MA563 Calculus of Variations

Prof. Elizabeth Mansfield
Office: E213 Institute of Mathematics and Statistics
Telephone: ×3057,
Email: E.L.Mansfield@kent.ac.uk

Exercise Sheet - Noether's Theorem 2

Q1. This question is about Kepler's problem, and asks you to fill in the missing details of the lecture notes. The Lagrangian is

$$\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\frac{k}{\sqrt{x^2+y^2+z^2}}\right]\mathrm{d}x.$$

1. Show that invariance under translation in time t leads to the conserved quantity
$$\mathcal{E}=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{k}{\sqrt{x^2+y^2+z^2}},$$
which is called the "energy" of the system.
2. We showed in lectures that if
$$\mathbf{x}=(x,y,z),\qquad \dot{\mathbf{x}}=(\dot{x},\dot{y},\dot{z})$$
then on solutions of the Euler Lagrange system,
$$\mathbf{c}=\mathbf{x}\wedge\dot{\mathbf{x}}$$
where $\mathbf{c}=(c_{yz},c_{zx},c_{xy})$ are three constants, known as the angular momenta of the system.
Let R be a rotation matrix, independent of time t, that takes $\mathbf{c}$ to $(0,0,\omega)^T$, ie, takes $\mathbf{c}$ to the z-axis, where $\omega=|\mathbf{c}|$. Assume $\omega\neq 0$. Let
$$\begin{pmatrix}X\\Y\\Z\end{pmatrix}=R\begin{pmatrix}x\\y\\z\end{pmatrix}$$
be the new co-ordinates. Show that on solutions of the transformed Euler Lagrange system, $Z=\dot{Z}\equiv 0$.
Hint: Show for any vectors $\mathbf{v}$, $\mathbf{w}$ and any rotation matrix R that $R(\mathbf{v}\wedge\mathbf{w})=R(\mathbf{v})\wedge R(\mathbf{w})$.

Suppose now we are in co-ordinates where $\mathbf{c}$ *lies on the* z*-axis so that solutions lie on the* (x,y)*-plane, ie* $z\equiv 0$*. We may now set* $z=0$ *into the Lagrangian, the expression for* $\mathcal{E}$ *and the Euler Lagrange equations.*

3. Let $\rho = x^2+y^2$. Find the ODE for ρ implied by the Euler Lagrange equations and the conservation of energy. Integrate it once to obtain one new constant of integration. What does this new constant mean physically?
4. Let $r = \sqrt{\rho}$ and set $x = r\cos\theta$, $y = r\sin\theta$. From $x\dot{y} - \dot{x}y = \omega$, find an expression for $\dot{\theta}$.
5. In this example it is easy to find the equation for θ, the parameter of the one-parameter group action involving x and y. Another way is to set

$$A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Write $\dot{A}A^{-1}$ as a function of ω, ρ and $\dot{\rho}$. Hence obtain a linear ODE system for x and y, and show it has the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f & g \\ -g & f \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for some f and g depending on t. Find the eigenvalues and eigenvectors of the matrix and hence solve for x and y in terms of the exponentials of $\int f\,\mathrm{d}t$ and $\int g\,\mathrm{d}t$. Hint: show the eigenvectors are independent of t.

Q2. We have studied first order several rotation invariant Lagrangians in the plane, where three invariants are $\rho = x^2 + y^2$, $\dot{\rho}$ and $\chi = x\dot{y} - \dot{x}y$. By dimensional considerations, we expect any invariant to be a function of these three. Show the invariant $\dot{x}^2 + \dot{y}^2$ can indeed be written as a function of them.

Q3. Generalise the previous question to three dimensions; write $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ in terms of $\rho = x^2 + y^2 + z^2$, $\dot{\rho}$, $\chi_{yz} = y\dot{z} - \dot{y}z$, $\chi_{zx} = x\dot{z} - \dot{x}z$ and $\chi_{xy} = x\dot{y} - \dot{x}y$.

Q4. It is possible to plot expressions involving solutions to DEs using Maple's `odeplot` once the numeric solution of the DEs has been calculated using `dsolve`, see the help page for `odeplot` for examples. Solve the ODEs for ρ and θ using `dsolve` with `method=numeric`; you will need to specify values for $\mathcal{E}$, k and ω as well as $\theta(0)$, $\rho(0)$ and $\rho'(0)$. Then plot the expression for the energy (in terms of ρ) versus time t. Does the numerical solution preserve energy? Now plot $(x(t), y(t))$. What do you observe? Is the numerical scheme trustworthy for this example?