Chapter 2: Markov Chains

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Let $X$ denote a Markov chain with state space $E$. 

Let $\pi$ denote a probability measure on $E$. If $P(X_0 = i) = \pi_i$ implies $P(X_n = i) = \pi_i$ for all $n \in \mathbb{N}$ and $i \in E$, then $\pi$ is called a stationary distribution for $X$. If $\pi$ is a stationary distribution, then $c \cdot \pi$ for any $c \geq 0$ is called a stationary measure.
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Let $\mathcal{X}$ denote a Markov chain with state space $E$ and transition matrix $P$.

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Proof of theorem 2.18

Let $P(X_0 = i) = \pi_i$ for all $i \in E$. 

The case $n = 1$ holds by assumption, and the induction step follows by induction hypothesis and the Markov property. The last statement is obvious.
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Example 2.19

Let the transition matrix of a Markov chain $X$ be given by

\[
P = \begin{pmatrix}
0.8 & 0.2 & 0 & 0 \\
0.2 & 0.8 & 0 & 0 \\
0 & 0 & 0.4 & 0.6 \\
0 & 0 & 0.6 & 0.4 \\
\end{pmatrix}
\]

Then $\pi = (0.5, 0.5, 0, 0)$, $\pi' = (0, 0, 0.5, 0.5)$ as well as any linear combination of them are stationary distributions for $X$. This shows that a stationary distribution does not need to be unique.
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Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

\[
P = \begin{pmatrix}
1 - p & p & 0 & 0 & \cdots \\
0 & 1 - p & p & 0 & \ddots \\
0 & 0 & 1 - p & p & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

Hence \( \pi P = \pi \) implies first \( \pi_0 (1 - p) = \pi_0 \Rightarrow \pi_0 = 0 \) since \( 0 < p < 1 \).

Assume that \( \pi_n = 0 \) for any \( n \in \mathbb{N}_0 \). This and the condition \( \pi P = \pi \) further imply for \( \pi_{n+1} \)

\[
\pi_n \cdot p + \pi_{n+1} \cdot (1 - p) = \pi_{n+1} \Rightarrow \pi_{n+1} = 0
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which completes an induction argument proving \( \pi_n = 0 \) for all \( n \in \mathbb{N}_0 \).

Hence the Bernoulli process does not have a stationary distribution.
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Example 2.21

The solution of \( \pi P = \pi \) and \( \sum_{j \in E} \pi_j = 1 \) is unique for

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Example 2.21

The solution of $\pi P = \pi$ and $\sum_{j \in E} \pi_j = 1$ is unique for

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with $0 < p < 1$. Thus there are transition matrices which have exactly one stationary distribution.
A transient Markov chain (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof:
Theorem 2.22

A transient Markov chain (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution $\pi$.
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Proof: Assume that $\pi P = \pi$ holds for some distribution $\pi$. Further let $E = \mathbb{N}$ without loss of generality.
A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution $\pi$. Further let $E = \mathbb{N}$ without loss of generality. Choose any state $m \in \mathbb{N}$ with $\pi_m > 0$. 

Since $\sum_{n=1}^{\infty} \pi_n = 1$ is bounded, there is an index $M > m$ such that $\sum_{n=M}^{\infty} \pi_n < \pi_m$. Set $\varepsilon := \pi_m - \sum_{n=M}^{\infty} \pi_n$. By theorem 2.17, there is an index $N \in \mathbb{N}$ such that $P_N(i, m) < \varepsilon$ for all $i \leq M$. Then the stationarity of $\pi$ implies $\pi_m = \sum_{i=1}^{\infty} \pi_i P_N(i, m) = M - 1 \sum_{i=1}^{\infty} \pi_i P_N(i, m) + \sum_{i=M}^{\infty} \pi_i P_N(i, m) < \varepsilon + \sum_{i=M}^{\infty} \pi_i = \pi_m$ which is a contradiction.
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A transient Markov chain (i.e. a Markov chain with transient states only) has no stationary distribution.

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\pi_m = \sum_{i=1}^{\infty} \pi_i P^N(i, m) = \sum_{i=1}^{M-1} \pi_i P^N(i, m) + \sum_{i=M}^{\infty} \pi_i P^N(i, m) < \varepsilon + \sum_{i=M}^{\infty} \pi_i = \pi_m
$$

which is a contradiction.
Define

\[ N_i(n) := \sum_{k=0}^{n} \mathbb{I}\{X_k=i\} \]

as the number of visits to state \( i \) until time \( n \).
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as the number of visits to state \( i \) until time \( n \). Further define for a recurrent state \( i \in E \) the mean time of return

\[ m_i := \mathbb{E}(\tau_i | X_0 = i) \]
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By definition \( m_i > 0 \) for all \( i \in E \). A recurrent state \( i \in E \) with \( m_i < \infty \) will be called **positive recurrent**, otherwise \( i \) is called **null recurrent**.

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Define

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By definition \( m_i > 0 \) for all \( i \in E \). A recurrent state \( i \in E \) with \( m_i < \infty \) will be called **positive recurrent**, otherwise \( i \) is called **null recurrent**.
The elementary renewal theorem (which will be proven in chapter 4) states that

\[
\lim_{n \to \infty} \frac{\mathbb{E}(N_i(n) | X_0 = j)}{n} = \frac{1}{m_i}
\]

for all recurrent \( i \in E \).
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Thus the asymptotic rate of visits to a recurrent state is determined by the mean recurrence time of this state.
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for all recurrent $i \in E$ and independently of $j \in E$ provided $j \leftrightarrow i$, with the convention of $1/\infty := 0$. Thus the asymptotic rate of visits to a recurrent state is determined by the mean recurrence time of this state.
Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.
Theorem 2.23

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Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$
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Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$ and $i$ is null recurrent.
Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Proof:
Assume that \( i \leftrightarrow j \) for two states \( i, j \in E \) and \( i \) is null recurrent. Thus there are numbers \( m, n \in \mathbb{N} \) with \( P^n(i, j) > 0 \) and \( P^m(j, i) > 0 \).
Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$ and $i$ is null recurrent. Thus there are numbers $m, n \in \mathbb{N}$ with $P^n(i, j) > 0$ and $P^m(j, i) > 0$. Because of the representation $E(N_i(k)|X_0 = i) = \sum_{l=0}^{k} P^l(i, i)$, we obtain
Proof of theorem 2.23 (contd.)

\[ 0 = \lim_{k \to \infty} \frac{\sum_{i=0}^{k} P^l(i, i)}{k} \]
Proof of theorem 2.23 (contd.)

\[ 0 = \lim_{k \to \infty} \frac{\sum_{l=0}^{k} P^l(i, i)}{k} \]

\[ \geq \lim_{k \to \infty} \frac{\sum_{l=0}^{k-m-n} P^l(j, j)}{k} \cdot P^n(i, j)P^m(j, i) \]

and thus \( m_j = \infty \), which signifies the null recurrence of \( j \).
Proof of theorem 2.23 (contd.)

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and thus \( m_j = \infty \), which signifies the null recurrence of \( j \).
Theorem 2.24

Let $i \in E$ be positive recurrent and define the mean first visit time $m_i := \mathbb{E}(\tau_i | X_0 = i)$. Then a stationary distribution $\pi_j$ is given by

$$\pi_j = m_i - 1 \cdot \sum_{n=0}^{\infty} P(X_n = j, \tau_i > n | X_0 = i)$$

for all $j \in E$. In particular, $\pi_i = m_i - 1$ and $\pi_k = 0$ for all states $k$ outside of the communication class belonging to $i$. 
Theorem 2.24

Let \( i \in E \) be positive recurrent and define the mean first visit time \( m_i := \mathbb{E}(\tau_i|X_0 = i) \). Then a stationary distribution \( \pi \) is given by

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\pi_j := m_i^{-1} \cdot \sum_{n=0}^{\infty} P(X_n = j, \tau_i > n|X_0 = i)
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Let $i \in E$ be positive recurrent and define the mean first visit time $m_i := \mathbb{E}(\tau_i|X_0 = i)$. Then a stationary distribution $\pi$ is given by

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First of all, $\pi$ is a probability measure
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Proof of theorem 2.24

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The particular statements in the theorem are obvious from the definition of $\pi$
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The particular statements in the theorem are obvious from the definition of $\pi$ and the fact that a recurrent communication class is closed.
The stationarity of $\pi$ is shown as follows.
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since $X_0 = X_{\tau_i} = i$ in the conditioning set $\{X_0 = i\}$. Further,
Proof of theorem 2.24 (contd.)

\[ P(X_n = j, \tau_i > n - 1|X_0 = i) = \frac{P(X_n = j, \tau_i > n - 1, X_0 = i)}{P(X_0 = i)} \]
Proof of theorem 2.24 (contd.)

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\[ = \sum_{k \in E} p_{kj} P(X_{n-1} = k, \tau_i > n - 1 | X_0 = i) \]
Hence we obtain

\[ \pi_j = m_i^{-1} \cdot \sum_{n=1}^{\infty} \sum_{k \in E} p_{kj} \mathbb{P}(X_{n-1} = k, \tau_i > n - 1 | X_0 = i) \]
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which completes the proof.
Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain.
Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof: Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let $\pi$ denote the stationary distribution as constructed in theorem 2.24 and $i$ the positive recurrent state that served as recurrence point for $\pi$. Further, let $\nu$ denote any stationary distribution for $\mathcal{X}$. Then there is a state $j \in E$ with $\nu_j > 0$ and a number $m \in \mathbb{N}$ with $P_m(j, i) > 0$, since $\mathcal{X}$ is irreducible.
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L. Breuer

Chapter 2: Markov Chains
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Hence we can multiply \(\nu\) by a factor \(c > 0\) such that \(c \cdot \nu_i = \pi_i = 1/m_i\). Denote \(\tilde{\nu} := c \cdot \nu\), i.e. \(\tilde{\nu}_k := c \cdot \nu_k\) for all \(k \in E\). Let \(\tilde{P}\) denote the transition matrix \(P\) without the \(i\)th column, i.e. \(\tilde{P} = (\tilde{p}_{hk})_{h,k \in E}\) with

$$\tilde{p}_{hk} = \begin{cases} p_{hk}, & k \neq i \\ 0, & k = i \end{cases}$$
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\[ \tilde{p}_{hk} = \begin{cases} p_{hk}, & k \neq i \\ 0, & k = i \end{cases} \]

Denote further the Dirac measure on \( i \) by \( \delta^i \), i.e.

\[ \delta^i_k = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases} \]
Then the stationary distribution $\pi$ can be represented by

$$\pi = m_i^{-1} \cdot \delta^i \sum_{n=0}^{\infty} \tilde{P}^n$$
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This is clear for the entry $\tilde{\nu}_i$ and easily seen for $\tilde{\nu}_k$ with $k \neq i$ because in this case

$$(\tilde{\nu} \tilde{P})_k = c \cdot (\nu P)_k = c \cdot \nu_k = \tilde{\nu}_k$$
Now we can proceed with the same argument to see that

\[ m_i\tilde{\nu} = \delta^i + (\delta^i + m_i\tilde{\nu}\tilde{P})\tilde{P} = \delta^i + \delta^i\tilde{P} + m_i\tilde{\nu}\tilde{P}^2 = \ldots \]
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Proof of theorem 2.25 (contd.)

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Hence \( \tilde{\nu} \) already is a probability measure and thus \( c = 1 \). This yields \( \nu = \tilde{\nu} = \pi \) and thus the statement.
Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain.

Proof:
Since all states in $E$ are positive recurrent, the construction in theorem 2.24 can be pursued for any initial state $j$. This yields $\pi_j = m^{-1}J$ for all $j \in E$. The statement now follows from the uniqueness of the stationary distribution.
Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then the stationary distribution $\pi$ of $\mathcal{X}$ is given by

$$\pi_j = m_j^{-1} = \frac{1}{\mathbb{E}(\tau_j|X_0 = j)}$$

for all $j \in E$. 

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For an irreducible, positive recurrent Markov chain,
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$$\lim_{n \to \infty} \frac{\mathbb{E}(N_j(n) | X_0 = i)}{n} = \pi_j$$

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For an irreducible, positive recurrent Markov chain, the stationary probability \( \pi_j \) of a state \( j \) coincides with its asymptotic rate of recurrence, i.e.

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for all \( j \in E \) and independently of \( i \in E \).
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The first statement immediately follows from the elementary renewal theorem.
Proof of theorem 2.28

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The first statement immediately follows from the elementary renewal theorem. For the second statement, it suffices to employ $\mathbb{E}(N_j(n) | X_0 = i) = \sum_{l=0}^{n} P^l(i, j)$. If an asymptotic distribution does exist,
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$$E(N_j(n)|X_0 = i) = \sum_{l=0}^{n} P^l(i, j).$$

If an asymptotic distribution does exist, then for any initial distribution $\nu$ we obtain

$$p_j = \lim_{n \to \infty} (\nu P^n)_j = \sum_{i \in E} \nu_i \lim_{n \to \infty} P^n(i, j)$$
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\]

\[= \pi_j\]
Example

Let $X$ denote a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $X$ has no asymptotic distribution, but a stationary distribution, namely $\pi = (1/2, 1/2)$. 
Example

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Let $\mathcal{X}$ denote a Markov chain with transition matrix

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Then $\mathcal{X}$ has no asymptotic distribution, but a stationary distribution, namely $\pi = (1/2, 1/2)$. 
Theorem 2.31

An irreducible Markov chain with finite state space $F$ is positive recurrent.

Proof: For all $n \in \mathbb{N}$ and $i \in F$ we have 
\[
\sum_{j \in F} P_n(i,j) = 1
\]
Hence it is not possible that $\lim_{n \to \infty} P_n(i,j) = 0$ for all $j \in F$.
Thus there is one state $h \in F$ such that
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\sum_{n=0}^{\infty} P_n(i,h) = r_{ih} = 0
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which means by corollary 2.15 that $h$ is recurrent and by irreducibility that the chain is recurrent.
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$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k(i, j) = 0$$

would hold for all $j \in F$, independently of $i$ because of irreducibility. Hence the chain must be positive recurrent.
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for all $j \in F$, which contradicts our first observation in this proof. Hence the chain must be positive recurrent.
Choose any parameters $0 < p, q < 1$. Let the arrival process be distributed as a Bernoulli process with parameter $p$ and the service times $(S_n : n \in \mathbb{N}_0)$ be iid according to the geometric distribution with parameter $q$. 
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Theorem 2.34 (memoryless property)

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$$P(S = k) = (1 - q)^{k-1}q$$
for all $k \in \mathbb{N}$. 

Proof:
$$P(S = k | S > k - 1) = \frac{P(S = k, S > k - 1)}{P(S > k - 1)} = \frac{P(S = k)}{P(S > k - 1)} = (1 - q)^{k-1}q (1 - q)^{k-1}q = q$$
indeedly of $k$. 

L. Breuer

Chapter 2: Markov Chains
Theorem 2.34 (memoryless property)

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$$p_{ij} = \begin{cases} p(1 - q), & j = i + 1 \\ pq + (1 - p)(1 - q), & j = i \\ q(1 - p), & j = i - 1 \end{cases}$$ for $i \geq 1$. 

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P = \begin{pmatrix}
1 - p & p & 0 & \cdots \\
q(1 - p) & pq + (1 - p)(1 - q) & p(1 - q) & \cdots \\
0 & q(1 - p) & pq + (1 - p)(1 - q) & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
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Abbreviate $p' := p(1 - q)$ and $q' := q(1 - p)$.
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$$

and

$$
\pi_n = \pi_{n-1} p' + \pi_n (1 - (p' + q')) + \pi_{n+1} q'
$$

for all $n \geq 2$. 
We try the geometric form

\[ \pi_{n+1} = \pi_n \cdot r \]

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and hence \( r = \frac{p'}{q'} \)
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and hence $r = p'/q' < 1 \iff p < q$. Further,

$$\pi_1 = \pi_0 \frac{p}{q'} = \pi_0 \frac{\rho}{1 - p}$$

with $\rho := p/q$. 
Stationary distribution - 2

and

$$\pi_2 = \frac{1}{q'} \left( \pi_1 (p' + q') - \pi_0 p \right)$$
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Normalisation of $\pi$ yields

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \left( 1 + \frac{p}{q'} \sum_{n=1}^{\infty} \left( \frac{p'}{q'} \right)^{n-1} \right)$$
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Verify this as an exercise!