MA310 MAPLE — Example Sheet 2

1 Solving Equations

*Maple* is able to solve a variety of mathematical equations.

<table>
<thead>
<tr>
<th>Maple Command</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>solve(f(x)=0,x)</td>
<td>solves the equation ( f(x) = 0 ) for ( x )</td>
</tr>
<tr>
<td>fsolve(f(x)=0,x)</td>
<td>numerically finds the real roots of ( f(x) = 0 )</td>
</tr>
<tr>
<td>fsolve(f(x)=0,x,complex)</td>
<td>numerically finds all roots of ( f(x) = 0 )</td>
</tr>
<tr>
<td>subs(x=a,expr)</td>
<td>replaces each ( x ) in expr with ( a )</td>
</tr>
</tbody>
</table>

**Example 1.** Use *Maple* to solve the equation \( 3x + 7 = 0 \).

\[
> \text{solve}(3*x+7,x);
\]

\[-\frac{7}{3}\]

**Example 2.** Use *Maple* to solve the equation \( 3x^2 - 5x + 1 = 0 \).

To obtain the solution in terms of radicals (i.e., square roots), we use the `solve` command.

\[
> \text{solve}(3*x^2-5*x+1,x);
\]

\[
\frac{5}{6} + \frac{1}{6}\sqrt{13}, \quad \frac{5}{6} - \frac{1}{6}\sqrt{13}
\]

Note that there are two solutions, separated by a comma. To obtain a numerical approximation to the solution we use the `evalf` command

\[
> \text{evalf}(%);
\]

1.434258546, 2.324081208

Alternatively we use the `fsolve` command.

\[
> \text{fsolve}(3*x^2-5*x+1,x);
\]

.2324081208, 1.434258546

(Note that the solution is given to 10 digits.)

**Example 3.** Use *Maple* to solve the equation \( x^2 - 2x + 2 = 0 \).

\[
> \text{solve}(x^2-2*x+2,x);
\]

\[
1 + I, 1 - I
\]

In *Maple*, the complex number \( i = \sqrt{-1} \) is represented by \( I \). *Maple* has obtained the complex roots \( 1 + i \) and \( 1 - i \) of the equation \( x^2 - 2x + 2 = 0 \).

**Example 4.** Use *Maple* to find the inverse of \( f(x) = \frac{2x - 3}{1 - 5x} \).

To find the inverse we need to solve the equation \( y = \frac{2x - 3}{1 - 5x} \) for \( x \).
> solve(y=(2*x-3)/(1-5*x),x);

\[ \frac{y + 3}{5y + 2} \]

Thus the inverse is given by \( f^{-1}(y) = \frac{y + 3}{5y + 2} \).

**Example 5.** Use Maple to solve \( x^3 - 1 = 0 \).

If we use the `solve` command then we obtain three solutions

\[
1, -\frac{1}{2} + \frac{1}{2}i\sqrt{3}, -\frac{1}{2} - \frac{1}{2}i\sqrt{3}
\]

However if we use the `fsolve` command (without the `complex` option) then we obtain only the real solution

\[
> \text{fsolve}(x^3-1,x);
\]

1.

If we use the `fsolve` command with the `complex` option then we obtain numerical approximations to all three solutions

\[
> \text{fsolve}(x^3-1,x,\text{complex});
\]

\[-.5000000000 - .8660254038I, -.5000000000 + .8660254038I, 1.\]

**Verifying solutions**

To check a solution, one can substitute it back into the original equation.

**Example 6.** Verify the solution of Example 1.

\[
> \text{sol:=solve}(3*x+7,x);
\]

\[
sol := \frac{-7}{3}
\]

> subs(x=sol,3*x+7);

0

**Example 7.** Verify the solution of Example 2.

\[
> \text{sol:=solve}(3*x^2-5*x+1,x);
\]

\[
sol := \frac{5}{6} + \frac{1}{6}\sqrt{13}, \frac{5}{6} - \frac{1}{6}\sqrt{13}
\]

> simplify(subs(x=sol[1],3*x^2-5*x+1));

0

> simplify(subs(x=sol[2],3*x^2-5*x+1));

0

2
Remarks

⋆ You might need to use simplify or expand in some cases to get 0.

⋆ sol[1] and sol[2] are respectively the first and second solutions in the set sol.

⋆ Note that the commands simplify and subs are nested. Maple performs subs first then simplify.

Exercises

1. Use Maple help to get more information about the Maple commands solve, fsolve, and subs.

2. Use the solve command find the solutions of:
   (i) $x^2 - x + 6 = 0$ and (ii) $x^4 - 4x^3 + 5x^2 - 2x - 6 = 0$.
   Verify your answers.

3. Use Maple to find approximate numerical solutions of:
   (i) $x^4 - 4x^3 + 5x^2 - 2x - 6 = 0$ and (ii) $\exp(-x) = x$.

4. Use Maple to solve $|4x - 5| = |x + 1|$, where $|x|$ is the absolute value of $x$.

5. Plot the following cubic equations and determine the number of real roots:
   (i) $x^3 - 4x + 1 = y$, (ii) $x^3 - 6x - 6 = y$, (iii) $x^3 - 7x - 7 = y$.
   Verify your answers by solving for the roots.

2 Differentiating

Calculus is one of the most important tools used in applied Mathematics. Maple provides many powerful tools for solving problems in calculus. These include taking limits, differentiation, integration and solving ordinary differential equations.

<table>
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<th>Maple Command</th>
<th>Meaning</th>
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<tr>
<td>limit(f,x=a)</td>
<td>compute the limit of $f$ as $x \to a$, i.e. $\lim_{x \to a} f$</td>
</tr>
<tr>
<td>diff(f,x)</td>
<td>differentiate $f$ with respect to $x$, i.e. $\frac{df}{dx}$</td>
</tr>
</tbody>
</table>

In the first example sheet we saw that Maple allows you to use the assignment command := to give a name to an expression. For example, we set $f = x^3$ as follows:

\[
> f := x^3;
\]

\[
f := x^3
\]
You can also define functions using Maple’s “arrow notation” \rightarrow. This allows Maple
know how to evaluate the function when it appears in Maple expressions. For example
we define the function \( f(x) = x^3 \) as follows:

\[
> f := x \rightarrow x^3;
\]

Then evaluating \( f \) at an argument produce the cube of \( f \)’s argument, for example

\[
> f(2);
8
\]

\[
> f(y+1);
(y + 1)^3
\]

**Limits**

*Maple* is useful for computing limits of functions.

**Example 8.** Obtain the limit of

(a) \( f(x) = \frac{x^2 - 1}{x^2 + 1} \) as \( x \to 2 \),
(b) \( f(x) = \frac{x^2 + 2x + 1}{x^4 + 3x^3 + 7x^2 + x + 2} \) as \( x \to 1 \).

\[
> f := x \rightarrow (x^2-1)/(x^2+1);
\]

\[
> \text{limit}(f(x),x=2);
3/5
\]

\[
> f := x \rightarrow (x^2+2*x+1)/(x^4+3*x^3+7*x^2+x+2);
\]

\[
> \text{limit}(f(x),x=1);
2/7
\]

*Maple* can evaluate the limit of a function \( f(x) \) which, although is undefined at \( x = a \),
possesses a limit as \( x \to a \).

**Example 9.** Obtain the limit of

(a) \( f(x) = \frac{x^2 - 4}{x - 2} \) as \( x \to 2 \),
(b) \( f(x) = \frac{\sin(x)}{x} \) as \( x \to 0 \).

\[
> f := x \rightarrow (x^2-4)/(x-2);
\]

\[
> \text{limit}(f(x),x=2);
\]
> limit(f(x), x=2);
4

> f := x -> sin(x)/x;
f := x → sin(x)/x

> limit(f(x), x=0);
1

It is also possible to take the limit of a function $f(x)$ at the point $x = a$ from either the positive or negative direction. These are often called "one-sided limits" and written as $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$, respectively.

**Example 10.** Calculate the limit of $\tan(x)$ as $x \to \pi/2$ from the left and right.

> limit(tan(x), x=Pi/2,left);
$\infty$

> limit(tan(x), x=Pi/2, right);
$-\infty$

However if one attempts to get Maple to calculate the limit of $\tan(x)$ as $x \to \pi/2$ without specifying from the left or right, then it gives the answer as "undefined".

> limit(tan(x), x=Pi/2);
undefined

**Differentiation**

The derivative $f'(x)$ of the function $f(x)$ with respect to $x$ is defined by the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists. Maple can be used to evaluate such limits.

**Example 11.** Let $f(x) = x^3$. Compute and simplify

(a), $\frac{f(x+h) - f(x)}{h}$ and (b), $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

> f := x -> x^3;
f := x → x^3

> g := (f(x+h)-f(x))/h;
g := (x + h)^3 - x^3

> simplify(g);
$3x^2 + 3xh + h^2$
Maple can symbolically calculate the derivative of any expression containing polynomial terms and functions “known” to Maple (such as exp, log, sin, cos, tan).

Example 12. Differentiate the following with respect to $x$:

(a) $\sin x$,  
(b) $3x^4 - 4x^2 - 5$,  
(c) $\sqrt{x^2 + 1}(x^3 + 2x - 1)$.

Higher derivatives can be calculated by repeating the variables. For example the second and third derivatives of $\sin x$ is calculated as follows

An alternative method of calculating the second and third derivatives of $\sin x$ is

Consequently the $N^{th}$ derivative of $f(x)$ with respect to $x$ is given by

Exercises

6. Use Maple help to get more information about the Maple commands limit and diff.

7. Using Maple’s arrow notation, define the following functions and determine $f(2)$:

(i) $f(x) = x^4 - 3x^3 - 2x^2 + 7$  
(ii) $f(x) = [x^3 - 2x \sin(\pi x)] \exp(-x^2)$
8. Let \( f(x) = \frac{x^2 - 2x + 1}{x^4 + 3x^3 - 7x^2 + x + 2} \). (a) Obtain the limit of \( f(x) \) as \( x \to 1 \).
   (b) What happens if one substitutes the value \( x = 1 \) into \( f(x) \)?
   (c) First factor \( f(x) \) then substitute the value \( x = 1 \).

9. Let \( f(x) = \frac{|x|}{x} \). Calculate \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \). Show that \( \lim_{x \to 0} f(x) \) is undefined.

10. Using the formal definition of the derivative, calculate the derivatives of the following:
    (i) \( f(x) = \sin x \)  
        (ii) \( f(x) = \ln x \)

11. Differentiate the following functions:
    (i) \( f(x) = \sqrt{\frac{1-x}{1+x}} \)  
        (ii) \( f(x) = x \cos \frac{1}{x} \)

12. Calculate and simplify
    (i) \( \frac{d^3}{dx^3} (e^{2x} \ln x) \),  
        (ii) \( \frac{d^4}{dx^4} (x^3 \exp(-x^2)) \),  
        (iii) \( \frac{d^5}{dx^5} \left( \frac{1}{x^2 + x - 6} \right) \).

13. Challenge Exercise. Let \( f(x) = \exp \left( -\frac{1}{x^2} \right) \). Calculate and simplify the first, second, third and fourth derivatives of \( f(x) \), i.e. \( f'(x) \), \( f''(x) \), \( f'''(x) \) and \( f''''(x) \).
    Evaluate the limit as \( x \to 0 \) of \( f(x) \), \( f'(x) \), \( f''(x) \), \( f'''(x) \) and \( f''''(x) \). Plot \( f(x) \) for \(-10 \leq x \leq 10\). What is \( \lim_{x \to 0} \frac{d^n f}{dx^n} \) for any \( n \)? Can you prove your conjecture?

3 Sets and lists in Maple

Being a computer system for doing mathematics, Maple supports a variety of data structures, such as lists, arrays or tables as well as mathematical objects like sets, polynomials, vectors or matrices. Of particular importance are sequences and lists. A sequence is just a collection of expressions, separated by commas:
\[
\texttt{> s1:= a,A,b,B,123,sin;} \\
\texttt{s1 := a, A, b, B, 123, sin}
\]

To create large sequences without typing in all the individual expressions, one can use the command ‘seq’. This works if the expressions follow some general rule:
\[
\texttt{> s2:= seq(i^2,i=1..10);} \\
\texttt{s2 := 1, 4, 9, 16, 25, 36, 49, 64, 81, 100}
\]
Sequences can be collected in lists and sets by using the brackets ‘[,’ and ‘{,’ respectively. As in mathematics, the major difference between lists and sets is that lists keep track of the order of the sequence and possible duplicates, whereas in sets the order is chosen arbitrarily and duplicates are discarded. The empty list or set is defined using empty brackets.

**Example 13.** Compare and explain the Maple output of the following commands:

- Elist:= [ ]; Alist:= [a,b,c,c,1,2]; Blist :=[c,b,2,a,1,a];
- Clist:= [1,b,2,1,3,4]; Slist:= [seq((-1)^i, i=1..10)];
- E:= { }; A:= {a,b,c,c,1,2}; B:= {c,b,2,a,1,a};
- C:= {1,b,2,1,3,4}; S:= {seq((-1)^i, i=1..10)};

Selection of elements from sets and lists is done with op(i, expr), which returns the i\textsuperscript{th} operand of expr:

- Alist:= [a,b,c,c,1,2]; A:= {a,b,c,c,1,2};

\[
Alist := [a, b, c, c, 1, 2] \]
\[
A := \{1, 2, a, b, c\}
\]
- x:= op(3, Alist); y:= op(3, A);

\[
x := c
\]
\[
y := b
\]

Notice that Maple changed the order of the set A, which is ok because the order of elements in a set is arbitrary.

**Operations on sets**

The basic operations with sets are:

<table>
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<th>Maple Command</th>
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<tbody>
<tr>
<td>A union B</td>
<td>returns the union of the sets A and B</td>
</tr>
<tr>
<td>A intersect B</td>
<td>returns the intersection of the sets A and B</td>
</tr>
<tr>
<td>A minus B</td>
<td>returns the difference set A\B</td>
</tr>
<tr>
<td>member(a,A)</td>
<td>returns ‘true’ if ( a \in A ) and ‘false’ otherwise.</td>
</tr>
</tbody>
</table>

**Example 14.**

- A; C; Auc:= A union C; AiC:= A intersect C; AmC:= A minus C; member(a,A);
- member(a,C);

\[
\{1, 2, a, b, c\}
\]
\[
\{1, 2, 3, 4, b\}\]
\[ AuC := \{1, 2, 3, 4, a, b, c\} \]
\[ AiC := \{1, 2, b\} \]
\[ AmC := \{a, c\} \]
\[ true \]
\[ false \]

Remember that sets themselves can be elements of other sets. Think about the output of the following MAPLE commands:

\[
> X := \{\}, \{a, b\}; Y := \{a, b\}; member(\{\}, X); member(\{\}, Y); member(X, Y); member(Y, X);
\]

The cardinality of a finite set can be determined in MAPLE with the \texttt{nops}:

\[
> A := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}; B := \{2, 4, 6, 7, 9\}; nops(A); nops(B);
> nops(A minus B); nops (B minus A);
\]

\[
\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}
\]

\[
\{2, 4, 6, 7, 9\}
\]

\[
10
\]

\[
5
\]

\[
5
\]

\[
0
\]

Recall that the \texttt{powerset} of a set \( A \) is the set whose elements are the subsets of \( A \). There is a MAPLE procedure to construct the powerset of any given set. To make it work, you have to load the MAPLE - combinatorics package 'combinat':

\[
> \text{with(combinat);}
> A := \{1, a\}; PA := \text{powerset}(A);
\]

\[
A := \{1, a\}
\]

\[
PA := \{\{\}, \{1\}, \{1, a\}, \{a\}\}
\]

If \( A \) is a set and \( k \) a number, then the Maple function \texttt{choose(A,k)} returns the set of subsets of \( A \) with cardinality \( k \). You might find it instructive to compare \texttt{choose(n,k)} and \texttt{binomial(n,k)} for integers \( n \) and \( k \).

\[
> A := \{a, 2, b, 1, 3\}; \text{choose}(A, 2);
\]

\[
A := \{1, 2, 3, a, b\}
\]

\[
\{\{1, a\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, b\}, \{2, a\}, \{2, b\}, \{3, a\}, \{3, b\}, \{a, b\}\}
\]
Exercises

14. Use the \texttt{seq} command to create the set consisting of the first 10 cubic numbers (i.e. third powers).

15. The rules of de Morgan state that for arbitrary sets $A$, $B$ and $C$ the following holds

\[ A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C), \quad A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C). \]

Verify these rules with \textit{Maple} by doing some examples with sets like $A := \{1,2,3,4,5,6,7,8,9,10\}$, $B := \{2,7,4,9,6\}$, $C := \{3,1,2,9,4,10\}$.

16. a) Find all positive integer numbers less or equal to 10000 that are squares and cubics (third powers) at the same time. Give a mathematical argument for your findings.

b) Create the set of all integers between 1 and 100, that are not divisible by any square number $n^2 > 1$. How many of these numbers exist?

c) Create the set of all integers between 1 and 100. From this set subtract all even numbers and all multiples of 3, 5 or 7. Characterize the remaining set of numbers.

17. Let $X := \{\{\}, \{\{\}\}, a\}$ and $Y$ the powerset of $X$. Is $\{\{\}\}$ an element of $X$ or of $Y$? Is $\{\{a\}, \{\}\}$ an element of $Y$ or of the powerset of $Y$? First try to decide for yourself. Then use MAPLE to check.

18. Use \textit{Maple} to determine the size of the powersets of the sets $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, \ldots. Derive from your results a formula for the size of the powerset of a finite set. Can you prove it?

19. Let $A$ be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f, g, h, i\}$. Create the set of all subsets of $A$ of size two, that contain at least one letter (e.g. $\{1, a\}$ or $\{c, d\}$). Try to determine the number of those subsets without \textit{Maple}. Do the same thing for the subsets of size three, i.e. find (the number of) all subsets of of size three in $A$ which contain at least one letter.

25 November 2008