[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1$ and all other pairs of generators commute.
- D_X is isomorphic to the algebra of linear differential operators with polynomial coefficients.
- Every element has the normal form

$$Q = \sum_{\alpha,\beta \in \mathbb{Z}^n} c_{\alpha\beta} \boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta,$$

  where finitely many of $c_{\alpha\beta} \in K$ are nonzero.

D-modules

- D_X is simple: only trivial two-sided ideals.
- Today we consider only left ideals and left D_X-modules.
- Examples of D-modules: $K[\boldsymbol{x}]$, $K[[\boldsymbol{x}]]$, $C^\infty(X)$.
- Another example: localization $K[\boldsymbol{x}, f^{-1}]$ where f is a nonzero polynomial:

$$x_i \cdot g f^{-j} = x_i g f^{-j},$$
$$\partial_i \cdot g f^{-j} = \left(\frac{\partial g}{\partial x_i} f - j g \right) f^{-j-1},$$

  for $1 \leq i \leq n$, $g \in K[\boldsymbol{x}]$, and $j \in \mathbb{Z}$.
- Software:
    - `kan/sm1` (Takayama)
    - `risa/asir` (Noro)
    - `dmod.lib, Singular` (Levandovsky et al.)
    - `D-modules, Macaulay2` (L., Tsai)

Gröbner bases

- D_X is Gröbner-friendly: D_X is an algebra of solvable type.
- Gröbner bases can be computed with respect to any w-compatible monomial order, where $w = (w_{\boldsymbol{x}}, w_{\boldsymbol{\partial}}) \in \mathbb{R}^{2n}$ satisfies $w_{\boldsymbol{x}} + w_{\boldsymbol{\partial}} \geq 0$ componentwise.

Definition (Characteristic ideal/variety)

$$\mathrm{cI}(I) = \mathrm{in}_w(I) \subset \mathrm{gr}_w(A_n) = K[x, \xi] = K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$$

Here $w = (0, e)$ assigns $w(x_i) = 0$ and $w(\partial_i) = 1$ for all i.

$$\mathrm{cV}(A_n/I) = \mathbb{V}(\mathrm{cI}(I)) \subset K^{2n}$$

Theorem (Fundamental theorem of algebraic analysis)

Let I be a nonzero left A_n-ideal, then $n \leq \dim(\mathrm{cV}(A_n/I)) \leq 2n$,

Holonomic D-modules

- An ideal (or a D-module) is called holonomic if its dimension equals n.
- Examples:
  - any nontrivial principal ideal in case $r = 1$;
  - D_X-module $R = K[\boldsymbol{x}]$;
  - D_X-module $K[\boldsymbol{\partial}]$;
  - localization $R_f = K[\boldsymbol{x}, f^{-1}]$.
- Theorem: Every holonomic D_X-module is cyclic.
- Every holonomic module M can be thought of as

$$M = D_X \cdot (\text{cyclic generator}) \cong D_X / \operatorname{Ann}_{D_X}(\text{cyclic generator}).$$

Localization

- $R_f = K[\boldsymbol{x}, f^{-1}] = D_X f^a$ for some $a \in \mathbb{Z}_{<0}$.
- Fact: largest possible a = largest root of the Bernstein-Sato polynomial $b(s) \in \mathbb{Q}[s]$, the monic polynomial of the smallest possible degree such that

$$b(s) f^s = Q(s, \boldsymbol{x}, \boldsymbol{\partial}) \cdot f^{s+1}, \text{ where } Q \in D_X[s].$$

- Ingredients of the algorithm for localization (Oaku 1997):
  1. Computation of $\operatorname{Ann}(f^s)$.
  2. Computation of $b(s)$ to determine a.
  3. Specialization: $\operatorname{Ann}(f^s)|_{s=a}$.
- Items 1 and 2 use elimination via Gröbner bases (expensive!).
- Annihilator of f^a in $K(\boldsymbol{x}) \otimes_{K[\boldsymbol{x}]} D_X$ is $\langle f\partial_i - a\frac{\partial f}{\partial x_i} \mid i = 1, \ldots, n\rangle$.

Example: planar curves

- Theorem (Kashiwara, Varchenko): a can be set to $-n+1$.
- For planar curves, $f \in K[x,y]$, localization $R_f = D_X \cdot f^{-1} = D_X / \operatorname{Ann}(f^{-1})$.
- Reiffen curves: $f = f_{p,q} = x^p + y^q + xy^{q-1} = 0$, where $p \geq 4, q \geq p+1$.

$$
\begin{aligned}
\operatorname{Ann}(f_{4,5}^{-1}) \quad = \quad & \langle 4x^2\partial_x + 5xy\partial_x + 3xy\partial_y + 4y^2\partial_y + 16x + 20y, \\
& 16xy^2\partial_x + 4y^3\partial_x + 12y^3\partial_y - 125xy\partial_x - 4x^2\partial_y + \\
& \ 5xy\partial_y - 100y^2\partial_y + 64y^2 - 500y, \\
& 16y^3\partial_x^2 - 16y^3\partial_x\partial_y + 125xy\partial_x^2 - 35xy\partial_x\partial_y + 100y^2\partial_x\partial_y + \\
& \ 12x^2\partial_y^2 - 2xy\partial_y^2 - 24y^2\partial_y^2 + 112xy\partial_x - 36y^2\partial_x + \\
& \ 84y^2\partial_y - 930x\partial_x + 625y\partial_x + 26x\partial_y - \\
& \ 893y\partial_y + 448y - 3720\rangle
\end{aligned}
$$

Iterative approach

Definition (Truncated annihilator)

$$\mathrm{Ann}^{(d)}(\ldots) = \langle Q \in D_X \, \mathrm{Ann}(\ldots) \mid \mathrm{ord}\, Q \le d \rangle$$

To compute, e.g., $\mathrm{Ann}^{(d)}(f^{-1})$ find R-syzygies for the vector of partial derivatives $(\partial^\alpha \cdot f^{-1})_\alpha$ where $|\alpha| \le d$.

Example ($\mathrm{Ann}^{(1)}(f^{-1})$ for $f = x^2 - y^3$)

$$\begin{array}{rcll} \partial_x \cdot f^{-1} & = & -2x & f^{-2} \\ \partial_y \cdot f^{-1} & = & 3y^2 & f^{-2} \\ 1 \cdot f^{-1} & = & (x^2 - y^3) & f^{-2} \end{array}$$

E.g., $3x(-2x) + 2y(3y^3) + 6(x^2 - y^3) = 0$, hence,

$$3x\partial_x + 2y\partial_y + 6 \in \mathrm{Ann}^{(1)}(f^{-1}).$$

Stopping criteria?

- $\mathrm{Ann}^{(d)}(f^{-1}) = \mathrm{Ann}(f^{-1})$ for some d. When to stop?
- Note: $\mathrm{Ann}^{(d)}(f^{-1}) = \mathrm{Ann}^{(d-1)}(f^{-1})$ does not imply $\mathrm{Ann}^{(d+1)}(f^{-1}) = \mathrm{Ann}^{(d)}(f^{-1})$.
- Project: study $\kappa(f)$ ...

Definition ("kappa" invariant)

$$\kappa(f) = \min\{d \mid \mathrm{Ann}^{(k)}(f^{-1}) = \mathrm{Ann}(f^{-1}) \text{ for all } k \geq d\}$$

κ(planar curve)

Input: $f \in R = K[x, y]$, a curve with the isolated singularity at the origin.
Output: $d = \kappa(f)$, $A = \mathrm{Ann}(f^{-1})$.

1: $d \leftarrow 0$.
2: **repeat**
3: $d \leftarrow d + 1$.
4: $A \leftarrow \mathrm{Ann}^{(d)}(f^{-1})$.
5: $\mathfrak{a} \leftarrow$ primary component of $\mathrm{cV}(A)$ corresponding to the origin.
6: **until** $\deg \mathfrak{a} =$ (multiplicity of f at the origin) $- 1$

Experiment: $\kappa(f_{p,q})$ seems to depend only on p; starting with $p = 4$, the sequence $\kappa(f_{p,*})$ is

$$2, 2, 3, 4, 4, 5, 6, 6, ...$$

Weyl closure

- Weyl closure of $I \subset D_X$ is

$$\mathrm{WeylCl}(I) = K(\boldsymbol{x}) \otimes_{K[\boldsymbol{x}]} I \cap D_X.$$

- Recall: Annihilator of f^{-1} in $K(\boldsymbol{x}) \otimes_{K[\boldsymbol{x}]} D_X$ is $\langle f\partial_i + \frac{\partial f}{\partial x_i} \,|\, i = 1, \ldots, n\rangle$.
- $\mathrm{Ann}(f^{-1}) = \mathrm{WeylCl}(\langle f\partial_i + \frac{\partial f}{\partial x_i} \,|\, i = 1, \ldots, n\rangle)$.
- Can we find the Weyl closure? Yes! (Harry Tsai, 2000)

Conclusion/Future

- Computation of the truncated annihilator could be cheaper...
- In case of a planar curve
  - there is a stopping criterion;
  - it needs to know the degree of the (embedded) component of the characteristic variety;
  - one can compute this degree numerically! (future)
- A stopping criterion in a more general setting? (future)
- Iterative algorithm for the Weyl closure? (future)