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FIRST PASSAGE TIMES FOR MARKOV–ADDITIVE PROCESSES

LOTHAR BREUER,* *University of Kent*

Abstract

The present paper generalises some results for spectrally negative Lévy processes to the setting of Markov–additive processes (MAPs). Among them are Laplace transforms for hitting times, the distribution of maxima, a Wiener–Hopf factorisation, and the stationary distribution of the reflected process. These all depend more or less on one matrix only, which can be regarded as the generalised inverse function of the cumulant matrix. A numerically stable iteration to compute this matrix is given. The theory is first developed for MAPS without positive jumps and then extended to include positive jumps having phase–type distributions, for which the overshoot distribution is given, too. Numerical examples show agreement with existing approaches in special cases.

Keywords: Lévy process; hitting times; maximum; Markov-additive process; Wiener-Hopf factor; reflected process; stationary distribution

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1. Introduction

Markov–modulation is the basis for many generalisations in stochastic processes. It leads from renewal processes to Markov renewal theory, which has enabled an analysis of the classical M/G/1 and GI/M/1 queues. Another important example for queueing theory is the Markov–modulated Poisson process, later generalised to Markovian arrival processes which are now standard use in current literature. On a more general level, Markov–additive processes, introduced by Çinlar [9] and by Ezhov and Skorokhod [10], are a powerful generalisation of Lévy processes. The use of Markov–additive processes (MAPs), or rather: sub–classes of MAPs, has become increasingly popular in stochastic modelling. For a recent textbook presentation see Asmussen [3], chapter XI.

Stochastic modelling based on MAPs requires a theory which is able to provide methods

* Postal address: Institute of Mathematics and Statistics, University of Kent, Canterbury CT2 7NF, UK

for the computation of key characteristics such as hitting times, maxima, or stationary distributions. On the level of Lévy processes, many results can be obtained for the spectrally negative variant, i.e. the class of Lévy processes without positive jumps. Standard references are Bertoin [7] or Bingham [8]. Recently, some results could be proven for Lévy processes with phase–type positive jumps as well, see Mordecki [11], Asmussen et al. [4], and Pistorius [12]. The present paper generalises some of these results first to the level of MAPs without positive jumps, whereas later an extension to MAPs with phase–type positive jumps is sketched.

Many arguments in this paper are straightforward generalisations of ideas from Asmussen [1], where the overall supremum of MAPs with continuous paths is investigated. As expected, the main method in generalising the existing theory of Lévy processes consists in finding a matrix equivalent of the inverse function of the cumulant.

As far as the author is aware, the only existing approach to determine the Laplace transform of first passage times for MAPs is the construction of suitable vector–valued martingales as proposed in Asmussen and Kella [5]. One advantage of the present paper is seen in the feature that here the classical theory of Lévy processes finds a natural extension. A switch of analytical methods is not required.

In section 2, notations and preliminary results shall be provided. Hitting times will be investigated in section 3. The relation between hitting times, maxima and the stationary distribution for the process reflected at the origin carries through to the MAP level and will shortly be described in section 4. Section 5 contains a Wiener–Hopf factorisation for MAPs. The question of numerically computing the Laplace exponent $U(\gamma)$, which enables a computation of all other characteristics, will be examined in section 6. Section 7 describes the relation between the martingale approach of Asmussen and Kella [5] and a diagonalisation of $U(\gamma)$. In section 8, the theory is extended to include positive jumps with phase–type distributions. The last section contains some numerical examples, which show in some special cases that the results obtained here are in agreement with results obtained by different approaches.

2. Preliminaries

2.1. Lévy processes

A Lévy process is a real–valued stochastic process with stationary and independent increments. The study of this class of processes goes back as far as the foundations of modern probability theory. A modern textbook is Bertoin [7], whereas Bingham [8] gives an extensive presentation for the fluctuation theory of spectrally negative Lévy processes. For ease of reference, we shall restate some classical results for these, along with the necessary notation, in the following.

Let $\mathcal{Y} = (Y_t : t \geq 0)$ be a one–dimensional real–valued Lévy process with parameters μ for the drift, σ^2 for the variation, and ν for its Lévy measure. Assume that $\mathcal{Y}$ has no positive jumps, i.e. that ν is concentrated on $]-\infty, 0]$. Then the Lévy–Khintchine formula yields $\mathbb{E}(\exp(\alpha Y_t)) = \exp(t\psi(\alpha))$ for $Re(\alpha) \geq 0$, where the function

$$\psi(\alpha) = \alpha\mu + \frac{1}{2}\sigma^2\alpha^2 + \int_{-\infty}^{0} (e^{\alpha x} - 1 - \alpha x \mathbf{1}_{\{x>-1\}})\, \nu(dx)$$

is called the Laplace exponent of $\mathcal{Y}$, cf. Bertoin [7], section VII.1. A subordinator is a Lévy process with increasing paths, implying that $\sigma^2 = 0$ and the Lévy measure to be concentrated on $[0, \infty[$. We assume that $\mathcal{Y}$ is not the negative of a subordinator. Further let $Y_0 = 0$ throughout this section.

Since the function ψ is strictly convex, there is a largest solution, called $\Phi(0)$, of the equation $\psi(\alpha) = 0$, $\alpha \geq 0$. As $\mathcal{Y}$ is not the negative of a subordinator, $\lim_{\alpha\to\infty} \psi(\alpha) = \infty$. Hence there is an inverse function of $\psi : [\Phi(0), \infty[\to [0, \infty[$. It shall be denoted by $\Phi : [0, \infty[\to [\Phi(0), \infty[$. Denote the running maximum of $\mathcal{Y}$ by $S_t := \sup\{Y_s : s \leq t\}$ for $t \geq 0$. The assumption that $\mathcal{Y}$ has no positive jumps yields

Lemma 1. *(Bertoin [7], theorem VII.1)*
Let $\tau(x) := \inf\{s \geq 0 : Y_s > x\}$ for $x \geq 0$ denote the first hitting time of the level x. Then $(\tau(x) : x \geq 0)$ is a subordinator with Laplace exponent Φ.

Lemma 2. *(Bertoin [7], corollary VII.2)*
Let $\tau \sim Exp(\lambda)$ be an exponentially distributed random variable, independent of $\mathcal{Y}$ and with $\lambda > 0$. Then S_τ has an exponential distribution with parameter $\Phi(\lambda)$.

Lemma 3. *(Bertoin [7], theorem VII.4)*
Let $\tau \sim Exp(q)$ be an exponentially distributed random variable, independent of $\mathcal{Y}$ and with $q > 0$. Further let $G_\tau = \sup\{t < \tau : X_t = S_t\}$. Then

$$\mathbb{E}(\exp(-\alpha G_\tau - \beta S_\tau)) = \frac{\Phi(q)}{\Phi(\alpha + q) + \beta}$$

$$\mathbb{E}(\exp(-\alpha(\tau - G_\tau) - \beta(S_\tau - Y_\tau)) = \frac{q \cdot (\Phi(\alpha + q) - \beta)}{\Phi(q) \cdot (\alpha + q - \psi(\beta))}$$

for $\alpha, \beta > 0$.

Remark 1. Later on, we shall use the second equation of lemma 3 with a matrix argument U instead of $-\beta$. Of course, convergence of the integral $\mathbb{E}(\exp(-\alpha(\tau - G_\tau)I + U(S_\tau - Y_\tau))$ can be expected only for matrices U with all eigenvalues having negative real part.

2.2. Markov–additive processes

Let $\mathcal{J} = (J_t : t \geq 0)$ be an irreducible Markov (jump) process with finite state space $E = \{1, \dots, m\}$ and infinitesimal generator matrix $Q = (q_{ij})_{i,j \in E}$. We call J_t the phase at time $t \geq 0$. Define the real–valued process $\mathcal{X} = (X_t : t \geq 0)$ as evolving like a Lévy process with parameters μ_i (drift), σ_i^2 (variation), and ν_i (Lévy measure) during intervals when the phase equals $i \in E$. Whenever $\mathcal{J}$ jumps from a state $i \in E$ to another state $j \in E$, this may be accompanied (with probability λ_{ij}^-) by a jump of $\mathcal{X}$ which has distribution function F_{ij}. Then the two–dimensional process $(\mathcal{X}, \mathcal{J})$ is called a Markov–additive process (or shortly MAP).

A MAP can also be defined by the following property (see Asmussen [3], section XI.2a): $(\mathcal{X}, \mathcal{J})$ is a Markov process such that

$$\mathbb{E}(f(X_{t+s} - X_t)g(J_{t+s})|\mathcal{F}_t, J_t = i) = \mathbb{E}(f(X_s)g(J_s)|X_0 = 0, J_0 = i)$$

holds for all $s, t > 0$ and $i \in E$, where f and g are measurable functions. We shall use this in the proof to theorem 1.

First we shall investigate only MAPs without positive jumps, for which the measures ν_i and F_{ij} have support on the negative real numbers only. Then the cumulant of the Lévy process X^i in phase i may be written as

$$\psi_i(\alpha) = \alpha\mu_i + \frac{1}{2}\sigma_i^2\alpha^2 + \int_{-\infty}^0 (e^{\alpha x} - 1 - \alpha x \mathbf{1}_{\{x > -1\}})\, \nu_i(dx)$$

for such $\alpha \in \mathbb{C}$ that allow an integration. For notational convenience we define the modified (defective) distribution functions $\tilde{F}_{ij}$ by $\tilde{F}_{ij}(0) = 1 - \lambda_{ij}^-$ and $\tilde{F}_{ij}(dx) = \lambda_{ij}^- F_{ij}(-dx)$ for $x > 0$ and $i, j \in E$. Note that $\lambda_{ii} = 0$ which implies $\tilde{F}_{ii} = \delta_0$, denoting the Dirac one–point measure on zero. We further assume that none of the Lévy processes X^i is the negative of a subordinator.

Denote the matrix–valued moment generating function of X_t, given $X_0 = 0$, by $\hat{X}_t(\alpha)$. Write $\hat{F}(s) = (\hat{F}_{ij}(s))_{i,j \in E}$ where $\hat{F}_{ij}(s) = \int_0^\infty e^{sx} d\tilde{F}_{ij}(x)$ is the moment generating function of $\tilde{F}_{ij}$. By proposition XI.2.2 in Asmussen [3], the (i,j)th entry of $\hat{X}_t(\alpha)$ is given as

$$\mathbb{E}(e^{\alpha X_t}; J_t = j | X_0 = 0, J_0 = i) = (e^{K(\alpha)t})_{ij} \tag{1}$$

with

$$K(\alpha) = diag(\psi_i(\alpha))_{i\in E} + Q \circ \hat{F}(-\alpha) \tag{2}$$

for arguments $\alpha \in \mathbb{C}$ that allow convergence of the integrals involved. Here the notation $A \circ B$ denotes entrywise multiplication of the matrices A and B having the same dimension, i.e. $(A \circ B)_{ij} := a_{ij} b_{ij}$.

3. First passage times

We first assume that $X_0 = x$ with $x < 0$ and denote the time of first passage to zero by $\tau := \inf\{t \geq 0 : X_t \geq 0\}$. Define the functions

$$f_{ij}(x) := \mathbb{E}_{ij}(e^{-\gamma\tau} | X_0 = x) := \mathbb{E}(e^{-\gamma\tau}, J_\tau = j | J_0 = i, X_0 = x)$$

for all $x < 0$ and an arbitrary but fixed $\gamma > 0$. Since the paths of $\mathcal{X}$ are upper semi–continuous and spatially homogeneous (conditioned on the phase process), we obtain the functional equation $f_{ij}(x+y) = \sum_{k\in E} f_{ik}(x) f_{kj}(y)$ for all $x, y < 0$. This yields the form

$$f_{ij}(x) = e_{ij}^{-U(\gamma)x} \tag{3}$$

for all $x < 0$ and $i, j \in E$, where $U(\gamma)$ is a matrix of dimension $E \times E$.

A perhaps more formal argument for this is given in the following theorem, which generalises lemma 1. A MAP with increasing paths shall be called a Markov–additive subordinator. Spatial homogeneity of $(\mathcal{X}, \mathcal{J})$ yields for $x > 0$

$$f_{ij}(-x) = \mathbb{E}(e^{-\gamma\tau(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0)$$

where $\tau(x) := \inf\{t \geq 0 : X_t > x\}$ denotes the first hitting time of the level x.

Theorem 1. *Let $(\mathcal{X}, \mathcal{J}) = ((X_t, J_t) : t \geq 0)$ be a MAP without positive jumps. Then the process $((\tau(x), J_{\tau(x)}) : x \geq 0)$ is a Markov–additive subordinator. In particular*

$$\mathbb{E}(e^{-\gamma\tau(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0) = e_{ij}^{U(\gamma)x}$$

for $\gamma, x > 0$ and $i, j \in E$, where $U(\gamma)$ is a matrix of order $E \times E$.

Proof. Clearly, $\tau(x)$ is increasing in x. Choose any numbers $x, y > 0$ and measurable functions $h : \mathbb{R}_+ \to \mathbb{R}$, $g : E \to \mathbb{R}$. Then the strong Markov property and spatial homogeneity yield

$$\begin{aligned}
&\mathbb{E}(h(\tau(x+y) - \tau(x)) \cdot g(J_{\tau(x+y)})|\mathcal{F}_{\tau(x)}, J_{\tau(x)} = i, X_0 = 0) \\
&\qquad = \mathbb{E}(h(\tau(x+y)) \cdot g(J_{\tau(x+y)})|J_0 = i, X_0 = x) \\
&\qquad = \mathbb{E}(h(\tau(y)) \cdot g(J_{\tau(y)})|J_0 = i, X_0 = 0)
\end{aligned}$$

where $\mathcal{F}_t := \sigma((X_s, J_s) : s \leq t)$ denotes the canonical filtration of the MAP $(\mathcal{X}, \mathcal{J})$. Hence the process $((\tau(x), J_{\tau(x)}) : x \geq 0)$ is a Markov–additive subordinator. The matrix $U(\gamma)$ is its Laplace exponent evaluated at $\gamma > 0$.

It remains to determine $U(\gamma)$, which is of course the main part of the paper. Since clearly $f_{ij}(x) \to 0$ as $x \to -\infty$ (for all $i, j \in E$), we see from (3) that all eigenvalues of $U(\gamma)$ must have negative real part.

Denote the semi-group of transition operators for $(\mathcal{X}, \mathcal{J})$ by $(P_t : t \geq 0)$ and define the matrix $f = (f_{ij})_{i,j \in E}$. By definition, $P_t f(x) = \mathbb{E}(e^{-\gamma(\tau - t)})$ for $x < 0$ and t small enough. This yields for the infinitesimal generator A of $(\mathcal{X}, \mathcal{J})$

$$\begin{aligned}
Af &= \lim_{t \downarrow 0} \frac{1}{t}(P_t f - f) = \lim_{t \downarrow 0} \frac{1}{t}(\mathbb{E}(e^{-\gamma(\tau - t)}) - \mathbb{E}(e^{-\gamma\tau})) \\
&= \mathbb{E}(e^{-\gamma\tau}) \lim_{t \downarrow 0} \frac{1}{t}(e^{\gamma\tau I} - I) = \mathbb{E}(e^{-\gamma\tau})\gamma I = \gamma I f
\end{aligned}$$

where $\mathbb{E}(e^{-\gamma\tau}) = (\mathbb{E}_{ij}(e^{-\gamma\tau}|X_0 = \cdot))_{i,j \in E}$. On the other hand, using equation (3), the ith row of Af is given by

$$\begin{aligned}
e_i' Af(x) &= e_i' A \, e^{-U(\gamma)x} \\
&= e_i' \left(\frac{1}{2}\sigma_i^2 U(\gamma)^2 - \mu_i U(\gamma) \right) e^{-U(\gamma)x} \\
&\quad + e_i' \int_{-\infty}^{0} (e^{-U(\gamma)z} - I + U(\gamma) z \mathbf{1}_{\{z > -1\}}) \, \nu_i(dz) \, e^{-U(\gamma)x} \\
&\quad + \sum_{j \neq i} q_{ij} e_j' \int_0^\infty e^{U(\gamma)z} \, d\tilde{F}_{ij}(z) \, e^{-U(\gamma)x} + q_{ii} e_i' e^{-U(\gamma)x}
\end{aligned}$$

where $\tilde{F}_{ij}(0) = 1 - \lambda_{ij}^-$ and $\tilde{F}_{ij}(dy) = \lambda_{ij}^- F_{ij}(-dy)$ and e_i' denotes the ith canonical row base vector. This leads to

$$e_i' \gamma I e^{-U(\gamma)x} = e_i' \gamma f(x) = e_i' Af(x) = e_i'(\psi_i(-U(\gamma)) + Q(\gamma)) e^{-U(\gamma)x}$$

for all $x < 0$, where $e_i' Q(\gamma) = \sum_{j \in E} q_{ij} e_j' \int_0^\infty e^{U(\gamma)z} \, d\tilde{F}_{ij}(z)$ after noting that $\tilde{F}_{ii}(0) = 1$. Thus a first result for $U(\gamma)$ is

$$e_i'(\psi_i(-U(\gamma)) + Q(\gamma) - \gamma I) = \mathbf{0} \tag{4}$$

for all $i \in E$. The problem of computing $U(\gamma)$ shall be taken up again in section 6.

Example 1. The case without jumps has been examined in Asmussen [1] for $\gamma = 0$. Here we obtain

$$\psi_i(-U(0)) = -\mu_i U(0) + \frac{\sigma_i^2}{2} U(0)^2 \qquad \text{and} \qquad Q(0) = Q$$

This specifies equation (4) to equation (4.1) in Asmussen [1], where the notation is $r_i = \mu_i$ and $U = U(0)$.

Example 2. Another special case appears in Pistorius [12], proposition 2(i). Here we have $\gamma = 0$, $\mu_i = \sigma(i)c + v(i)$, $\sigma_i^2 = \sigma(i)^2 s^2$, and $\nu_i = \sigma(i)\nu$. Further there are no jumps upon epochs of phase changes.

4. Maxima and the stationary distribution of the reflected process

The hitting times are closely related to other important characteristics such as maxima and the stationary distribution of the reflected process. This shall be described shortly in the present section. As a preparation the following observation is useful. By definition,

$$e^{-U(\gamma)x} = f(x) = \mathbb{E}(e^{-\gamma\tau} | X_0 = x) = \int_0^\infty \gamma e^{-\gamma t} \mathbb{P}(\tau \in dt | X_0 = x)$$

for all $x < 0$. For the maximum $M_t := \sup\{X_s : s \le t\}$ the relation

$$\mathbb{P}(\tau \le t | X_0 = x) = \mathbb{P}(M_t \ge -x | X_0 = 0)$$

holds. This implies $e^{-U(\gamma)x} = \mathbb{P}(M_{T(\gamma)} \ge -x | X_0 = 0)$ for all $x < 0$, where $T(\gamma)$ is an independent random variable with exponential distribution of rate γ. Hence we obtain

$$\mathbb{P}(M_{T(\gamma)} \ge x | X_0 = 0) = e^{U(\gamma)x} \tag{5}$$

for all $x > 0$. Thus the maximum before an exponential killing time $T(\gamma)$ has a phase–type distribution with rate matrix $U(\gamma)$. This generalises lemma 2.

Let π denote the stationary distribution of the phase process $\mathcal{J}$, i.e. $\pi Q = \mathbf{0}$, where $\mathbf{0}$ is the row vector with all entries being zero. Further define the drift of the Lévy process in phase i by

$m_i := \mathbb{E}(X_1^i)$ and write $m = (m_i)_{i \in E}$ as a column vector. If the mean drift πm is negative, then the overall supremum $M_\infty := \sup\{X_t : t \geq 0\}$ is finite a.s. In the limit $\gamma \to 0$ we obtain

Theorem 2. *If the mean drift πm is negative, then the overall supremum M_∞ has a phase–type distribution with parameters $(\beta, U(0))$, where β is the distribution of J_0.*

Now define the reflected process $X_t^r := X_t - \min\{X_s : s \leq t\}$. If $\pi m < 0$, then the process $(\mathcal{X}^r, \mathcal{J})$ has a stationary distribution. Denote its steady–state variables by (X^r, J). Denote a time–reversed version of $(\mathcal{X}, \mathcal{J})$ by $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$, where the generator matrix for $\tilde{\mathcal{J}}$ is given by $\tilde{Q} := \Delta_{1/\pi} Q \Delta_\pi$ and $\tilde{\mathcal{X}}$ has the same parameters as $\mathcal{X}$ but depends on $\tilde{\mathcal{J}}$. Finally write $\tilde{M}_\infty := \sup\{\tilde{X}_t : t \geq 0\}$. Then proposition 2.1 in Asmussen [1] yields

$$\mathbb{P}(X^r > x, J = i) = \pi_i \mathbb{P}(\tilde{M}_\infty > x | \tilde{J}_0 = i, \tilde{X}_0 = 0) = \pi_i e_i' e^{\tilde{U}(0)x} \mathbf{1}$$

where $\tilde{U}(0)$ pertains to $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ and $\mathbf{1}$ denotes the column vector with all entries being one.

5. Wiener–Hopf factorisation

Once the matrix $U(\gamma)$ is determined, it gives rise to a Wiener-Hopf factorisation for Markov-additive processes. Let $T(\gamma)$ denote an independent random variable with exponential distribution of rate γ. By (1) the moment generating function of $X_{T(\gamma)}$, given $X_0 = 0$, is determined as

$$\begin{aligned}\hat{X}_{T(\gamma)}(\alpha) &= \mathbb{E} \int_0^\infty \gamma e^{-\gamma t} e^{\alpha X_t} \, dt = \gamma \int_0^\infty e^{-\gamma t} \mathbb{E}(e^{\alpha X_t}) \, dt = \gamma \int_0^\infty e^{(K(\alpha) - \gamma I)t} \, dt \\ &= \gamma(\gamma I - K(\alpha))^{-1}\end{aligned}$$

for all suitable $\alpha \in \mathbb{C}$. Now we consider the maximum $M_{T(\gamma)}$ of the process $\mathcal{X}$ before $T(\gamma)$. By equation (5), we obtain

$$\begin{aligned}\mathbb{E}(e^{\alpha M_{T(\gamma)}} | X_0 = 0) &= \int_0^\infty e^{\alpha x} \mathbb{P}(M_{T(\gamma)} \in dx | X_0 = 0) = \int_0^\infty e^{\alpha x} (-U(\gamma)) e^{U(\gamma)x} dx \\ &= -U(\gamma)(-U(\gamma) - \alpha I)^{-1}\end{aligned}$$

This may be regarded as the left Wiener–Hopf factor in a factorisation of $\hat{X}_t(\alpha)$. For the right Wiener–Hopf factor we obtain

$$\begin{aligned}\mathbb{E}\left(e^{\alpha(X_{T(\gamma)} - M_{T(\gamma)})}\right) &= \mathbb{E}(e^{\alpha M_{T(\gamma)}} | X_0 = 0)^{-1} \hat{X}_{T(\gamma)}(\alpha) \\ &= \gamma(-U(\gamma) - \alpha I)(-U(\gamma))^{-1}(\gamma I - K(\alpha))^{-1}\end{aligned}$$

Comparing this with the Wiener–Hopf factors for spectrally negative Lévy processes (see e.g. theorem 4a in Bingham [8], with $s = \alpha$ and $\sigma = \gamma$), we see that the function K takes the role of the cumulant function ψ for a Lévy process, while $-U$ generalises the inverse function ψ^{-1}, which is denoted by ϕ in the present paper.

This interpretation is supported by the following observation. The ith row in (2) is

$$e_i' K(\alpha) = e_i' \psi_i(\alpha) + (q_{ij} \hat{F}_{ij}(-\alpha))_{j \in E} = e_i' \psi_i(\alpha) + \sum_{j \in E} q_{ij} e_j' \hat{F}_{ij}(-\alpha)$$

Plugging in $-U(\gamma)$ instead of α yields

$$e_i' K(-U(\gamma)) = e_i' \psi_i(-U(\gamma)) + \sum_{j \in E} q_{ij} e_j' \hat{F}_{ij}(U(\gamma)) = \gamma e_i'$$

for all $i \in E$, where the last equality is due to (4). In this sense $-U$ is the inverse function of the cumulant matrix K.

6. Computation of the Laplace exponent

Although (4) looks like a nice equation to determine $U(\gamma)$, there is no obvious numerical approach to solve it. The next theorems provide an iteration which seems numerically stable.

For all phases $i \in E$, let $\phi_i(0)$ be the largest solution of the equation $\psi_i(\alpha) = 0$, $\alpha \geq 0$, cf. section 2.1. Denote the inverse functions of $\psi_i : [\phi_i(0), \infty[\to [0, \infty[$ by ϕ_i.

Theorem 3. *The matrix $U(\gamma)$ is given by*

$$e_i' U(\gamma) = -\phi_i(q_i + \gamma) e_i' + \phi_i(q_i) \sum_{k \in E} p_{ik} e_k' \hat{F}_{ik}(U(\gamma)) L_i(-U(\gamma))$$

for all $i \in E$, where

$$\hat{F}_{ik}(U(\gamma)) = \int_0^\infty e^{U(\gamma) y} \, d\tilde{F}_{ik}(y)$$

with $\tilde{F}_{ik}(0) = 1 - \lambda_{ik}^-$ and $\tilde{F}_{ik}(dy) = \lambda_{ik}^- F_{ik}(-dy)$, and further

$$L_i(-U(\gamma)) = \frac{q_i}{\phi_i(q_i)} \cdot (\phi_i(q_i + \gamma) I + U(\gamma)) \cdot ((q_i + \gamma) I - \psi_i(-U(\gamma)))^{-1}$$

for all $i, k \in E$.

Proof. Let $J_0 = i$ for some $i \in E$. Define the random variable $S^i := \exp(-\gamma G^i) \cdot M^i$, where M^i is the maximum and G^i is the time of maximum before the first phase change. The first equation in lemma 3 shows that the (defective) density function of S^i is given by

$$f_{S^i}(x) = \phi_i(q_i)e^{-\phi_i(q_i+\gamma)\cdot x}$$

Now define the random variable $Y^i := \exp(-\gamma(\tau_i - G^i)) \cdot (M^i - X_{\tau_i-})$, where τ_i is the time until the first phase change. Once the maximum M^i is reached the discounted height process $e^{-\gamma t}X_t$ descends by y units according to the distribution function F_{Y^i}, whereafter $\mathcal{J}$ jumps to another phase k with probability p_{ik}. This may be accompanied by another downward jump (z units) having distribution function $\tilde{F}_{ik}$. From there the ascend goes beyond M^i again with probability $e_k' e^{U(\gamma)(y+z)}$. Altogether this leads to the equations

$$\begin{aligned} U(\gamma)_{ij} &= -\phi_i(q_i+\gamma)\delta_{ij} + \phi_i(q_i)\int_0^\infty dF_{Y^i}(y)\sum_{k\in E} p_{ik}\int_0^\infty d\tilde{F}_{ik}(z)\; e_k' e^{U(\gamma)(y+z)}e_j \\ &= -\phi_i(q_i+\gamma)\delta_{ij} + \phi_i(q_i)\sum_{k\in E} p_{ik}e_k' \int_0^\infty e^{U(\gamma)z}d\tilde{F}_{ik}(z)\int_0^\infty e^{U(\gamma)y}dF_{Y^i}(y)e_j \end{aligned}$$

for $i,j \in E$. Since $U(\gamma)$ is a negative definite matrix, the second equation in lemma 3 can be applied to the argument $-U(\gamma)$ instead of the scalar $\beta > 0$. This yields

$$U(\gamma)_{ij} = -\phi_i(q_i+\gamma)\delta_{ij} + \phi_i(q_i)\sum_{k\in E} p_{ik}e_k'\hat{F}_{ik}(U(\gamma))L_i(-U(\gamma))e_j$$

for all $i,j \in E$. The statement is just the row vector version of the above equalities.

Example 3. The proof above is a straightforward generalisation of theorem 4.1 in Asmussen [1]. Taking up the case treated there (without jumps and $\gamma = 0$, see example 1), we obtain

$$L_i(-U(0)) = \frac{q_i}{\phi_i(q_i)} \cdot (\phi_i(q_i)I + U(0)) \cdot (q_i I - \psi_i(-U(0)))^{-1}$$

In order to ease the comparison, we shall use the same notations as in Asmussen [1], theorem 4.1, i.e. we set $\omega_i = \phi_i(q_i)$, $r_i = \mu_i$, and $\lambda_i = q_i$. Further

$$\psi_i(-U(0)) = -r_i U(0) + \frac{\sigma_i^2}{2}U(0)^2$$

The relations $\omega_i\eta_i = 2\lambda_i/\sigma_i^2$ and $\eta_i - \omega_i = 2r_i/\sigma_i^2$ lead to the factorisation

$$\begin{aligned} \lambda_i I - \psi_i(-U(0)) &= -\frac{\sigma_i^2}{2}U^2 + r_iU(0) + \lambda_i I = \frac{\sigma_i^2}{2}\left(-U(0)^2 + (\eta_i - \omega_i)U(0) + \omega_i\eta_i I\right) \\ &= \frac{\sigma_i^2}{2}(-U(0) + \eta_i I)(U(0) + \omega_i I) \end{aligned}$$

such that, if $U(0) + \omega_i I$ is invertible,

$$L_i(-U(0)) = \frac{\lambda_i}{\omega_i}\frac{2}{\sigma_i^2}(\eta_i I - U(0))^{-1} = \eta_i(\eta_i I - U(0))^{-1}$$

which specifies the exponential distribution for Y^i.

Remark 2. By assumption the phase process $\mathcal{J}$ is irreducible. This implies that no entry of any row $e_i' U(\gamma)$ equals zero, except in one case: If there are no jumps and no diffusion component (i.e. $\sigma_i = 0$, $\nu_i = \mathbf{0}$, and $F_{ij} = \delta_0$ for all $j \in E$), then $Y^i = 0$ a.s. since the Lévy process X^i is not a negative subordinator. Hence

$$\sum_{k\in E} p_{ik} e_k' \int_0^\infty e^{U(\gamma)z} d\tilde{F}_{ik}(z) \int_0^\infty e^{U(\gamma)y} dF_{Y^i}(y) e_j = \sum_{k\in E} p_{ik} e_k' I e_j = \sum_{k\in E} p_{ik}\delta_{kj} = p_{ij}$$

Thus for such rows we obtain $U(\gamma)_{ij} = \phi_i(q_i)p_{ij}$ for $i \neq j$ and $U(\gamma)_{ii} = -\phi_i(q_i + \gamma)$. Furthermore, the form of ϕ_i is particularly simple for such phases, namely $\phi_i(s) = s/\mu_i$ for all $s \geq 0$.

Theorem 4. *The matrix $U(\gamma)$ can be computed as the limit of the sequence $(U_n : n \geq 0)$ which is defined by $U_0 := -\Delta_{\phi(q+\gamma)}$ and $U_{n+1} := g(U_n)$ where we define the diagonal matrix $\Delta_{\phi(q+\gamma)} := diag(\phi_i(q_i + \gamma))_{i\in E}$ and the ith row of $g(U_n)$ is given by*

$$e_i' g(U_n) := -\phi_i(q_i + \gamma)e_i' + \phi_i(q_i) \sum_{k\in E} p_{ik} e_k' \hat{F}_{ik}(U_n) L_i(-U_n)$$

for all $i \in E$.

Proof. If U_n is a sub–generator matrix, then

$$v_n = \sum_{k\in E} p_{ik} e_k' \hat{F}_{ik}(U_n) L_i(-U_n)$$

is a sub–probability vector. Since further $\phi_i(q_i) < \phi_i(q_i + \lambda)$ for all $i \in E$, we see that $U_{n+1} = g(U_n)$ is a sub–generator matrix, too.

The inequalities $U_n \leq U_{n+1}$ hold entrywise as v_n is an increasing function of U_n within the domain of sub–generator matrices. This further implies $U_n \leq U(\gamma)$ for all $n \geq 0$, since $U_0 \leq U(\gamma)$ and $U(\gamma) = g(U(\gamma))$. Hence the sequence $(U_n : n \geq 0)$ converges monotonically towards a sub–generator matrix $U_\infty \leq U(\gamma)$.

It remains to show that $U(\gamma) \leq U_\infty$. Let N denote the number of phase changes before the maximum $M_{T(\gamma)} := \max\{X_t : t \leq T(\gamma)\}$ is reached, where $T(\gamma)$ is an independent random variable with exponential distribution of rate γ. Then

$$\mathbb{P}(M_{T(\gamma)} > x, N \leq n | X_0 = 0) = e^{U_n x}$$

for all $n \geq 0$, since every application of the function g adds another possible phase change. Hence

$$U_\infty = \lim_{n\to\infty} U_n = U(\gamma)$$

which is the statement.

Remark 3. For a phase $i \in E$ with $\sigma_i = 0$, $\mu_i = 1$, $\nu_i = \mathbf{0}$, and $F_{ij} = \delta_0$ for all $j \in E$, the singular case $(q_i + \gamma)I - \psi_i(-U_0) = \mathbf{0}$ arises. Remark 2 shows how to deal with this. We can compute the values for $e_i' U(\gamma)$ beforehand and put them as initial values into U_0, knowing that no iteration will change them.

Remark 4. To illustrate the relation between theorem 3 and equation (4), we show that the two statements are equivalent, given that the matrices $\phi_i(q_i+\gamma)I+U(\gamma)$ and $(q_i+\gamma)I-\psi_i(-U(\gamma))$ are invertible. Define the matrix $P(\gamma)$ by $e_i' P(\gamma) := \sum_{k\in E} p_{ik} e_k' \hat{F}_{ik}(U(\gamma))$ for $i \in E$. Starting from theorem 3, we obtain first

$$\begin{aligned}
e_i' U(\gamma)((q_i + \gamma)I - \psi_i(-U(\gamma))) &= e_i'\phi_i(q_i)P(\gamma)\frac{q_i}{\phi_i(q_i)} \cdot (\phi_i(q_i+\gamma)I + U(\gamma)) \\
&\quad - e_i'\phi_i(q_i+\gamma)((q_i+\gamma)I - \psi_i(-U(\gamma))) \\
&= e_i'\phi_i(q_i+\gamma)(Q(\gamma) - \gamma I) + e_i'\phi_i(q_i+\gamma)\psi_i(-U(\gamma)) \\
&\quad + e_i' q_i P(\gamma) U(\gamma)
\end{aligned}$$

for all $i \in E$, after observing that $e_i' Q(\gamma) = e_i' q_i (P(\gamma) - I)$. This is equivalent to

$$\begin{aligned}
\mathbf{0} &= e_i'\phi_i(q_i+\gamma)(\psi_i(-U(\gamma)) + Q(\gamma) - \gamma I) \\
&\quad + e_i' Q(\gamma)U(\gamma) - e_i'\gamma U(\gamma) + e_i' U(\gamma)\psi_i(-U(\gamma)) \\
&= e_i'(\psi_i(-U(\gamma)) + Q(\gamma) - \gamma I)(\phi_i(q_i+\gamma)I + U(\gamma))
\end{aligned}$$

since $U(\gamma)$ and $\psi_i(-U(\gamma))$ are commutative.

7. Diagonalisation and the Asmussen–Kella martingale

One possibility to determine the Laplace transform of the first hitting times $\tau(x)$ is the use of appropriate (vector–valued) martingales as proposed in Asmussen and Kella [5]. In this section we shall explore the relation between this approach and a diagonalisation of $U(\gamma)$.

Lemma 4. *Let X be a solution to equation (4), i.e.*

$$e_i'\psi_i(-X) + \sum_{k\in E} q_{ik} e_k' \int_0^\infty e^{Xy} d\tilde{F}_{ik}(y) = \gamma e_i'$$

for all $i \in E$. If the matrices $(q_i+\gamma)I - \psi_i(-X)$ are invertible, then X is uniquely determined as $X = U(\gamma)$.

Proof. The assumption implies

$$e_i'(\psi_i(-X) + Q(X) - \gamma I)(\phi_i(q_i+\gamma)I - X) = \mathbf{0}$$

for all $i \in E$, where $e_i'Q(X) = \sum_{k\in E} q_{ik} e_k' \int_0^\infty e^{Xy} d\tilde{F}_{ik}(y)$. Applying the arguments of remark 4 backwards, we arrive at

$$e_i'X = \phi_i(q_i) \sum_{k\in E} p_{ik} e_k' \hat{F}_{ik}(X) L_i(-X) - \phi_i(q_i+\gamma) e_i'$$

provided the matrices $(q_i+\gamma)I - \psi_i(-X)$ are invertible. Then theorem 4 shows that $X = U(\gamma)$ is the limit of a uniquely defined sequence of matrices.

Denote the moment generating function of $\tilde{F}_{ij}$ by $\hat{F}_{ij}(s) = \int_0^\infty e^{sx} d\tilde{F}_{ij}(x)$ and further define the matrix $\hat{F}(s) = (\hat{F}_{ij}(s))_{i,j\in E}$ for all complex numbers $s \in \mathbb{C}$ for which the entries $\hat{F}_{ij}(s)$ are defined.

Theorem 5. *If there are complex numbers $s_1, \ldots, s_m$ with $Re(s_j) > 0$ for all $j \in E$ and respective linearly independent vectors $h^1, \ldots, h^m$ such that*

$$e_i'(\psi_i(s_j)\cdot I + Q \circ \hat{F}(-s_j) - \gamma I)h^j = 0 \tag{6}$$

for all $i, j \in E$, then the matrix $U(\gamma)$ has a diagonalisation of the form $U(\gamma) = -H\Delta_s H^{-1}$, where $H = (h^1, \ldots, h^m)$ and $\Delta_s = diag(s_1, \ldots, s_m)$.

Proof. By lemma 4 it suffices to show that $U = -H\Delta_s H^{-1}$ solves equations (4). Since the vectors $h^1, \ldots, h^m$ build a base for $\mathbb{R}^m$, this is equivalent to

$$e_i'\psi_i(H\Delta_s H^{-1})h^j + \sum_{k\in E} q_{ik} e_k' \int_0^\infty e^{-H\Delta_s H^{-1} y} d\tilde{F}_{ik}(y) h^j = \gamma h_{ij}$$

for all $i, j \in E$, where h_{ij} is the (i,j)th entry of H.

The form of the ψ_i shows that $\psi_i(H\Delta_s H^{-1}) = H\psi_i(\Delta_s)H^{-1}$. Denoting the ith row vector of H by h_i, this yields for the first term

$$e_i'\psi_i(H\Delta_s H^{-1})h^j = h_i\psi_i(\Delta_s)e_j = h_i\psi_i(s_j)e_j = h_{ij}\psi_i(s_j)$$

For the second term we obtain

$$\begin{aligned}\sum_{k\in E} q_{ik}e_k' \int_0^\infty e^{-H\Delta_s H^{-1}y}d\tilde{F}_{ik}(y)h^j &= \sum_{k\in E} q_{ik}h_k \int_0^\infty e^{-\Delta_s y}d\tilde{F}_{ik}(y)e^j \\ &= \sum_{k\in E} q_{ik}h_{kj} \int_0^\infty e^{-s_j y}d\tilde{F}_{ik}(y) \\ &= e_i'(Q\circ \hat{F}(-s_j))h^j \\ &= \gamma h_{ij} - e_i'\psi_i(s_j)h^j\end{aligned}$$

where the last equality is due to the assumption stated in (6). Since $e_i'\psi_i(s_j)h^j = h_{ij}\psi_i(s_j)$, the proof is complete.

Theorem 6. *If the matrix $U(\gamma)$ has a diagonalisation of the form $U(\gamma) = -H\Delta_s H^{-1}$, with $H = (h^1, \dots, h^m)$ and $\Delta_s = diag(s_1, \dots, s_m)$, then $Re(s_j) > 0$ and*

$$e_i'(\psi_i(s_j)\cdot I + Q\circ \hat{F}(-s_j) - \gamma I)h^j = 0$$

for all $i, j \in E$.

Proof. This follows by the same arguments as in the proof to the theorem above, together with the observation that all eigenvalues of $U(\gamma)$ must have negative real part.

8. Positive jumps of phase–type and the distribution of the overshoot

An extension of the classical fluctuation theory for spectrally negative Lévy processes is the incorporation of positive jumps having phase–type distributions. The phase–type assumption is almost without loss of generality, due to the two following results. Asmussen et al. [4], proposition 1, shows that Lévy processes with phase–type jumps are dense within the class of one–dimensional Lévy processes (equipped with the Prokhorov distance). Further, Pistorius [12], proposition 1, shows that the joint distribution of first passage times and the respective overshoots (equipped with convergence in distribution) are continuous functionals of Lévy processes (equipped with convergence in law).

In the present section, we shall sketch an analogous extension for MAPs. Assume now that the Lévy measures ν_i have a positive part (i.e. with support on $]0, \infty[$), which is a compound Poisson process of jump intensity λ_i^+ and jump sizes of phase–type distribution with parameters $(\alpha^{(ii)}, T^{(ii)})$ for $i \in E$. Without loss of generality we assume that the distributions $PH(\alpha^{(ii)}, T^{(ii)})$ have no atom (which would be on zero anyway). Further we assume that a phase change from i to j may trigger a positive jump of distribution $PH(\alpha^{(ij)}, T^{(ij)})$ with probability λ_{ij}^+. Note that in connection with the spectrally negative part of the model described before this section the restriction $\lambda_{ij}^+ + \lambda_{ij}^- \leq 1$ applies. Again we may assume without loss of generality that the distributions $PH(\alpha^{(ij)}, T^{(ij)})$ have no atom, which means that a phase change from i to j without accompanying jump occurs with probability $1 - \lambda_{ij}^+ - \lambda_{ij}^- \leq 1$. Denote the respective exit vectors by $\eta^{(ij)} := -T^{(ij)}\mathbf{1}$, where $\mathbf{1}$ denotes the column vector (of appropriate dimension) with all entries being one.

Our aim is again an expression for the (matrix–valued) Laplace transform $\mathbb{E}(e^{-\gamma\tau(x)})$ of the first passage time $\tau(x) := \min\{t \geq 0 : X_t \geq x\}$ for $x > 0$. The possibility of positive jumps now generates a new random variable, the overshoot $o(x) := X_{\tau(x)} - x$. This is non–negative for all $x > 0$. We shall derive the distribution of $o(x)$, too. Since the most useful formula is the implicit equation of theorem 3, we will focus our attention on extending this.

The main idea is to spread out the phase–type positive jumps as a succession of linear pieces (each with positive slope one) of exponential duration, cf. Asmussen [2] or Badescu et al. [6]. Then the new process is again a MAP without positive jumps, to which we can apply the methods derived before. To account for the jump nature of the new phase–type upward movements, we simply omit to discount the time, i.e. we treat these phases with $\gamma = 0$. This is the same method as applied in Badescu et al. [6] for a fluid flow model. Thus we obtain again a phase–type form

$$\mathbb{E}(e^{-\gamma\tau(x)}|X_0 = 0) = e^{U'(\gamma)x}$$

for $x > 0$. Now the matrix $U'(\gamma)$ is of a larger order than its equivalent $U(\gamma)$ for the spectrally negative case, as we need additional phases to model the new upward movements.

Denote the order of the distribution $PH(\alpha^{(ij)}, T^{(ij)})$, which is the dimension of the vector $\alpha^{(ij)}$, by m_{ij}. We add the set

$$E_+ := \{(i, j, k) : i, j \in E, 1 \leq k \leq m_{ij}\}$$

of new phases to the old phase space E and denote the new phase space by $E' := E \cup E_+$, where it is understood that $E \cap E_+ = \emptyset$. Considering remark 2, the entries of $U'(\gamma)$ for a jump phase $p \in E_+$ can be computed immediately, i.e. without the iteration given in theorem 4. Altogether this yields the following extension of theorem 3.

Theorem 7. *Let $(\mathcal{X}, \mathcal{J})$ be a MAP with phase–type positive jumps. Then the matrix $U'(\gamma)$ governing the first passage times $\tau(x)$ is given by*

$$e_p' U'(\gamma) = \sum_{l=1}^{m_{ij}} T_{kl}^{(ij)} e'_{(i,j,l)} + \eta_k^{(ij)} e_j'$$

for $p = (i,j,k) \in E_+$, and

$$\begin{aligned} e_i' U'(\gamma) = -\phi_i(q_i + \gamma) e_i' + \phi_i(q_i) \sum_{j \in E} p_{ij} e_j' \hat{F}_{ij}(U'(\gamma)) L_i(-U'(\gamma)) \\ + \phi_i(q_i) \sum_{j \in E} p_{ij} \lambda_{ij}^+ \sum_{k=1}^{m_{ij}} \alpha_k^{(ij)} e'_{(i,j,k)} L_i(-U'(\gamma)) \end{aligned}$$

for all $i \in E$. Here

$$\hat{F}_{ij}(U'(\gamma)) = \int_0^\infty e^{U'(\gamma) y} \, d\tilde{F}_{ij}(y)$$

with $\tilde{F}_{ij}(0) = 1 - \lambda_{ij}^- - \lambda_{ij}^+$ and $\tilde{F}_{ij}(dy) = \lambda_{ij}^- F_{ij}(-dy)$, and further

$$L_i(-U'(\gamma)) = \frac{q_i}{\phi_i(q_i)} \cdot (\phi_i(q_i + \gamma) I + U'(\gamma)) \cdot ((q_i + \gamma) I - \psi_i(-U'(\gamma)))^{-1}$$

for all $i, j \in E$.

Proof. Instead of a completely new proof we explain only the new components, which do not appear in theorem 3. The first equation, i.e. the pth row for $p = (i,j,k) \in E_+$ is immediate from remark 2: The phase–type upward movement stays in phase (i,j,k) an exponential time of parameter $-T_{kk}^{(ij)}$ and switches to phase (i,j,l) with rate $T_{kl}^{(ij)}$ After an exit with rate $\eta_k^{(ij)}$ the upward movement is over, and the spectrally negative part begins again with phase $j \in E$. Note that there is no time discount in this first equation.

In the second equation only the last term requires a new explanation. It corresponds to the following case: After moving downwards from the maximum M^i by an amount of Y^i units, the phase changes from i to j with probability p_{ij}. This phase change is accompanied by a positive jump with probability λ_{ij}^+. This jump will now be modelled by a phase–type upward movement without time discount, for which we have introduced the new phases $(i,j,k) \in E_+$, $k = 1, \ldots, m_{ij}$. These phases are entered according to the probability vector $\alpha^{(ij)}$. After this, we need to climb again by an amount of y units in order to go beyond M^i. Starting from phase (i,j,k) and given y, this is done with probability $e'_{(i,j,k)}exp(U'(\gamma)y)$. Integrating over y by dF_{Y^i} yields $L_i(-U'(\gamma))$ as the last factor.

The conditional memoryless property of the phase–type distribution immediately yields the distribution of the overshoot $o(x)$. It has an atom at zero which is the probability of reaching the level x continuously. This can happen only in phases $j \in E$, whence we obtain

$$\mathbb{P}(o(x)=0, \tau(x)<\infty) = e'_{J_0} e^{U'(0)x} \mathbf{1}_E$$

where $\mathbf{1}_E = \sum_{j\in E} e_j$ denotes the column vector with all entries of index $j \in E$ being one, all other entries being zero. If the level x is passed by a positive phase–type jump, the overshoot is positive and has distribution

$$\mathbb{P}(o(x)>y, \tau(x)<\infty) = \sum_{i,j\in E} \sum_{k=1}^{m_{ij}} e'_{J_0} e^{U'(0)x} e_{(i,j,k)} \cdot e_k^{(ij)} e^{T^{(ij)}y}$$

where $e_k^{(ij)}$ is the m_{ij}–dimensional kth row base vector.

9. Numerical examples

Example 4. The topic of the first example is the equivalence of the martingale approach and the diagonalisation of $U(\gamma)$. We will first compute $U(\gamma)$ via the Asmussen–Kella martingale and then via the iteration in theorem 4. Set $E=\{1,2\}$, $q_1=9$, $q_2=21$, $\sigma_i=2$ for $i=1,2$, $\mu_1=0$, $\mu_2=-8$, and $\nu_i=\mathbf{0}$, $F_{ij}=\delta_0$ for $i,j\in E$. Further set $\gamma=2$.

For the martingale approach we need to find $s\in\mathbb{C}$ with positive real part such that

$$\det \begin{pmatrix} 2s^2-11 & 9 \\ 21 & 2s^2-8s-23 \end{pmatrix} = 0$$

The two values $s_1 = 1.5530$ and $s_2 = 6.1205$ satisfy these conditions. Two respective eigenvectors in equation (6) are $h^1 = (0.82452, 0.56584)'$ and $h^2 = (-0.13942, 0.99023)'$. With $H=(h^1,h^2)$ this yields

$$U(2) = -H\Delta_s H^{-1} = \begin{pmatrix} -1.95545 & 0.58644 \\ 2.85833 & -5.71805 \end{pmatrix}$$

A direct computation of $U(2)$ via the iteration in theorem 4 yields

$$U(2) = \begin{pmatrix} -1.95545 & 0.58643 \\ 2.85832 & -5.71803 \end{pmatrix}$$

which may be regarded as the same result.

Example 5. Let $\sigma_i^2 = 0$ and $\mu_i = 1$ for all $i \in E$. The Lévy measures shall have the simple form $\nu_i(-dx) = \lambda_i \eta_i e^{-\eta_i x} dx$, with some given parameters $\lambda_i, \eta_i > 0$. If the parameters are chosen such that the process has a negative mean drift, then the overall supremum M_∞ will be finite a.s. and $\mathbb{P}(M_\infty > x) = \exp(U(0)x)$, c.f. theorem 2. Since the time aspect here is neglected ($\gamma = 0$), we can compute $U = U(0)$ by a fluid flow model, too. This yields a numerical test.

First we specify the necessary ingredients for theorem 4:

$$\begin{aligned} \psi_i(-U) &= -U + \lambda_i \left(\int_0^\infty e^{Ux} \eta_i e^{-\eta_i x} dx - I \right) = -U + \lambda_i \left(\eta_i (\eta_i I - U)^{-1} - I \right) \\ \phi_i(q_i) &= \frac{\lambda_i - \eta_i + q_i}{2} + \sqrt{\left(\frac{\lambda_i - \eta_i + q_i}{2} \right)^2 + \eta_i q_i} \\ L_i(-U) &= \frac{q_i}{\phi_i(q_i)} \cdot (\phi_i(q_i) I + U) \cdot (q_i I - \psi_i(-U))^{-1} \end{aligned}$$

Let there be $m = 2$ phases and $q_1 = 1$, $q_2 = 2$. Further set $\eta_1 = 3$, $\lambda_1 = 4$, $\eta_2 = 4$, $\lambda_2 = 6$. Then theorem 4 yields

$$U = \begin{pmatrix} -2.5801 & 1.4238 \\ 3.5470 & -5.1301 \end{pmatrix}$$

after 15 iterations. However, the initial value U_0 had to be slightly perturbed to a new value $U_0 = -\Delta_{\phi(q)} + [0, 0.1; 0.1, 0]$ in order to avoid a singular matrix in the first iteration (cf. remark 3).

This result can be compared with the following fluid flow model (see Asmussen [1]): Set $E_+ = \{1, 2\}$, $E_- = \{3, 4\}$, $r_1 = r_2 = 1$, $r_3 = r_4 = -1$, and

$$\Lambda = \begin{pmatrix} -5 & 1 & 4 & 0 \\ 2 & -8 & 0 & 6 \\ 3 & 0 & -3 & 0 \\ 0 & 4 & 0 & -4 \end{pmatrix}$$

After 30 iterations according to theorem 3.1 in Asmussen [1], the obtained values for $\alpha^{(-+)}$ and $U = T^{(++)} + T^{(+-)}\alpha^{(-+)}$ are

$$\alpha^{(-+)} = \begin{pmatrix} 0.60496 & 0.10594 \\ 0.25783 & 0.47832 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} -2.5802 & 1.4238 \\ 3.5470 & -5.1301 \end{pmatrix}$$

which can be regarded as the same result.

Example 6. Set again $\gamma = 0$ and $U = U(0)$. In the previous example, the simple form of the Lévy measure led to a closed form expression of the matrices $\psi_i(-U)$, which greatly simplified numerical evaluations. In the case of phase–type jumps (say with parameters α^i, T^i and exit vector $\eta^i = -T^i\mathbf{1}$) such a closed form is not obvious, and we need to evaluate the integrals

$$\int_0^\infty e^{Ux}\alpha^i e^{T^i x}\eta^i dx$$

numerically. Furthermore, the inverse functions ϕ_i need to be evaluated numerically.

This example serves to demonstrate that this is possible from the view point of numerical stability. Again we let $\sigma_i^2 = 0$ and $\mu_i = 1$ for all $i \in E$. The Lévy measures shall have the form $\nu_i(-dx) = \lambda_i \alpha^i e^{T^i x}\eta^i dx$. Let there be $m = 2$ phases and $q_1 = 1$, $q_2 = 2$. Further set $\lambda_1 = 4$, $\lambda_2 = 6$ as well as

$$T^1 = \begin{pmatrix} -6 & 6 \\ 0 & -6 \end{pmatrix} \qquad \text{and} \qquad T^2 = \begin{pmatrix} -8 & 8 \\ 0 & -8 \end{pmatrix}$$

with $\alpha^1 = \alpha^2 = (1, 0)$. After 15 iterations according to theorem 4, starting with $U_0 = \Delta_{\phi(q)}$, we arrive at

$$U = \begin{pmatrix} -2.9148 & 1.4329 \\ 3.6593 & -5.7093 \end{pmatrix}$$

Here the integration has been performed in a naive manner by a Riemann upper sum with step size 10^{-4} until the tail is bounded by 10^{-4}. The inverse function can be determined easily, since the functions $\psi_i : [\phi_i(0), \infty[\to [0, \infty[$ are strictly increasing. This has been done to an accuracy of 10^{-5}.

This result can again be compared against a fluid flow case. Its specification is $E_+ = \{1, 2\}$,

$E_- = \{3,4,5,6\}$, $r_1 = r_2 = 1$, $r_i = -1$ for $i \in E_-$, and

$$\Lambda = \begin{pmatrix} -5 & 1 & 4 & 0 & 0 & 0 \\ 2 & -8 & 0 & 0 & 6 & 0 \\ 0 & 0 & -6 & 6 & 0 & 0 \\ 6 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 8 \\ 0 & 8 & 0 & 0 & 0 & -8 \end{pmatrix}$$

After 30 iterations according to theorem 3.1 in Asmussen [1], the values for $\alpha^{(-+)}$ and $U = T^{(++)} + T^{(+-)}\alpha^{(-+)}$ are

$$\alpha^{(-+)} = \begin{pmatrix} 0.521216 & 0.108179 \\ 0.708560 & 0.086688 \\ 0.276425 & 0.381737 \\ 0.202654 & 0.604703 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} -2.9151 & 1.4327 \\ 3.6586 & -5.7096 \end{pmatrix}$$

Given the crudeness of the numerical integration procedure, the accuracy is reasonable. However, the 30 iterations for the fluid flow case were computed much quicker than the 15 iterations for the jump case, which of course is due to simpler iteration rules.

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