METHODS OF COMPUTATION AND INVARIANTS OF DIFFERENCE-DIFFERENTIAL DIMENSION POLYNOMIALS

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Let K be a difference-differential field of zero characteristic with basic sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivation operators and automorphisms of K, respectively (any two mappings of the set $\Delta \bigcup \sigma$ commute). Let Λ be the free commutative semigroup of all elements of the form $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ ($k_i \in \mathbf{N}$, $l_j \in \mathbf{Z}$), let the order of such an element be defined as $ord\,\lambda = \sum_{i=1}^{m} k_i + \sum_{j=1}^{n} |l_j|$, and let $\Lambda(r) = \{\lambda \in \Lambda \,|\, ord\,\lambda \leq r\}$ ($r \in \mathbf{N}$).

Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a difference-differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$.
As a field, $L = K(\{\lambda \eta_j | \lambda \in \Lambda, 1 \leq j \leq s\})$.

The following is a unified version of E. Kolchin's theorem on differential dimension polynomial and the speaker's theorem on the dimension polynomial of a difference field extension.

Theorem 1. With the above notation, there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that

(i) $\phi_{\eta|K}(r) = trdeg_K K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \leq j \leq s\})$ for all sufficiently large $r \in \mathbf{Z}$;

(ii) $\deg \phi_{\eta|K} \leq m + n$ and $\phi_{\eta|K}(t)$ can be written as $\phi_{\eta|K}(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$ where $a_0, \dots, a_{m+n} \in \mathbf{Z}$ and $2^n | a_{m+n}$.

(iii) $d = \deg \phi_{\eta|K}$, a_{m+n} and a_d do not depend on the set of difference-differential generators η of L/K ($a_d \neq a_{m+n}$ iff $d < m + n$). Moreover, $\dfrac{a_{m+n}}{2^n}$ is equal to the difference-differential transcendence degree of L over K (denoted by Δ-σ-$trdeg_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the family $\{\lambda\xi_i | \lambda \in \Lambda, 1 \leq i \leq k\}$ is algebraically independent over K.

The next result is an essential generalization of Theorem 1; it shows the existence of a dimension polynomial associated with any subextension of a finitely generated Δ-σ-field extension.

Theorem 2. (L., 2010) Let $L = K\langle\eta_1, \dots, \eta_s\rangle$, F an intermediate Δ-σ-field of L/K, and for any $r \in \mathbf{N}$, let

$$F_r = F \bigcap K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \leq j \leq s\}).$$

Then there exists a polynomial $\phi_{K,F,\eta}(t) \in \mathbf{Q}[t]$ such that

(i) $\phi_{K,F,\eta}(r) = trdeg_K F_r$ for all sufficiently large $r \in \mathbf{Z}$;

(ii) $\deg \phi_{K,F,\eta} \leq m + n$ and $\phi_{K,F,\eta}(t)$ can be written as $\phi_{K,F,\eta}(t) = \sum_{i=0}^{m+n} b_i \binom{t+i}{i}$ where $b_0, \dots, b_{m+n} \in \mathbf{Z}$ and $2^n | b_{m+n}$.

(iii) $d = \deg \phi_{K,F,\eta}(t)$, b_{m+n} and b_d do not depend on the set of Δ-σ-generators η. Furthermore, $\dfrac{b_{m+n}}{2^n} = \Delta\text{-}\sigma\text{-}trdeg_K F$.

Note that if an intermediate field E of the extension L/K is not difference-differential, there might be no polynomial whose values at sufficiently large $r \in \mathbf{Z}$ are equal to

$tr.deg_K(E \bigcap K(\{\lambda\eta_j | \lambda \in \Lambda(r), 1 \leq j \leq s\}))$

Indeed, let K be an ordinary differential field with one basic derivation δ ($\sigma = \emptyset$) , let $L = K\langle y \rangle$ be the differential field of fractions of one differential indeterminate y over K, and let $E = K(\delta^2 y, \delta^4 y, \dots, \delta^{2^k} y, \dots)$. Then

$$\omega_{K,E,y}(r) = [\log_2 r].$$

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Theorem 2 allows one to assign a dimension polynomial to every intermediate Δ-σ-field of L/K and, therefore, to a system of algebraic difference-differential equations whose solutions should be invariant with respect to the action of any (not necessarily commutative) group G commuting with basic derivations and translations. (We mean that $\tau G = G\tau$ for any $\tau \in \Delta \bigcup \sigma$ and $g(a) = a$ for any $g \in G$, $a \in K$.)
In this case the fixed field F of the group G is an intermediate Δ-σ-field of the corresponding Δ-σ-field extension L/K, and the polynomial $\phi_{K,F,\eta}(t)$ expresses A. Einstein's strength of the system.

Switching from descending chains of intermediate Δ-σ-fields to decreasing chains of the corresponding dimension polynomials, one can obtain a characterization of some invariants of a Δ-σ-dimension polynomial of L/K in the spirit of Krull-type dimension.

Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ and let $\mathfrak{U}$ denote the set of all intermediate Δ-σ fields of the extension L/K.

Let

$$\mathfrak{B}_{\mathfrak{U}} = \{(F, E) \in \mathfrak{U} \times \mathfrak{U} \mid F \supseteq E\}$$

and let $\bar{\mathbf{Z}}$ denote the ordered set $\mathbf{Z} \bigcup \{\infty\}$ (where the natural order on $\mathbf{Z}$ is extended by the condition $a < \infty$ for any $a \in \mathbf{Z}$).

Lemma 1. With the above notation, there exists a unique mapping
$\mu_{\mathfrak{U}} : \mathfrak{B}_{\mathfrak{U}} \to \bar{\mathbf{Z}}$ such that

(i) $\mu_{\mathfrak{U}}(F, E) \geq -1$ for any $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$;

(ii) If $d \in \mathbf{N}$, then $\mu_{\mathfrak{U}}(F, E) \geq d$ if and only if $tr.deg_E F > 0$ and there exists an infinite descending chain of intermediate Δ-σ-fields

$$F = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_r \supseteq \cdots \supseteq E$$

such that

$$\mu_{\mathfrak{U}}(F_i, F_{i+1}) \geq d - 1 \quad (i = 0, 1, \ldots).$$

Note that $\mu_{\mathfrak{U}}(F, E) = -1$ if and only if the field extension F/E is algebraic.

With the notation of Lemma 1, we define the Δ-σ-**transcendental type** of L/K, denoted by Δ-σ-$tr.type(L/K)$, as

$$\sup\{\mu_{\mathfrak{U}}(F,E) \mid (F,E) \in \mathfrak{B}_{\mathfrak{U}}\}.$$

Furthermore, we define the Δ–σ-**transcendence dimension** of the extension L/K as

$$\sup\{q \in \mathbf{N} \mid \text{ there exists a descending chain}$$

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_q \text{ such that } F_i \in \mathfrak{U} \text{ and}$$

$$\mu_{\mathfrak{U}}(F_{i-1},F_i) = \Delta\text{-}\sigma\text{-}tr.type(L/K) \text{ for } i = 1,\dots,q\}.$$

It is denoted by Δ-σ-$tr.dim(L/K)$.

Clearly, if Δ-σ-$tr.type(L/K) < \infty$, then Δ-σ-$tr.dim(L/K) > 0$.

Theorem 3. Let K be a Δ-σ-field of zero characteristic, $\operatorname{Card}\Delta = m$, and $\operatorname{Card}\sigma = n$. Let L be a finitely generated Δ-σ-field extension of K. Then

(i) $\Delta\text{-}\sigma\text{-}tr.type(L/K) \leq \Delta\text{-}\sigma\text{-}type_K L \leq m + n$.

(ii) If $\Delta\text{-}\sigma\text{-}tr.deg_K L > 0$, then

$\Delta\text{-}\sigma\text{-}tr.type(L/K) = m + n$ and

$\Delta\text{-}\sigma\text{-}tr.dim(L/K) = \Delta\text{-}\sigma\text{-}tr.deg_K L$.

(iii) If $\Delta\text{-}\sigma\text{-}tr.deg_K L = 0$, then $\Delta\text{-}\sigma\text{-}tr.type(L/K) < m + n$.

Let K be a difference-differential field of zero characteristic whose basic sets of derivations and automorphisms, Δ and σ, are unions of p and q disjoint sets, respectively ($p, q \geq 1$):

$$\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p$$

where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$) and

$$\sigma = \sigma_1 \bigcup \cdots \bigcup \sigma_q$$

where $\sigma_i = \{\alpha_{i1}, \dots, \alpha_{in_i}\}$ ($i = 1, \dots, q$).
In other words, we fix partitions of the sets Δ and σ.

$$(\sum_{i=1}^{p} m_i = m = Card\,\Delta,\ \sum_{j=1}^{q} n_j = m = Card\,\sigma.)$$

If $\lambda = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \alpha_{11}^{l_{11}} \dots \alpha_{1n_1}^{l_{1n_1}} \alpha_{21}^{l_{21}} \dots \alpha_{qn_q}^{l_{qn_q}} \in \Lambda$ ($k_{ij} \in \mathbf{N}$, $l_{ij} \in \mathbf{Z}$),

then the orders of λ with respect to the sets Δ_a and σ_b ($1 \leq a \leq p$, $1 \leq b \leq q$) are defined as follows:

$$\Delta\text{-}ord_a\,\lambda = \sum_{j=1}^{m_a} k_{aj}, \quad \sigma\text{-}ord_b\,\lambda = \sum_{j=1}^{n_b} |l_{bj}|.$$

If $r_1, \dots, r_{p+q} \in \mathbf{N}$, we set

$$\Lambda(r_1, \dots, r_{p+q}) = \{\lambda \in \Lambda \,|\, \Delta\text{-}ord_i\,\lambda \leq r_i,\ \sigma\text{-}ord_j\,\lambda \leq r_{p+j}\ (1 \leq i \leq p,\ 1 \leq j \leq q)\}.$$

For any permutation $(j_1, \dots, j_{p+q})$ of the set $\{1, \dots, p+q\}$, we define the lexicographic order $<_{j_1,\dots,j_{p+q}}$ on $\mathbf{N}^{p+q}$ as follows: $(r_1, \dots, r_{p+q}) <_{j_1,\dots,j_{p+q}} (s_1, \dots, s_{p+q})$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbf{N}$, $1 \leq k \leq p+q-1$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$.

In what follows, if $\Sigma \subseteq \mathbf{N}^{p+q}$, then Σ' denotes the set $\{e \in \Sigma | e$ is a maximal element of Σ with respect to one of the $(p+q)!$ lexicographic orders $<_{j_1,\dots,j_{p+q}}\}$.

Example.
Let $\Sigma = \{(3,0,2), (2,1,1), (0,1,4), (1,0,3), (1,1,6), (3,1,0), (1,2,0)\} \subseteq \mathbf{N}^3$.

Then $\Sigma' = \{(3,0,2), (3,1,0), (1,1,6), (1,2,0)\}$.

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Theorem 4. Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-σ-field extension generated by a set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\Phi_\eta(t_1, \dots, t_{p+q}) \in \mathbf{Q}[t_1, \dots, t_{p+q}]$ such that

(i) $\Phi_\eta(r_1, \dots, r_{p+q}) = trdeg_K K(\bigcup_{j=1}^{s} \Lambda(r_1, \dots, r_{p+q})\eta_j)$

for all sufficiently large $(r_1, \dots, r_{p+q}) \in \mathbf{N}^{p+q}$ (i. e., there exist $s_1, \dots, s_{p+q} \in \mathbf{N}$ such that the last equality holds for all $(r_1, \dots, r_{p+q}) \in \mathbf{N}^{p+q}$ with $r_1 \geq s_1, \dots, r_{p+q} \geq s_{p+q}$);

(ii) $deg_{t_i} \Phi_\eta \leq m_i$ $(1 \leq i \leq p)$ and $deg_{t_{m+j}} \Phi_\eta \leq n_j$ $(1 \leq j \leq q)$, so that $deg\, \Phi_\eta \leq m + n$ and $\Phi_\eta(t_1, \dots, t_{p+q})$ can be represented as

$$\Phi_\eta(t_1, \dots, t_{p+q}) =$$

$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n_1} \cdots \sum_{i_{p+q}=0}^{n_q} a_{i_1 \dots i_{p+q}} \binom{t_1 + i_1}{i_1} \cdots \binom{t_{p+q} + i_{p+q}}{i_{p+q}}$$

where $a_{i_1 \dots i_{p+q}} \in \mathbf{Z}$ and $2^n \,|\, a_{m_1 \dots m_p n_1 \dots n_q}$.

(iii) Let $E_\eta = \{(i_1, \dots, i_{p+q}) \in \mathbf{N}^{p+q} \,|\, 0 \le i_k \le m_k,\, 0 \le i_{p+j} \le n_j$ $(1 \le k \le p,\, 1 \le j \le q)$ and $a_{i_1 \dots i_{p+q}} \neq 0\}$.

Then $d = deg\,\Phi_\eta$, $a_{m_1 \dots m_p n_1 \dots n_q}$, elements $(k_1, \dots, k_{p+q}) \in E'_\eta$, the corresponding coefficients $a_{k_1 \dots k_{p+q}}$ and the coefficients of the terms of total degree d do not depend on the choice of the system of Δ-σ-generators η.

Also, $\dfrac{a_{m_1 \dots m_p n_1 \dots n_q}}{2^n} = \Delta\text{-}\sigma\text{-}tr.deg_K L$.

There are two main methods of computation of difference-differential dimension polynomials of Δ-σ-field extensions. One of them is based on constructing a characteristic set of the defining prime Δ-σ-ideal of the extension; the other approach is the computation of the dimension polynomial of the associated module of Käller differentials via (generalized) Gröbner basis method. Both approaches use total term orderings with respect to several orders defined by the partitions of the basic sets of derivations and automorphisms.

Let us consider $p+q$ total orderings $<_1, \dots, <_{p+q}$ of the set Λ such that

$$\lambda = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \alpha_{11}^{l_{11}} \dots \alpha_{1n_1}^{l_{1n_1}} \alpha_{21}^{l_{21}} \dots \alpha_{qn_q}^{l_{qn_q}} <_i$$

$$\lambda' = \delta_{11}^{k'_{11}} \dots \delta_{1m_1}^{k'_{1m_1}} \delta_{21}^{k'_{21}} \dots \delta_{pm_p}^{k'_{pm_p}} \alpha_{11}^{l'_{11}} \dots \alpha_{1n_1}^{l'_{1n_1}} \alpha_{21}^{l'_{21}} \dots \alpha_{qn_q}^{l'_{qn_q}}$$

$(1 \leq i \leq p)$ if and only if

$$(\Delta\text{-}ord_i\lambda, ord\,\lambda, \Delta\text{-}ord_1\lambda, \dots, \Delta\text{-}ord_{i-1}\lambda, \Delta\text{-}ord_{i+1}\lambda, \dots,$$

$$\Delta\text{-}ord_p\lambda, \sigma\text{-}ord_1\lambda, \dots, \sigma\text{-}ord_q\lambda, k_{i1}, \dots, k_{im_i}, k_{11}, \dots, k_{pm_p}$$

$$(\text{except for} k_{i1}, \dots, k_{im_i}), \dots, |l_{11}|, \dots, |l_{qn_q}|, l_{11}, \dots, l_{qn_q})$$

is less than the corresponding vector associated with λ' with respect to the lexicographic order on $\mathbf{N}^{m+n+p+q+1}$.

The orders $<_{p+j}$ $(1 \leq j \leq q)$ are defined in a similar way, the corresponding $(m+n+p+q+1)$-tuple begins with

$(\sigma\text{-}ord_j \lambda, ord\, \lambda, \sigma\text{-}ord_1 \lambda, \dots,)$.

Also, we say that λ divides λ' and write $\lambda \mid \lambda'$ if the n-tuples $(l_{11}, \dots, l_{qn_q})$ and $(l'_{11}, \dots, l'_{qn_q})$ belong to the same of the 2^n orthants of $\mathbf{Z}^n$ (in this case we say that λ and λ' are *similar* and write $\lambda \sim \lambda'$) and $\lambda' = \lambda\lambda''$ where $\lambda'' \sim \lambda$.

Let $L = K\langle \eta_1, \dots, \eta_s \rangle$, let $K\{y_1, \dots, y_s\}$ be the ring of Δ-σ-polynomials over K, and let ΛY denote the set of all *terms* λy_i $(\lambda \in \Lambda,\ 1 \leq i \leq s)$.

Note that as a ring, $K\{y_1, \dots, y_s\} = K[\Lambda Y]$.

Consider $p + q$ orders $<_1, \ldots, <_{p+q}$ on the set ΛY that correspond to the orders on the semigroup Λ (we use the same symbols $<_i$ for the orders on Λ and ΛY). These orders are defined as follows: $\lambda y_j <_i \lambda' y_k$ if and only if $\lambda <_i \lambda'$ in Λ or $\lambda = \lambda'$ and $j < k$.

The order of a term $u = \lambda y_k$ and its orders with respect to the sets Δ_r and σ_s ($1 \leq r \leq p$, $1 \leq s \leq q$) are defined as the corresponding orders of λ.

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If $A \in K\{y_1, \dots, y_s\} \setminus K$ and $1 \le k \le p+q$, then the highest with respect to $<_k$ term that appears in A is called the *k-leader* of the Δ-σ-polynomial A. It is denoted by $u_A^{(k)}$.

If A is written as a polynomial in $u_A^{(1)}$,

$A = I_d(u_A^{(1)})^d + I_{d-1}(u_A^{(1)})^{d-1} + \dots + I_0$
(none of the I_ks contain $u_A^{(1)}$), then I_d is called a *initial* of A; it is denoted by I_A.

If $A, B \in K\{y_1, \dots, y_s\}$, then A is said to have lower rank than B (we write $rk\, A < rk\, B$) if either $A \in K$, $B \notin K$, or $(u_A^{(1)}$, $deg_{u_A^{(1)}} A, ord_2 u_A^{(2)}, \dots, ord_{p+q} u_A^{(p+q)}) <_{lex}$ $(u_B^{(1)}, deg_{u_B^{(1)}} B, ord_2 u_B^{(2)}, \dots, ord_{p+q} u_B^{(p+q)})$ ($u_A^{(1)}$ and $u_B^{(1)}$ are compared with respect to $<_1$).

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If the vectors are equal (or $A, B \in K$) we say that A and B are of the same rank and write $rk\, A = rk\, B$.

If $A, B \in K\{y_1, \ldots, y_s\}$, then B is said to be *reduced with respect to A* if
(i) B does not contain terms $\lambda u_A^{(1)}$ ($\lambda \neq 1$) with $ord_i(\lambda u_A^{(i)}) \leq ord_i u_B^{(i)}$ ($2 \leq i \leq p + q$).

(ii) If B contains $u_A^{(1)}$, then either there exists j, $2 \leq j \leq p + q$, such that

$$ord_j u_B^{(j)} < ord_j u_A^{(j)} \text{ or } ord_j u_A^{(j)} \leq ord_j u_B^{(j)}$$

for all $j = 2, \ldots, p + q$ and $deg_{u_A^{(1)}} B < deg_{u_A^{(1)}} A$.

If $B \in K\{y_1, \ldots, y_s\}$, then B is said to be *reduced with respect to a set* $\Sigma \subseteq K\{y_1, \ldots, y_s\}$ if B is reduced with respect to every element of Σ.

A set $\Sigma \subseteq K\{y_1, \dots, y_s\}$ is called *autoreduced* if $\Sigma \bigcap K = \emptyset$ and every element of Σ is reduced with respect to any other element of this set.

Lemma 2. Every autoreduced set is finite.

Lemma 3. Let $\Sigma = \{A_1, \dots, A_d\}$ be an autoreduced set in $K\{y_1, \dots, y_s\}$ and let I_k denote the initial of A_k, and let $I(\Sigma) = \{B \in K\{y_1, \dots, y_s\} \mid B = 1$ or B is a product of finitely many elements of the form $\lambda(I_k)\}$. Then for any Δ-σ-polynomial B, there exist $B_0 \in K\{y_1, \dots, y_s\}$ and $J \in I(\Sigma)$ such that B_0 is reduced with respect to Σ, B_0 has lower rank than B and $JB \equiv B_0\,(mod[\Sigma])$ (that is, $JB - B_0 \in [\Sigma]$).

Assuming that the elements of autoreduced sets are arranged in order of increasing rank, we say that an autoreduced set $\Sigma = \{A_1, \ldots, A_r\}$ has lower rank than an autoreduced set $\Sigma' = \{B_1, \ldots, B_s\}$ if one of the following two cases holds

(1) There exists $k \in \mathbf{N}$ such that $k \leq \min\{r, s\}$, $rk\, A_i = rk\, B_i$ for $i = 1, \ldots, k-1$ and $rk\, A_k < rk\, B_k$.

(2) $r > s$ and $rk\, A_i = rk\, B_i$ for $i = 1, \ldots, s$.
If $r = s$ and $rk\, A_i = rk\, B_i$ for $i = 1, \ldots, r$, then Σ is said to have the same rank as Σ'.

Lemma 3. Every nonempty family of autoreduced sets contains an autoreduced set of lowest rank.

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If Q is an ideal of the ring $K\{y_1, \dots, y_s\}$, then an autoreduced subset of Q of lowest rank is called a **characteristic set** of this ideal.

Let P be the defining Δ-σ-ideal of a Δ-σ-field extension $L = K\langle\eta_1, \dots, \eta_s\rangle$, that is, $P = Ker(K\{y_1, \dots, y_s\} \to L)$, $y_i \mapsto \eta_i$, and let $\Sigma = \{A_1, \dots, A_d\}$ be a characteristic set of P. For any $r_1, \dots, r_{p+q} \in \mathbf{N}$, let

$U_{r_1 \dots r_{p+q}} = U'_{r_1 \dots r_{p+q}} \bigcup U''_{r_1 \dots r_{p+q}}$ where

$U'_{r_1 \dots r_{p+q}} = \{u \in \Lambda Y \,|\, ord_i u \leq r_i$ for $i = 1, \dots, p+q$ and $u_A^{(1)} \nmid u$ for all $A \in \Sigma\}$ and

$U''_{r_1 \dots r_{p+q}} = \{u \in \Lambda Y \,|\, ord_i u \leq r_i$ for $i = 1, \dots, p+q$ and for every $\lambda \in \Lambda, A \in \Sigma$ such that $u = \lambda u_A^{(1)}$, $(\lambda \sim u_A^{(1)})$, there exists $i \in \{2, \dots, p+q\}$ such that $ord_i(\lambda u_A^{(i)}) > r_i\}$.

Theorem 5. $\overline{U}_{r_1\dots r_{p+q}} = \{u(\eta) | u \in U_{r_1\dots r_{p+q}}\}$ is a transcendence basis of the field
$K(\{\lambda\eta_j \,|\, \tau \in \Lambda(r_1, \dots, r_{p+q}),\ 1 \le j \le s\})$ over K.
Thus, it remains to evaluate *Card* $U'_{r_1\dots r_{p+q}}$ and *Card* $U''_{r_1\dots r_{p+q}}$.
The first of these two numbers is the value of a numerical polynomial in $p+q$ variables associated with a subset of $\mathbf{N}^m \times \mathbf{Z}^n$ consisting of $(p+q)$-tuples $(k_1, \dots, k_p, l_1, \dots, l_q)$ such that a term of the form $\delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} y_i$ $(1 \le i \le s)$ is the leader of some $A \in \Sigma$.
Algorithms of computation of such numerical polynomials are given in [Kondrateva, M.; Levin, A.; Mikhalev, A.; Pankratev, E. Differential and Difference Dimension Polynomials. *Kluwer Acad. Publ.* 1998, Chapter 2].

As to *Card* $U''_{r_1,\dots,r_{p+q}}$, one can use the combinatorial method of inclusion and exclusion to express it as a linear combination with integer coefficients of numerical polynomials of $r_1, \dots, r_{p+q}$ of the form $\binom{t_i + m_i - a_{ij}}{m_i}, \binom{t_i + m_i - b_{ij}}{m_i}, \quad 1 \le i \le p,$

or $\binom{t_i + n_{i-p} - a_{ij}}{n_{i-p}}, \binom{t_i + n_{i-p} - b_{ij}}{n_{i-p}}, \; p+1 \le i \le p+q,$

where $a_{ij} = ord_i u^{(1)}_{A_j}$ and $b_{ij} = ord_i u^{(i)}_{A_j}$ $(1 \le j \le d)$.

Another approach to the computation of the dimension polynomial of a finitely generated Δ-σ-field extension $L = K\langle\eta_1, \dots, \eta_s\rangle$ is based on the consideration of the corresponding module of Kähler differentials $\Omega_{L|K}$.
This L-vector space can be equipped with the structure of a left module over the ring of Δ-σ-operators $\mathcal{D}$ over L consisting of all finite sums $\sum_{\lambda\in\Lambda} a_\lambda\lambda$ where $a_\lambda \in L$ and the multiplication satisfies the conditions $\delta_i a = a\delta_i + \delta_i(a)$, $\alpha_j a = \alpha_j(a)\alpha_j$, and $\alpha_j^{-1}a = \alpha_j^{-1}(a)\alpha_j^{-1}$ ($1 \le i \le m$, $1 \le j \le n$).
The corresponding action of the elements of $\Delta\bigcup\sigma$ on $\Omega_{L|K}$ is defined in such a way that $\delta(d\zeta) = d\delta(\zeta)$ and $\alpha(d\zeta) = d\alpha(\zeta)$ for any $\zeta \in L$, $\delta \in \Delta$, $\alpha \in \sigma$.

The ring $\mathcal{D}$ has a natural $(p+q)$-dimensional filtration $\{\mathcal{D}_{r_1\dots r_{p+q}} | (r_1,\dots,r_{p+q}) \in \mathbf{Z}^{p+q}\}$ where $\mathcal{D}_{r_1\dots r_{p+q}}$ is the vector L-subspace of $\mathcal{D}$ generated by $\Lambda(r_1,\dots,r_{p+q})$ if all $r_i \geq 0$, and 0 otherwise. The $\mathcal{D}$-module $M = \Omega_{L|K} = \sum_{i=1}^{s} \mathcal{D} d\eta_i$ has a natural $(p+q)$-dimensional filtration $\{M_{r_1\dots r_{p+q}} | (r_1,\dots,r_{p+q}) \in \mathbf{Z}^{p+q}\}$ where $M_{r_1\dots r_{p+q}}$ $(r_i \geq 0)$ is the vector L-subspace of $\Omega_{L|K}$ generated by the set $\{d\eta | \eta \in K(\{\lambda\eta_j \,|\, \lambda \in \Lambda(r_1,\dots,r_{p+q}), 1 \leq j \leq s\})$ and $M_{r_1\dots r_{p+q}} = 0$ whenever at least one r_i is negative. This is an excellent filtration of M in the sense that every its component is a finitely dimensional vector L-space and

$$\mathcal{D}_{r_1\dots r_{p+q}} M_{s_1\dots s_{p+q}} = M_{r_1+s_1,\dots,r_{p+q}+s_{p+q}}$$

for all sufficiently large $(s_1,\dots s_{p+q})$ and nonnegative r_i.

Also,

$$trdeg_K K(\bigcup_{j=1}^{s} \Lambda(r_1, \dots, r_{p+q})\eta_j) = dim_L(M_{r_1 \dots r_{p+q}}) \text{, for such } r_i, s_j.$$

It follows that the computation of the difference-differential dimension polynomial of the extension L/K can be reduced to the computation of the dimension polynomial associated with the filtration $\{M_{r_1 \dots r_{p+q}}\}$; the existence of such a polynomial is established by the following theorem.

Theorem 6. Let $\{M_{r_1 \dots r_{p+q}}\}$ be an excellent $(p+q)$-dimensional filtration of a left $\mathcal{D}$-module M. Then there exists a polynomial $\phi(t_1, \dots, t_{p+q}) \in \mathbf{Q}[t_1, \dots, t_{p+q}]$ such that

(i) $\phi(r_1, \dots, r_{p+q}) = dim_L M_{r_1 \dots r_{p+q}}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$.

(ii) $deg_{t_i}\phi \le m_i$ $(1 \le i \le p)$, $deg_{t_{p+j}}\phi \le n_j$ $(1 \le j \le q)$, so that $deg\,\phi \le m+n$ and $\phi(t_1, \dots, t_{p+q})$ can be represented as

$$\phi = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n_1} \dots \sum_{i_{p+q}=0}^{n_q} a_{i_1 \dots i_{p+q}} \binom{t_1+i_1}{i_1} \dots \binom{t_{p+q}+i_{p+q}}{i_{p+q}}$$

where $a_{i_1 \dots i_{p+q}} \in \mathbf{Z}$ and $2^n \mid a_{m_1 \dots m_p n_1 \dots n_q}$.

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(iii) Let $A = \{(i_1, \ldots, i_{p+q}) \in \mathbf{N}^{p+q} \,|\, 0 \leq i_k \leq m_k,\ 0 \leq i_{p+j} \leq n_j$ $(1 \leq k \leq p,\ 1 \leq j \leq q)$ and $a_{i_1 \ldots i_p} \neq 0\}$.

Then $d = deg\, \phi_\eta$, $a_{m_1 \ldots m_p n_1 \ldots n_q}$, elements $(k_1, \ldots, k_{p+q}) \in A'$, the corresponding coefficients $a_{k_1 \ldots k_{p+q}}$ and the coefficients of the terms of total degree d do not depend on the choice of the excellent filtration.

Also, $\dfrac{a_{m_1 \ldots m_p n_1 \ldots n_q}}{2^n}$ is equal to the maximal number of elements of M linearly independent over $\mathcal{D}$.

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The polynomial $\phi(t_1, \ldots, t_{p+q})$ can be computed via the technique of generalized Gröbner bases based on a type of reduction similar to the reduction considered in the above theory of characteristic sets involving several term orderings (this approach was explored in [2] and [3, Chapter 3]) or on a generalization of this approach suggested by F. Winkler and M. Zhou [4]; the corresponding algorithms were developed by Christian Dönch in [1].

1. Dönch, Christian. Bivariate Difference-Differential Dimension Polynomials and their Computation in MAPLE. *Technical Report* no. 09-19, RISC Report Series. J. Kepler University. Linz, Austria, 2009, 1- 29.

2. Levin, Alexander. Gröbner bases with respect to several orderings and multivariable dimension polynomials. *J.Symb. Comput.*, 42 (2007), no. 5, 561-578.

3. Levin, Alexander. *Difference Algebra*. Springer, 2008.

4. Zhou, Meng; Winkler, Franz. Computing difference-differential dimension polynomials by relative Gröbner bases in difference-differential modules. *Journal of Symb. Comput.*, 43 (2008), no. 10, 726-745.

Example. Let K be a differential field with a basic set $\Delta = \{\delta_1, \delta_2, \delta_3\}$ and let $L = K\langle\eta\rangle$ be a Δ-field extension with the defining equation

$$\delta_1^{a+c}\delta_2^{b}\eta + \delta_2^{a+b}\delta_3^{c}\eta + \delta_1^{a}\delta_3^{b+c}\eta = 0$$

where a, b, and c are positive integers. (It means that the defining Δ-ideal of η over K is the Δ-ideal of $K\{y\}$ generated by the Δ-polynomial $f = \delta_1^{a+c}\delta_2^{b}y + \delta_2^{a+b}\delta_3^{c}y + \delta_1^{a}\delta_3^{b+c}y$. Since f is linear, the Δ-ideal $[f]$ is prime.

Let $\Phi_\eta(t_1, t_2, t_3)$ be the dimension polynomial associated with the partition

$$\Delta = \{\delta_1\}\bigcup\{\delta_2\}\bigcup\{\delta_3\}.$$

Applying Theorem 5 (and the method of evaluation of the sets $U'_{r_1\dots r_{p+q}}$ and $U''_{r_1\dots r_{p+q}}$) we obtain that

$\Phi_\eta(t_1,t_2,t_3) = [\binom{t_1+1}{1}\binom{t_2+1}{1}\binom{t_3+1}{1} - \binom{t_1+1-(a+c)}{1}$
$\binom{t_2+1-b}{1}\binom{t_3+1}{1}] + [(t_1-(a+c)+1)a(t_3+1) + (t_1-(a+c)+$
$1)(t_2-b+1)(b+c) - (t_1-(a+c)+1)a(b+c)] =$
$(b+c)t_1t_2 + (a+b)t_1t_3 + (a+c)t_2t_3 +$ *terms of total degree* ≤ 1.
Note that the corresponding Kolchin differential dimension polynomial, which describes $tr.deg_K K(\Lambda(r)\eta)$, is as follows:

$$\phi_\eta(t) = \binom{t+3}{3} - \binom{t+3-(a+b+c)}{3} = \frac{a+b+c}{2}t^2 + \text{ terms of degree} \leq 1.$$

In this case the polynomial $\phi_\eta(t)$ carries just one differential birational invariant $a+b+c$ while $\Phi_\eta(t_1,t_2,t_3)$ determines three such invariants, $a+b, a+c$, and $b+c$, that is, Φ_η determines all three parameters of the defining equation while $\phi_\eta(t)$ gives just the sum of these parameters.

Thanks!

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