Chapter 3: Homogeneous Markov Processes on Discrete State Spaces

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From now on we shall convene on the technical assumption

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which holds for all applications that we will examine. Then a Markov process \( \mathcal{Y} \) is called **irreducible**, **transient**, **recurrent** or **positive recurrent** if its embedded Markov chain \( \mathcal{X} \) is.
An initial distribution $\pi$ is called **stationary**
An initial distribution $\pi$ is called \textit{stationary} if the process $\mathcal{Y}^\pi$ is stationary,
An initial distribution $\pi$ is called stationary if the process $Y^\pi$ is stationary, i.e. if

$$\mathbb{P}(Y_{t_1}^\pi = j_1, \ldots, Y_{t_n}^\pi = j_n) = \mathbb{P}(Y_{t_1+s}^\pi = j_1, \ldots, Y_{t_n+s}^\pi = j_n)$$

for all $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n$, and states $j_1, \ldots, j_n \in E$, and $s \geq 0$. 
A distribution $\pi$ on $E$ is stationary if and only if $\pi G = 0$ holds.
Theorem 3.9

A distribution $\pi$ on $E$ is stationary if and only if $\pi G = 0$ holds.

Proof:
First we obtain

$$\pi P(t) = \pi e^{G \cdot t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \pi G^n = \pi I + \sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = \pi + 0 = \pi$$

for all $t \geq 0$, with $0$ denoting the zero measure on $E$. 
Proof of theorem 3.9 (contd.)

With this, theorem 3.8 yields

\[ P(Y_{t_1} = j_1, \ldots, Y_{t_n} = j_n) \]

\[ = \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1}) \]
With this, theorem 3.8 yields

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\[ = \pi_{j_1} P_{j_1,j_2}(t_2 - t_1) \cdots P_{j_{n-1},j_n}(t_n - t_{n-1}) \]

for all times \( t_1 < \ldots < t_n \) with \( n \in \mathbb{N} \), and states \( j_1, \ldots, j_n \in E \).

Hence the process \( Y^{\pi} \) is stationary.
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\[ = \sum_{i \in E} \pi_i P_{i,j_1}(t_1 + s) P_{j_1,j_2}(t_2 - t_1) \cdots P_{j_{n-1},j_n}(t_n - t_{n-1}) \]
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for all times $t_1 < \ldots < t_n$ with $n \in \mathbb{N}$, and states $j_1, \ldots, j_n \in E$. 

Hence the process $Y^\pi$ is stationary.
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With this, theorem 3.8 yields

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$$\pi P(t) = \pi e^{G \cdot t} = \pi$$

for all $t \geq 0$. 

As above, this means

$$\sum_{n=1}^{\infty} t^n \pi G^n = 0$$

for all $t \geq 0$, which yields $\pi G = 0$ because of the uniqueness of the zero power series.
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because of the uniqueness of the zero power series.
Balance equations

The equation $\pi G = 0$ is equivalent to an equation system

$$\sum_{i \neq j} \pi_i g_{ij} = -\pi_j g_{jj} \iff \sum_{i \neq j} \pi_i g_{ij} = \pi_j \sum_{i \neq j} g_{ji}$$

for all $j \in E$. 

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The generator of the Poisson process with parameter \( \lambda \) is given by

\[
G = \begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & \cdots \\
0 & -\lambda & \lambda & 0 & \cdots \\
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\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]
Example: Poisson process

The generator of the Poisson process with parameter $\lambda$ is given by

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This process has no stationary distribution, which can be seen as follows.
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$$\pi_0 \lambda = 0 \quad \text{and} \quad \pi_i \lambda = \pi_{i-1} \lambda$$

for all $i \geq 1$. 
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for all $i \geq 1$. These are solvable only by $\pi_i = 0$ for all $i \in E$, which means that there is no stationary distribution $\pi$. 
Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$.
Theorem 3.11

Let $Y$ be a Markov process with embedded Markov chain $X$. Let $X$ be irreducible and positive recurrent.

Proof:

According to theorems 2.25 and 2.18, the transition matrix $P$ of $X$ admits a unique stationary distribution $\nu$ with $\nu P = \nu$. The generator $G$ is defined by $G = \Lambda(P - I)$ with $\Lambda = \text{diag}(\lambda_i : i \in E)$. L. Breuer

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Proof:
According to theorems 2.25 and 2.18, the transition matrix \( P \) of \( \mathcal{X} \) admits a unique stationary distribution \( \nu \) with \( \nu P = \nu \).
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Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent. Further assume that 
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\[ \pi_j := \frac{\mu_j}{\sum_{i \in E} \mu_i} = \frac{\nu_j/\lambda_j}{\sum_{i \in E} \nu_i/\lambda_i} \]

for all \( j \in E \)
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for all \( j \in E \) yields a stationary distribution for \( \mathcal{Y} \). This is unique because \( \nu \) is unique.
We define a **skip–free Markov process** by

$$g_{ij} = 0 \quad \text{for all states} \quad i, j \in E \subset \mathbb{N}_0 \quad \text{with} \quad |i - j| > 1$$
Skip-free Markov processes

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Denote the remaining infinitesimal transition rates by

$$\lambda_i := g_{i,i+1} \quad \text{and} \quad \mu_i := g_{i,i-1}$$

for all possible values of $i$. 
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for all possible values of \( i \). The rates \( \lambda_i \) and \( \mu_i \) are called arrival rates and departure rates, respectively.
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for all possible values of \( i \). The rates \( \lambda_i \) and \( \mu_i \) are called **arrival rates** and **departure rates**, respectively. The state transition graph of such a process assumes the form
Its balance equations are given by

\[ \lambda_0 \pi_0 = \mu_1 \pi_1 \]
Skip-free Markov processes

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for all \( i \in \mathbb{N} \). By induction on \( i \) it is shown that these are equivalent to the equation system

\[ \lambda_{i-1} \pi_{i-1} = \mu_i \pi_i \]

for all \( i \in \mathbb{N} \).
This system is solved by successive elimination

\[\pi_i = \pi_0 i - 1 \prod_{j=0}^{\lambda_i - 1} \mu_j + 1 = \pi_0 \lambda_0 \lambda_1 \cdots \lambda_{i-1} \mu_1 \mu_2 \cdots \mu_i \]

for all \(i \geq 1\). The solution \(\pi\) is a probability distribution if and only if it can be normalized, i.e. if

\[\sum_{n \in E} \pi_n = 1.\]

This condition implies

\[1 = \sum_{n \in E} \pi_0 n - 1 \prod_{j=0}^{\lambda_i - 1} \lambda_j \mu_j + 1 = \pi_0 \sum_{n \in E} n - 1 \prod_{j=0}^{\lambda_i - 1} \lambda_j \mu_j + 1 \]

with the empty product being defined as one. This means that \(\pi_0 = \left(\sum_{n \in E} n - 1 \prod_{j=0}^{\lambda_i - 1} \lambda_j \mu_j + 1 \right)^{-1}\) and thus \(\pi\) is a probability distribution if and only if the series in the brackets converges.
Skip-free Markov processes

This system is solved by successive elimination with a solution of the form

\[ \pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_j+1} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \]

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$$1 = \sum_{n \in E} \pi_0 \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_j+1} = \pi_0 \sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_j+1}$$

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\[ \pi_0 = \left( \sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} \right)^{-1} \]
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for all $i \geq 1$. The solution $\pi$ is a probability distribution if and only if it can be normalized, i.e. if $\sum_{n \in E} \pi_n = 1$. This condition implies

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and thus $\pi$ is a probability distribution if and only if the series in the brackets converges.
Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ denote two independent random variables.
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$$\min(X, Y) \sim \text{Exp}(\lambda + \mu)$$

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}$$

$$\mathbb{P}(X = Y) = 0$$
Let $Z := \min(X, Y)$. 

Independence of $X$ and $Y$ yields 

$$P(Z > t) = P(X > t) \cdot P(Y > t)$$ 

and thus

$$P(\min(X, Y) > t) = e^{-\lambda t} e^{-\mu t} = e^{-(\lambda + \mu) t}$$

for all $t \geq 0$. 

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$$Z > t \iff X > t \text{ and } Y > t$$
Proof - 1

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\[ Z > t \iff X > t \quad \text{and} \quad Y > t \]

Independence of \( X \) and \( Y \) yields

\[ P(Z > t) = P(X > t) \cdot P(Y > t) \]

and thus

\[ P(\min(X, Y) > t) = e^{-\lambda t} e^{-\mu t} = e^{-(\lambda+\mu)t} \]

for all \( t \geq 0 \).
Conditioning on $X \in dt$ yields

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\mathbb{P}(X < Y) = \int_{0}^{\infty} \lambda e^{-\lambda t} \mathbb{P}(Y > t) \, dt
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P(X = Y) = \lim_{h \to 0} \int_0^\infty \lambda e^{-\lambda t} P(Y \in [t, t + h]) \, dt
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Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$
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State space $E = \mathbb{N}_0$
The M/M/1 queue as a Markov process

Holding time in state 0:

Holding time in state $i \geq 1$:

$$H_i = \min(A, S), \text{ where } A \sim \exp(\lambda) \text{ and } S \sim \exp(\mu)$$

Hence, $H_i \sim \exp(\lambda + \mu)$ for $i \geq 1$.

Further:

$$p_{ij} = \begin{cases} 
    P(A < S) = \frac{\lambda}{\lambda + \mu}, & j = i + 1 \\
    P(S < A) = \frac{\mu}{\lambda + \mu}, & j = i - 1 
\end{cases}$$

for $i \geq 1$. 

L. Breuer
Chapter 3: Homogeneous Markov Processes on Discrete State Spaces
The M/M/1 queue as a Markov process

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Hence,

\[ G = \begin{pmatrix} 
-\lambda & \lambda & 0 & 0 & \cdots \\
\mu & -\lambda - \mu & \lambda & 0 & \cdots \\
0 & \mu & -\lambda - \mu & \lambda & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix} \]
Stationary distribution

$Y = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$ with arrival rates $\lambda_i$ and departure rates $\mu_i$. Thus the stationary distribution $\pi$ is given by

$$\pi_0 = \left( \sum_{n \in E} (n-1) \prod_{j=0} \lambda_j \mu_j + 1 \right)^{-1} = \left( \infty \sum_{n=0} \rho^n \right)^{-1} = (1 - \rho)$$

if $\rho := \lambda/\mu < 1$ and $\pi_i = \pi_0 \rho^i$ for $i \geq 1$. For $\rho \geq 1$ there is no stationary distribution.
\( \mathcal{Y} = (Y_t : t \geq 0) \) is a skip-free Markov process on \( E = \mathbb{N}_0 \)
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