Lecture Notes on Risk Theory

February 20, 2010

Contents

Chapter 1

Introduction and basic definitions

An insurance company needs to pay claims from time to time, while collecting premiums from its customers continuously over time. We assume that it starts with an **initial (risk) reserve** $u \geq 0$ and the premium income is linear with some slope $c > 0$. At times T_n, $n \in \mathbb{N}$, a claim occurs. Thus we call T_n the nth **claim time**. Naturally, we assume that $T_1 > 0$ and $T_{n+1} > T_n$ for $n \in \mathbb{N}$. The sequence of claim times forms a counting process $\mathcal{N} = (N_t : t \geq 0)$, defined by $N_t := \max\{n \in \mathbb{N} : T_n \leq t\}$, where we define $\max \emptyset := 0$. This process $\mathcal{N}$ is called the **claim process**. The nth **claim size** is denoted by U_n, $n \in \mathbb{N}$. Then the insurance company's **risk reserve** at any time t is given by is given by

$$R_t = u + c \cdot t - \sum_{k=1}^{N_t} U_k \tag{1.1}$$

where the empty sum is defined as zero, i.e. $\sum_{k=1}^{0} U_k = 0$. Since N_t and U_k are random, $(R_t : t \geq 0)$ is a stochastic process. A typical path looks like figure 1.

We see that at time T_4 something special happens: The risk reserve R_{T_4} is negative, implying of course the ruin of the insurance company. Therefore, the stopping time

$$\tau(u) := \min\{t \geq 0 : R_t \leq 0\}$$

is called the **time of ruin**. The following questions arise:

1. The only random term on the right–hand side of equation 1.1 is the sum $\sum_{k=1}^{N_t} U_k$ representing the accumulated claims until time t. We shall study this in chapter 2.

2. An insurance company wants to insure itself against unusually high amounts of claims in any period. Some general results on this so–called reinsurance will be presented in chapter 3.

3. An obvious criterion for the viability of an insurance policy is the **ruin probability** $\psi(u) := \mathbb{P}(\tau(u) < \infty)$.

Chapter 2

Accumulated claims in a fixed time interval

Define $X := \sum_{k=1}^{N_t} U_k$ for some time $t > 0$ that shall be fixed throughout this chapter. We assume that the claim sizes U_n, $n \in \mathbb{N}$, are iid and further that N_t and all U_n are independent. Denote

$$p_k := \mathbb{P}(N_t = k)$$

for $k \in \mathbb{N}_0$ and write

$$\delta_0(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

for the distribution function of the Dirac measure on zero. Let Q denote the distribution function of U_1. Further denote the distribution function of the sum of a fixed number of claims by

$$R_k(x) := \mathbb{P}(U_1 + \ldots + U_k \leq x)$$

for $k \in \mathbb{N}$ and $x \geq 0$. In order to obtain R_k in terms of Q, we define

$$Q^{*0} := \delta_0, \qquad Q^{*1} := Q, \qquad \text{and} \qquad Q^{*n+1}(x) := \int_0^x Q^{*n}(x-y)dQ(y)$$

for $n \in \mathbb{N}$. The distribution function Q^{*n} is called the nth **convolutional power** of Q.

Lemma 2.1 $R_k = Q^{*k}$ *for all* $k \in \mathbb{N}$.

Proof: For $k = 1$ the statement holds by definition. The induction step from k to $k + 1$ is straightforward:

$$\begin{aligned} R_{k+1}(x) &= \mathbb{P}(U_1 + \ldots + U_{k+1} \leq x) \\ &= \int_0^x \mathbb{P}(U_1 + \ldots + U_k \leq x - y) dQ(y) \quad \text{by conditioning on } U_{k+1} = y \\ &= \int_0^x Q^{*k}(x-y) dQ(y) \qquad \text{by induction hypothesis} \\ &= Q^{*k+1}(x) \qquad \text{by definition} \end{aligned}$$

□

Theorem 2.1 $\mathbb{P}(X \leq x) = \sum_{k=0}^{\infty} p_k Q^{*k}(x)$

Proof:

$$\begin{aligned} \mathbb{P}(X \leq x) &= \sum_{k=0}^{\infty} \mathbb{P}(X \leq x, N_t = k) = \sum_{k=0}^{\infty} p_k \mathbb{P}(X \leq x | N_t = k) \\ &= \sum_{k=0}^{\infty} p_k Q^{*k}(x) \end{aligned}$$

by definition of $X = \sum_{k=1}^{N_t} U_k$ and lemma 2.1.
□

With $p := 1 - p_0$ and

$$\bar{Q} := \sum_{k=1}^{\infty} \frac{p_k}{p} Q^{*k}$$

we can write

$$\mathbb{P}(X \leq x) = (1-p)\delta_0(x) + p\bar{Q}(x)$$

for $x \geq 0$. The problem remains of course to determine a simple expression for $\bar{Q}(x)$. This is possible in the following

Example 2.1 Assume that N_t has a geometric distribution, i.e. $p_k = (1-p)p^k$ for all $k \in \mathbb{N}_0$, and $Q = Exp(\mu)$. Then $Q^{*k} = Erl_k(\mu)$, with density function

$$f_k(x) = \mu \frac{(\mu x)^{k-1}}{(k-1)!} e^{-\mu x}$$

for $x \geq 0$ (proof as an exercise). Further, $\bar{Q}$ has the density function

$$\begin{aligned} f(x) = \sum_{k=1}^{\infty} \frac{p_k}{p} \mu \frac{(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} &= \sum_{k=1}^{\infty} (1-p) p^{k-1} \mu \frac{(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} \\ &= (1-p)\mu e^{-\mu x} \sum_{k=1}^{\infty} \frac{(p\mu x)^{k-1}}{(k-1)!} = (1-p)\mu e^{-\mu x} e^{p\mu x} \\ &= (1-p)\mu e^{-(1-p)\mu x} \end{aligned}$$

for all $x \geq 0$. Hence $\bar{Q} = Exp((1-p)\mu)$ and

$$\mathbb{P}(X \leq x) = (1-p)\delta_0(x) + p(1 - e^{-(1-p)\mu x}) = 1 - pe^{-(1-p)\mu x}$$

for $x \geq 0$.

Theorem 2.2 $\mathbb{E}(X) = \mathbb{E}(N_t) \cdot \mathbb{E}(U_1)$

Proof:

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(\sum_{k=1}^{N_t} U_k \right) = p_0 \cdot 0 + \sum_{k=1}^{\infty} p_k \cdot \mathbb{E}(U_1 + \ldots + U_k) = \sum_{k=1}^{\infty} p_k \cdot k\mu \\ &= \mathbb{E}(N_t) \cdot \mathbb{E}(U_1) \end{aligned}$$

□

Theorem 2.3

$$\begin{aligned} \mathbb{V}ar(X) &= \mathbb{E}(N_t) \cdot \mathbb{V}ar(U_1) + \mathbb{V}ar(N_t) \cdot \mathbb{E}(U_1)^2 \\ &\leq \mathbb{E}(U_1^2) \cdot \max(\mathbb{E}(N_t), \mathbb{V}ar(N_t)) \end{aligned}$$

Proof:

$$\mathbb{V}ar(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sum_{k=1}^{\infty} p_k \cdot \mathbb{E}\left(\left(\sum_{n=1}^{k} U_n \right)^2 \right) - \mathbb{E}(N_t)^2 \cdot \mathbb{E}(U_1)^2$$

according to theorem 2.2. For the expectation in the first term, we obtain

$$\mathbb{E}\left(\left(\sum_{n=1}^{k} U_n \right)^2 \right) = \mathbb{E}\left(\sum_{n=1}^{k} U_n^2 + \sum_{i \neq j} U_i U_j \right) = \sum_{n=1}^{k} \mathbb{E}(U_n^2) + \sum_{i \neq j} \mathbb{E}(U_i U_j)$$

Since all U_n are iid, we arrive at

$$\mathbb{E}\left(\left(\sum_{n=1}^{k} U_n\right)^2\right) = k \cdot \mathbb{E}(U_1^2) + k(k-1)\mathbb{E}(U_1)^2$$

Hence

$$\begin{aligned}
\mathbb{V}ar(X) = &\sum_{k=1}^{\infty} p_k \cdot k \cdot \left(\mathbb{E}(U_1^2) - \mathbb{E}(U_1)^2\right) + \sum_{k=1}^{\infty} p_k \cdot k^2 \cdot \mathbb{E}(U_1)^2 \\
&- \mathbb{E}(N_t)^2 \cdot \mathbb{E}(U_1)^2 \\
= &\ \mathbb{E}(N_t) \cdot \mathbb{V}ar(U_1) + \mathbb{V}ar(N_t) \cdot \mathbb{E}(U_1)^2
\end{aligned}$$

The inequality follows from here via $\mathbb{E}(U_1^2) = \mathbb{V}ar(U_1) + \mathbb{E}(U_1)^2$.
□

Example 2.2 If $\mathcal{N}$ is a Poisson process, then $\mathbb{E}(N_t) = \mathbb{V}ar(N_t)$ and $\mathbb{V}ar(X) = \mathbb{E}(N_t) \cdot \mathbb{E}(U_1^2)$.

Chapter 3

Reinsurance

Let X denote the random variable of the accumulated claims after some fixed time $t > 0$ (e.g. after one year). X is also called the **risk** at time t. Let F denote the distribution function of X.

A **reinsurance policy** is a function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $0 \leq R(x) \leq x$ for all $x \in \mathbb{R}_+$. In the event $\{X = x\}$ the insurance company pays $R(x)$ of the claims and the resinsurer pays the rest, $x - R(x)$.

We assume first that the reinsurance premium is the **fair net premium**

$$P = P(R) = \mathbb{E}(X - R(X))$$

without safety loading. A special reinsurance policy is the **stop loss reinsurance** at price P, defined as

$$R^*(x) = \begin{cases} x, & x \leq M \\ M, & x > M \end{cases}$$

where M is the unique solution of

$$P = \int_M^\infty (x - M) dF(x)$$

M is called the **deductible**.

Example 3.1 Assume that $X \sim F := (1-p) \cdot \delta_0 + p \cdot Exp((1-p)\mu)$ as in example 2.1. Set $\beta := (1-p)\mu$. Then $dF(x) = p \cdot \beta e^{-\beta x} dx$ for $x > 0$ and

$$P = \int_M^\infty (x - M) dF(x) = p \cdot \left(\int_M^\infty x \beta e^{-\beta x} dx - M \int_M^\infty \beta e^{-\beta x} dx \right)$$

Integration by parts yields for the first integral

$$\left[x \cdot (-1) \cdot e^{-\beta x}\right]_M^\infty + \int_M^\infty e^{-\beta x} dx = M \cdot e^{-\beta M} + \frac{1}{\beta} e^{-\beta M}$$

As the second integral equals $e^{-\beta M}$, we obtain

$$P = p \cdot \frac{1}{\beta} e^{-\beta M}$$

from which the deductible is determined as

$$M = -\frac{1}{\beta} \ln \left(P \cdot \frac{\beta}{p} \right)$$

Thus the deductible has a meaningful (i.e. positive) value if and only if the condition

$$P < p \cdot \frac{1}{\beta} = \mathbb{E}(X)$$

holds. This is reasonable, as for larger premiums $P > \mathbb{E}(X)$ the insurance company would do better to cover the risk itself.

Theorem 3.1 *For a fixed fair net premium P the stop loss reinsurance yields the smallest variance $\mathbb{V}ar(R(X))$ of all reinsurance policies.*

Proof: Let R denote any reinsurance policy with fair net premium P. Define $P_1 := \mathbb{E}(X) - P = \mathbb{E}(R(X))$ and note that $P_1 = \mathbb{E}(R^*(X))$, too. Then

$$\begin{aligned} \mathbb{V}ar(R(X)) + P_1^2 &= \int_0^\infty R^2(x) dF(x) \\ &= \int_0^\infty (R(x) - M)^2 dF(x) + 2MP_1 - M^2 \\ &\geq \int_0^M (R(x) - M)^2 dF(x) + 2MP_1 - M^2 \end{aligned}$$

Over the range $0 < x < M$ we obtain for the integrand

$$(R(x) - M)^2 = (M - R(x))^2 \geq (M - x)^2 = (M - R^*(x))^2 = (R^*(x) - M)^2$$

As $R^*(x) = M$ for $x > M$, we further get

$$\mathbb{V}ar(R(X)) + P_1^2 \geq \int_0^\infty (R^*(x) - M)^2 dF(x) + 2MP_1 - M^2 = \mathbb{V}ar(R^*(X)) + P_1^2$$

which implies the statement.
□

Now assume that the reinsurance premium is of the form

$$P'(R) = \mathbb{E}(X - R(X)) + f(\mathbb{V}ar(X - R(X)))$$

where f is some positive increasing function. The term $f(\mathbb{V}ar(X - R(X)))$ is called the **safety loading**.

Theorem 3.2 *Given a fixed value $V > 0$ and the condition $\mathbb{V}ar(R(X)) = V$, the cheapest reinsurance policy R^* for the premium P' is given by*

$$R^*(x) = \sqrt{\frac{V}{\mathbb{V}ar(X)}} \cdot x$$

for all $x \geq 0$.

Remark 3.1 The term "cheapest" is to be understood in the following sense: After the time period $[0, t]$, the insurance company will have to pay the premium $P'(R)$ to the reinsurer and $R(X)$ as its own contribution to cover the accumulated claims. Altogether, the expected amount of this is

$$\mathbb{E}(X - R(X)) + f(\mathbb{V}ar(X - R(X))) + \mathbb{E}(R(X)) = \mathbb{E}(X) + f(\mathbb{V}ar(X - R(X)))$$

The goal is to minimise this expectation.

Proof: By the above remark it suffices to minimise $f(\mathbb{V}ar(X - R(X)))$ and hence $\mathbb{V}ar(X - R(X))$, as f is increasing. For this we can write

$$\mathbb{V}ar(X - R(X)) = \mathbb{V}ar(X) + \mathbb{V}ar(R(X)) - 2 \cdot Cov(X, R(X))$$

which shows that we need to maximise the covariance $Cov(X, R(X))$. This is achieved by the form $R(X) = a \cdot X$ for a constant a. Now the condition $\mathbb{V}ar(R(X)) = V$ yields

$$V = a^2 \cdot \mathbb{V}ar(X), \qquad \text{i.e.} \quad a = \sqrt{\frac{V}{\mathbb{V}ar(X)}}$$

which is the statement.
□

Chapter 4

Risk processes in discrete time

Let X_n denote the accumulated claims in the time interval $]n-1,n]$, $n \in \mathbb{N}$ (e.g. the nth year). We assume that the random variables X_n, $n \in \mathbb{N}$, are iid. As in chapter 1, the initial reserve and the rate of premium income are denoted by $u \geq 0$ and $c > 0$. The random variable

$$K_n := u + c \cdot n - \sum_{k=1}^{n} X_k$$

is called the **risk reserve** at time $n \in \mathbb{N}$. Then the **survival probability** until time m (with initial reserve u) is defined as

$$\bar{\psi}(u,m) := \mathbb{P}(K_n \geq 0 \; \forall n \leq m)$$

Remark 4.1 Let $X_1 \sim F$. Conditioning on $\{X_1 = x\}$ yields the recursive formula

$$\bar{\psi}(u,m+1) = \int_0^{u+c} \bar{\psi}(u+c-x,m) dF(x)$$

for all $u > 0$ and $m \in \mathbb{N}$. However, this is in general numerically intractable.

The **survival probability** over infinite time is defined as

$$\bar{\psi}(u) := \lim_{m \to \infty} \bar{\psi}(u,m)$$

Clearly, $\bar{\psi}(u,m)$ is decreasing in m and bounded below by zero, such that the limit $\bar{\psi}(u)$ does exist.

If we define $Y_n := X_n - c$ as the net claim in $]n-1,n]$ and $S_n := \sum_{k=1}^{n} Y_k$ for all $n \in \mathbb{N}_0$, then the Y_n are iid (because the X_n are so and c is constant) and thus $\mathcal{S} := (S_n : n \in \mathbb{N}_0)$ is a **random walk**. Y_n is called the nth **increment** of the random walk. For technical reasons (needed in the third statement of 4.1), we exclude the trivial case by assuming that $\mathbb{P}(X_n = c) < 1$, i.e. $\mathbb{P}(Y_n = 0) < 1$, for all $n \in \mathbb{N}$.

Theorem 4.1 *Let $(S_n : n \in \mathbb{N}_0)$ be a random walk with increments Y_n. Then*

$$\begin{aligned}
\mathbb{E}(Y_1) > 0 \quad &\Rightarrow \quad \lim_{n\to\infty} S_n = \infty \\
\mathbb{E}(Y_1) < 0 \quad &\Rightarrow \quad \lim_{n\to\infty} S_n = -\infty \\
\mathbb{E}(Y_1) = 0 \quad &\Rightarrow \quad \limsup_{n\to\infty} S_n = \infty, \; \liminf_{n\to\infty} S_n = -\infty
\end{aligned}$$

where the implications hold almost certainly, i.e. with probability one.

Proof: The first two statements are immediate consequences of the strong law of large numbers. For a proof of the third statement, see Rolski et al. [3], p.234f.
□

We say that the random walk has a positive (resp. negative) **drift** if $\mathbb{E}(Y_1) > 0$ (resp. $\mathbb{E}(Y_1) < 0$), and that it is **oscillating** if $\mathbb{E}(Y_1) = 0$. For the remainder of this chapter we assume a negative drift, i.e. the **net profit condition** $\mathbb{E}(X_1) < c$. Define

$$M := \sup\{S_n : n \in \mathbb{N}_0\}$$

and note that theorem 4.1 states that $M < \infty$ almost certainly. The crucial connection to our risk process is now the observation

$$\bar{\psi}(u) = \mathbb{P}(M \leq u)$$

Define the (first strong) ascending **ladder epoch** by

$$\nu^+ := \min\{n \in \mathbb{N} : S_n > 0\}$$

with the understanding that $\min \emptyset := \infty$. The random variable ν^+ is a **stopping time** with respect to the sequence $\mathcal{Y} = (Y_n : n \in \mathbb{N})$, i.e.

$$\mathbb{P}(\nu^+ = n | \mathcal{Y}) = \mathbb{P}(\nu^+ = n | Y_1, \ldots, Y_n)$$

for all $n \in \mathbb{N}$. This means we do not need any future information to determine the event $\{\nu^+ = n\}$.

The corresponding random variable

$$L^+ := \begin{cases} S_{\nu^+}, & \nu^+ < \infty \\ \infty & \text{otherwise} \end{cases}$$

is called the (first strong) ascending **ladder height**. Denote the distribution function of L^+ by $G^+(x) := \mathbb{P}(L^+ \le x)$ and note that

$$G^+(\infty) := \lim_{x\to\infty} G^+(x) = \mathbb{P}(L^+ < \infty)$$

may be strictly less than one.

Theorem 4.2 *If* $\mathbb{E}(Y_1) < 0$*, then* $\mathbb{E}(\nu^+) = \infty$.

Proof: Assume that $\mathbb{E}(\nu^+) < \infty$. Then Wald's lemma (see lemma 4.1 below) applies to

$$\mathbb{E}(L^+) = \mathbb{E}\left(\sum_{n=1}^{\nu^+} Y_n\right) = \mathbb{E}(\nu^+) \cdot \mathbb{E}(Y_1)$$

But $\mathbb{E}(L^+) > 0$, while $\mathbb{E}(\nu^+) > 0$ and $\mathbb{E}(Y_1) < 0$, which leads to a contradiction. Hence $\mathbb{E}(\nu^+) = \infty$.
□

Lemma 4.1 Wald's Lemma *Let* $\mathcal{Y} = (Y_n : n \in \mathbb{N})$ *be a sequence of iid random variables and assume that* $\mathbb{E}(|Y_1|) < \infty$*. Let* S *be a stopping time for the sequence* $\mathcal{Y}$ *with* $\mathbb{E}(S) < \infty$*. Then*

$$\mathbb{E}\left(\sum_{n=1}^{S} Y_n\right) = \mathbb{E}(S) \cdot \mathbb{E}(Y_1)$$

Proof: For all $n \in \mathbb{N}$ define the random variables $I_n := 1$ on the set $\{S \ge n\}$ and $I_n := 0$ else. Then

$$\mathbb{E}\left(\sum_{n=1}^{S} Y_n\right) \le \mathbb{E}\left(\sum_{n=1}^{S} |Y_n|\right) = \mathbb{E}\left(\sum_{n=1}^{\infty} I_n|Y_n|\right) = \sum_{n=1}^{\infty} \mathbb{E}(I_n|Y_n|)$$

by monotone convergence, as $I_n|Y_n| \ge 0$ for all $n \in \mathbb{N}$. Since S is a stopping time for $\mathcal{Y}$, we obtain $\mathbb{P}(S \ge 1) = 1$ and further

$$\mathbb{P}(S \ge n|\mathcal{Y}) = 1 - \mathbb{P}(S \le n-1|\mathcal{Y}) = 1 - \mathbb{P}(S \le n-1|Y_1, \ldots, Y_{n-1})$$

for all $n \geq 2$. Since the Y_n are independent, the equality above implies that I_n and $|Y_n|$ are independent, too. This yields

$$\mathbb{E}(I_n|Y_n|) = \mathbb{E}(I_n) \cdot \mathbb{E}(|Y_n|) = \mathbb{P}(S \geq n) \cdot \mathbb{E}(|Y_1|)$$

for all $n \in \mathbb{N}$. Now the relation $\sum_{n=1}^{\infty} \mathbb{P}(S \geq n) = \mathbb{E}(S)$ yields

$$\mathbb{E}\left(\sum_{n=1}^{S} Y_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(S \geq n) \cdot \mathbb{E}(|Y_1|) = \mathbb{E}(S) \cdot \mathbb{E}(|Y_1|) < \infty$$

Now we can use dominated convergence to obtain

$$\begin{aligned}\mathbb{E}\left(\sum_{n=1}^{S} Y_n\right) &= \mathbb{E}\left(\sum_{n=1}^{\infty} I_n Y_n\right) = \sum_{n=1}^{\infty} \mathbb{E}(I_n Y_n) = \sum_{n=1}^{\infty} \mathbb{E}(I_n)\mathbb{E}(Y_n) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(S \geq n) \cdot \mathbb{E}(Y_1) = \mathbb{E}(S) \cdot \mathbb{E}(Y_1)\end{aligned}$$

□

In the case $\nu^+ < \infty$ of a finite ladder epoch, we can define a new random walk starting at (ν^+, L^+) by $\mathcal{S}' = ((S_{\nu^+ + n} - L^+) : n \in \mathbb{N}_0)$. As the increments are iid, $\mathcal{S}'$ is independent from $(S_1, \ldots, S_{\nu^+})$ and the random walks $\mathcal{S}$ and $\mathcal{S}'$ have the same distribution. Doing this iteratively, we arrive at the following definitions:

Let $\nu_0^+ := 0$ and $\nu_{n+1}^+ := \min\{k > \nu_n^+ : S_k > S_{\nu_n^+}\}$ for all $n \in \mathbb{N}_0$, where $\min \emptyset := \infty$. The stopping time ν_n^+ is called the nth (strong ascending) ladder epoch. Correspondingly, we define the nth (strong ascending) ladder height as

$$L_n^+ := \begin{cases} S_{\nu_n^+} - S_{\nu_{n-1}^+}, & \nu_n^+ < \infty \\ \infty & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. By construction the ladder heights are iid. Defining now $N := \max\{n \in \mathbb{N} : \nu_n^+ < \infty\}$ we obtain the representation

$$M = \sum_{k=1}^{N} L_k^+$$

Remark 4.2 The net profit condition $\mathbb{E}(Y_1) < 0$ implies that $M < \infty$ almost certainly, see theorem 4.1. Since the $(L_k^+ : k \in \mathbb{N})$ are iid, this entails $N < \infty$ almost certainly. Hence we can conclude that $G^+(\infty) < 1$.

Theorem 4.3 *If* $\mathbb{E}(Y_1) < 0$*, then*

$$\mathbb{P}(M \le u) = (1-p) \cdot \sum_{k=0}^{\infty} (G^+)^{*k}(u) = \sum_{k=0}^{\infty} (1-p) \cdot p^k G_0^{*k}(u)$$

with $p := G^+(\infty)$ *and*

$$G_0(u) := \frac{1}{p} G^+(u) = \mathbb{P}(L^+ \le u | \nu^+ < \infty)$$

for all $u \ge 0$*.*

Proof: By definition, $p = \mathbb{P}(\nu^+ < \infty)$. The construction of the ladder epochs yields

$$\mathbb{P}(N = k) = \mathbb{P}(\nu_n^+ < \infty \;\; \forall\, n \le k, \nu_{k+1}^+ = \infty) = p^k \cdot (1-p)$$

Hence

$$\mathbb{P}(M \le u) = \mathbb{P}\left(\sum_{n=1}^{N} L_n^+ \le u \right) = \sum_{k=0}^{\infty} (1-p) \cdot p^k \cdot \mathbb{P}\left(\sum_{n=1}^{k} L_n^+ \le u \,\middle|\, \nu_k^+ < \infty \right)$$

by conditioning on $\{N = k\}$. The statement follows with the observation that

$$G_0^{*k}(u) = \mathbb{P}\left(\sum_{n=1}^{k} L_n^+ \le u \,\middle|\, \nu_k^+ < \infty \right)$$

□

Likewise, the ruin probability is given as $\psi(u) = \mathbb{P}(M > u)$ and thus

Corollary 4.1 *If the net profit condition* $\mathbb{E}(Y_1) < 0$ *holds, then*

$$\psi(u) = \sum_{k=1}^{\infty} (1-p) \cdot p^k \bar{G_0^{*k}}(u)$$

where $\bar{G_0^{*k}}(u) = 1 - G_0^{*k}(u)$*.*

This corollary is known as **Beekman's formula** in risk theory and as the **Pollaczek–Khinchin formula** in queueing theory.

Exercise 4.1 Show that $\psi(0) = p$ and $\lim_{u \to \infty} \psi(u) = 0$.

Chapter 5

The Adjustment Coefficient

Let P denote a probability measure with $\mathbb{E}(P) := \int x dP(x) \in]-\infty, 0[$. Assume that the moment generating function $m_P(\alpha) := \int e^{\alpha x} dP(x)$ satisfies $m_P(\alpha) < \infty$ for some $\alpha > 0$ and $\lim_{\alpha \to s-} m_P(\alpha) \geq 1$ for $s := \sup\{\alpha > 0 : m_P(\alpha) < \infty\}$. Then the solution $\gamma > 0$ of $m_P(\gamma) = 1$ is called the **adjustment coefficient** of P.

Remark 5.1 The adjustment coefficient does exist under the given assumptions, because $m_P(0) = 1$, $m'_P(0) < 0$, and $m_P(\alpha)$ is strictly convex as $m''_P(\alpha) > 0$ for all $\alpha < s$. The number γ is the only positive number such that $e^{\gamma x} dP(x)$ is a probability measure.

Theorem 5.1 *Lundberg inequality*
Let $(Y_n : n \in \mathbb{N})$ denote a sequence of independent random variables with identical distribution P, and define

$$\psi(u) := \mathbb{P}\left(\sup\left\{ \sum_{i=1}^{m} Y_i : m \in \mathbb{N} \right\} > u \right)$$

for $u \geq 0$. If $\mathbb{E}(Y_1) < 0$ and γ is the adjustment coefficient of P, then

$$\psi(u) \leq e^{-\gamma u}$$

for all $u \geq 0$.

Proof: Define

$$\psi_n(u) := \begin{cases} \mathbb{P}\left(\sup\left\{\sum_{i=1}^{m} Y_i : m \leq n\right\} > u\right), & u \geq 0 \\ 1, & u < 0 \end{cases}$$

for all $n \in \mathbb{N}$. We will show that $\psi_n(u) \leq e^{-\gamma u}$ for all $u \in \mathbb{R}$ by induction on n. Since $\psi(u) = \lim_{n\to\infty} \psi_n(u)$, this will yield the statement.

For $u \leq 0$ and all $n \in \mathbb{N}$, the bound holds by definition. Let $n = 1$ and $u > 0$. Then (see exercise below)

$$\begin{aligned}\psi_1(u) &= \mathbb{P}(Y_1 > u) = \mathbb{P}(\gamma Y_1 > \gamma u)\\ &\leq e^{-\gamma u}\mathbb{E}(e^{\gamma Y_1}) = e^{-\gamma u}\end{aligned}$$

Independence of the Y_n, $n \in \mathbb{N}$, yields further

$$\begin{aligned}\psi_{n+1}(u) &= 1 - \int_{-\infty}^{u} 1 - \psi_n(u-y)\, dP(y)\\ &= 1 - P(]-\infty, u]) + \int_{-\infty}^{u} \psi_n(u-y)\, dP(y)\\ &= \int_{-\infty}^{\infty} \psi_n(u-y)\, dP(y)\\ &\leq \int_{-\infty}^{\infty} e^{-\gamma(u-y)}\, dP(y) = e^{-\gamma u}\end{aligned}$$

where the third equality holds by definition of ψ_n, the inequality by induction hypothesis, and the last equality by the fact that $\mathbb{E}(e^{\gamma Y_1}) = 1$.
□

Exercise 5.1 Show that for any real–valued random variable X, a real number $a \in R$, and a positive and increasing function f the inequality

$$\mathbb{P}(X > a) \leq \frac{\mathbb{E}(f(X))}{f(a)}$$

holds.

Remark 5.2 A distribution P is called **heavy–tailed** if its moment generating function does not exist, i.e. if $\int e^{\alpha x} dP(x)$ does not converge for any $\alpha > 0$. If P is heavy–tailed, then it does not have an adjustment coefficient and Lundberg's inequality does not apply.

Chapter 6

Some risk processes in continuous time

6.1 The Sparre Andersen model

Our basic assumptions for this chapter are:

1. The claim sizes U_n, $n \in \mathbb{N}$, are iid with common distribution function F.
2. The claim arrival process $\mathcal{N}$ is an ordinary renewal process, which means that the inter–claim times $W_n := T_n - T_{n-1}$, $n \in \mathbb{N}$, with $T_0 := 0$, are iid with common distribution function G.
3. Claim sizes and inter–claim times are independent.

As in chapter 1, we denote the rate of premium income by $c > 0$. Define

$$Y_n := U_n - cW_n \qquad \text{and} \qquad S_n := \sum_{k=1}^{n} Y_k$$

for $n \in \mathbb{N}$. Then the process $\mathcal{S} = (S_n : n \in \mathbb{N})$ is a random walk (cf. chapter 4). Since $c > 0$, ruin can occur only at claim epochs T_n, whence we obtain for the ruin time

$$\tau(u) = \min\{t > 0 : R_t < 0\} = \min\{T_n : R_{T_n} = u - S_n < 0\}$$

This implies for the ruin probability

$$\psi(u) = \mathbb{P}\left(M := \sup_{n\in\mathbb{N}} S_n > u\right)$$

meaning that all results from chapter 4 apply to the Sparre Andersen model. Further results using a random walk analysis can be found in [2].

In order to determine the drift of the random walk $\mathcal{S}$, we observe that

$$\begin{aligned} & \mathbb{E}(Y_1) = \mathbb{E}(U_1) - c \cdot \mathbb{E}(W_1) < 0 \\ \Leftrightarrow \quad & \rho := \frac{\mathbb{E}(U_1)}{c \cdot \mathbb{E}(W_1)} < 1 \\ \Leftrightarrow \quad & \eta := \frac{1}{\rho} - 1 = \frac{1}{\mathbb{E}(U_1)} \cdot (c \cdot \mathbb{E}(W_1) - \mathbb{E}(U_1)) > 0 \end{aligned}$$

The number η is called the **relative safety loading**, while ρ is called the **system load**, a name that originated in queueing theory. The inequality

$$\mathbb{E}(U_1) - c \cdot \mathbb{E}(W_1) < 0$$

is called the **net profit condition**. Clearly, the random walk $\mathcal{S}$ has a negative drift if and only if the net profit condition holds.

6.2 The compound Poisson model

This is a special case of the Sparre Andersen model with $G(t) := 1 - e^{-\lambda t}$ for $t \geq 0$, where $\lambda > 0$. Thus the claim arrival process $\mathcal{N}$ is a Poisson process with intensity λ.

Remark 6.1 It can be shown (see Doob [1]) that $\mathcal{N}$ is necessarily a Poisson process if we postulate the following properties:

1. $N_0 = 0$ with probability one.

2. $\mathcal{N}$ has independent increments, i.e. $N_t - N_s$ and $N_v - N_u$ are independent for all $0 \leq s < t \leq u < v$.

3. $\mathcal{N}$ has stationary increments, i.e. $N_{s+t} - N_t$ and N_t have the same distribution for all $s > 0$.

4. There are no double events. More exactly, we postulate that

$$\mathbb{P}(N_h \geq 2) = o(h), \quad \text{i.e.} \quad \frac{1}{h}\mathbb{P}(N_h \geq 2) \to 0 \quad \text{as} \quad h \downarrow 0$$

Now we derive a (defective) renewal equation for the ruin probability $\psi(u)$. Define the complementary distribution function $\bar{F}$ by $\bar{F}(x) := 1 - F(x)$ for all $x \geq 0$.

Theorem 6.1 *For any initial reserve $u \geq 0$,*

$$\psi(u) = \frac{\lambda}{c} \int_u^\infty \bar{F}(x)\, dx + \frac{\lambda}{c} \int_0^u \psi(u-x)\bar{F}(x)\, dx$$

Proof: We condition on the time $T_1 = t$ of the first claim. As $T_1 \sim Exp(\lambda)$, we obtain for the survival probability $\bar{\psi}(u) = 1 - \psi(u)$

$$\begin{aligned}
\bar{\psi}(u) &= \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} \bar{\psi}(u+ct-s)dF(s)dt \\
&= \frac{\lambda}{c} \int_u^\infty e^{-\frac{\lambda}{c}(y-u)} \int_0^y \bar{\psi}(y-s)dF(s)dy
\end{aligned}$$

after the substitution $y = u + ct$ which entails $dy = c\ dt$ and $t = 0 \Leftrightarrow y = u$. This shows that the function

$$\bar{\psi}(x) = \frac{\lambda}{c} e^{\frac{\lambda}{c}x} \int_x^\infty e^{-\frac{\lambda}{c}y} \int_0^y \bar{\psi}(y-s)dF(s)dy \tag{6.1}$$

is differentiable on $x \in]0, \infty[$. Denote $g(x) := \int_x^\infty e^{-\frac{\lambda}{c}y} \int_0^y \bar{\psi}(y-s)dF(s)dy$. Then the derivative $\bar{\psi}'(x)$ is given by

$$\bar{\psi}'(x) = \frac{d}{dx}\left(\frac{\lambda}{c} e^{\frac{\lambda}{c}x} g(x)\right) = \frac{\lambda}{c}\left(\frac{\lambda}{c} e^{\frac{\lambda}{c}x} g(x) + e^{\frac{\lambda}{c}x} g'(x)\right)$$

As $g'(x) = -e^{-\frac{\lambda}{c}x} \int_0^x \bar{\psi}(x-s)dF(s)$, we obtain further

$$\begin{aligned}
\bar{\psi}'(x) &= \frac{\lambda}{c}\Bigg(\frac{\lambda}{c} e^{\frac{\lambda}{c}x} \int_x^\infty e^{-\frac{\lambda}{c}y} \int_0^y \bar{\psi}(y-s)dF(s)dy \\
&\qquad - e^{\frac{\lambda}{c}x} e^{-\frac{\lambda}{c}x} \int_0^x \bar{\psi}(x-s)dF(s)\Bigg) \\
&= \frac{\lambda}{c}\left(\bar{\psi}(x) - \int_0^x \bar{\psi}(x-s)dF(s)\right)
\end{aligned}$$

using the representation (6.1). We will determine $\bar{\psi}(u)$ via its derivative as

$$\begin{aligned}\bar{\psi}(u) - \bar{\psi}(0) &= \int_0^u \bar{\psi}'(x)dx \\ &= \frac{\lambda}{c}\left(\int_0^u \bar{\psi}(x)dx - \int_0^u \int_0^x \bar{\psi}(x-s)dF(s)dx\right)\end{aligned}$$

The second integral transforms as

$$\begin{aligned}\int_{x=0}^u \int_{s=0}^x \bar{\psi}(x-s)\ dF(s)\ dx &= \int_{s=0}^u \int_{x=s}^u \bar{\psi}(x-s)\ dx\ dF(s) \\ &= \int_{s=0}^u \int_{x=0}^{u-s} \bar{\psi}(x)\ dx\ dF(s) \\ &= \int_{x=0}^u \int_{s=0}^{u-x} dF(s)\ \bar{\psi}(x)\ dx \\ &= \int_{x=0}^u F(u-x)\bar{\psi}(x)\ dx\end{aligned}$$

Thus we can write

$$\begin{aligned}\bar{\psi}(u) &= \bar{\psi}(0) + \frac{\lambda}{c}\int_0^u (1 - F(u-x))\bar{\psi}(x)\ dx \\ &= \bar{\psi}(0) + \frac{\lambda}{c}\int_0^u \bar{\psi}(u-x)\bar{F}(x)\ dx\end{aligned}$$

We can determine $\bar{\psi}(0)$ if we let $u \to \infty$. Since $\bar{\psi}(u-x) \to 0$ and $\bar{F}(x) \to 0$ as $x \to \infty$, we obtain

$$\bar{\psi}(0) = 1 - \frac{\lambda}{c}\int_0^\infty \bar{F}(x)\ dx = 1 - \frac{\lambda}{c}\mathbb{E}(U_1)$$

This leads to

$$\begin{aligned}\psi(u) = 1 - \bar{\psi}(u) &= \frac{\lambda}{c}\int_0^\infty \bar{F}(x)\ dx - \frac{\lambda}{c}\int_0^u \bar{\psi}(u-x)\bar{F}(x)\ dx \\ &= \frac{\lambda}{c}\int_0^\infty \bar{F}(x)\ dx - \frac{\lambda}{c}\int_0^u (1 - \psi(u-x))\bar{F}(x)\ dx \\ &= \frac{\lambda}{c}\int_u^\infty \bar{F}(x)\ dx + \frac{\lambda}{c}\int_0^u \psi(u-x)\bar{F}(x)\ dx\end{aligned}$$

completing the proof.

□

Exercise 6.1 Show that

$$\lim_{u \to \infty} \int_0^u \bar{\psi}(u-x)\bar{F}(x)\, dx = \mathbb{E}(U_1)$$

Remark 6.2 The proof shows further that the system load

$$p = \psi(0) = \frac{\lambda}{c}\mathbb{E}(U_1) = \rho$$

depends on U_1 only by its mean. This is called **insensitivity** with respect to the shape of F, the distribution of U_1.

Remark 6.3 If c depends on λ by $c = (1+\eta)\lambda\mathbb{E}(U_1)$, then $\psi(u)$ is constant in λ.

Remark 6.4 The equation in the above theorem 6.1 is called a (defective) **renewal equation** for $\psi(u)$.

Theorem 6.2 *The Laplace transform of $\psi(u)$ is given by*

$$L_\psi(s) := \int_0^\infty e^{-su}\psi(u)\, du = \frac{1}{s} - \frac{c - \lambda\mathbb{E}(U_1)}{cs - \lambda(1 - L_F(s))}$$

where $L_F(s) = \int_0^\infty e^{-su}\, dF(u)$ is the Laplace transform of the claim size distribution.

Proof: Using theorem 6.1, we obtain

$$\begin{aligned}
L_\psi(s) &= \frac{\lambda}{c}\left(\int_0^\infty e^{-su}\int_u^\infty \bar{F}(x)\, dx\, du + \int_0^\infty e^{-su}\int_0^u \psi(u-x)\bar{F}(x)\, dx\, du\right) \\
&= \frac{\lambda}{c}\int_0^\infty \bar{F}(x)\int_0^x e^{-su}\, du\, dx \\
&\qquad + \frac{\lambda}{c}\int_0^\infty e^{-sx}\bar{F}(x)\int_x^\infty e^{-s(u-x)}\psi(u-x)\, du\, dx \\
&= \frac{\lambda}{c}\int_0^\infty \bar{F}(x)\frac{1}{s}\left(1 - e^{-sx}\right)\, dx + \frac{\lambda}{c}\int_0^\infty e^{-sx}\bar{F}(x)L_\psi(s)\, dx
\end{aligned}$$

Isolating $L_\psi(s)$ yields

$$L_\psi(s) = \frac{\lambda}{cs}\left(\int_0^\infty \bar{F}(x)\, dx - \int_0^\infty e^{-sx}\bar{F}(x)\, dx\right)\cdot\left(1 - \frac{\lambda}{c}\int_0^\infty e^{-sx}\bar{F}(x)\, dx\right)^{-1}$$

Integration by parts yields

$$\begin{aligned}\int_0^\infty e^{-sx}\bar{F}(x)\,dx &= \frac{1}{s} - \int_0^\infty e^{-sx}F(x)\,dx \\ &= \frac{1}{s} + \left[\frac{1}{s}e^{-sx}F(x)\right]_0^\infty - \frac{1}{s}\int_0^\infty e^{-sx}dF(x) \\ &= \frac{1}{s}\left(1 - L_F(s)\right)\end{aligned}$$

whence we obtain

$$\begin{aligned}L_\psi(s) &= \frac{\lambda}{cs}\left(\mathbb{E}(U_1) - \frac{1}{s}(1 - L_F(s))\right)\cdot\left(1 - \frac{\lambda}{cs}(1 - L_F(s))\right)^{-1} \\ &= \frac{\lambda\mathbb{E}(U_1) - \frac{\lambda}{s}(1 - L_F(s))}{cs - \lambda(1 - L_F(s))} \\ &= \frac{1}{s} - \frac{c - \lambda\mathbb{E}(U_1)}{cs - \lambda(1 - L_F(s))}\end{aligned}$$

$\square$

Example 6.1 If $U_1 \sim Exp(\mu)$, then $L_F(s) = \mu\cdot(\mu+s)^{-1}$ and

$$\begin{aligned}L_\psi(s) &= \frac{\frac{\lambda}{\mu} - \frac{\lambda}{s}(1 - \frac{\mu}{\mu+s})}{cs - \lambda(1 - \frac{\mu}{\mu+s})} = \frac{\frac{\lambda}{\mu} - \frac{\lambda}{\mu+s}}{cs - \frac{\lambda s}{\mu+s}} = \frac{\frac{\lambda(\mu+s)}{\mu} - \lambda}{cs(\mu+s) - \lambda s} \\ &= \frac{\lambda s}{\mu}\left(s\cdot(c\mu + cs - \lambda)\right)^{-1} = \frac{\lambda}{\mu c}\left(\mu - \frac{\lambda}{c} + s\right)^{-1} \\ &= \frac{\lambda}{\mu c}\int_0^\infty e^{-su}e^{-(\mu-\lambda/c)u}\,du\end{aligned}$$

Now uniqueness of the Laplace transform yields

$$\psi(u) = \frac{\lambda}{\mu c}e^{-(\mu-\lambda/c)u}$$

for $u \geq 0$.

Bibliography

[1] J. Doob. *Stochastic processes*. New York: Wiley, 1953.

[2] S. M. Pitts and K. Politis. The joint density of the surplus before and after ruin in the Sparre Andersen model. *J. Appl. Prob.*, 44:695–712, 2007.

[3] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels. *Stochastic Processes for Insurance and Finance*. Wiley, Chichester etc., 1999.