MA563 Calculus of Variations

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Exercise Sheet - Noether's Theorem

Q1. Consider the following one-parameter group action,

$$\begin{pmatrix} \alpha \cdot x \\ \alpha \cdot u \end{pmatrix} = \begin{pmatrix} \frac{2}{3}e^{2\alpha} + \frac{1}{3}e^{-\alpha} & \frac{2}{3}\left(e^{2\alpha} - e^{-\alpha}\right) \\ \frac{1}{3}\left(e^{2\alpha} - e^{-\alpha}\right) & \frac{1}{3}e^{2\alpha} + \frac{2}{3}e^{-\alpha} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Show that this is indeed a one-parameter group action. Hint: let the 2×2 matrix be called $A(\alpha)$ and use Maple to show $A(\beta)A(\alpha) = A(\alpha+\beta)$.
Find the infinitesimals ξ and ϕ.
Solve the ODE system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi(x,u), \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \phi(x,u)$$

with initial data $(x(0), u(0))$ and compare the result to the one-parameter group action. What do you observe?
Show that $\alpha \cdot (x+u) = e^{2\alpha}(x+u)$ and that $\alpha \cdot (2u - x) = e^{-\alpha}(2u-x)$ and conclude that the zeroth order invariant is $(x+u)(2u-x)^2$.
Plot the solutions of the ODE system in the plane, and compare the result to plots of $(x+u)(2u-x)^2 = c$ for various constants c. What do you observe?
Note: the point of this question is to illustrate how one-parameter actions are actually time $= \alpha$ *flow maps for autonomous vector fields (AKA solutions of autonomous first order ODE systems); autonomous means that* t *(or* α*) does not appear explicitly in the differential system to be solved. The one-parameter group action condition at* $\alpha = 0$ *is the initial condition, while the condition* $\beta \cdot (\alpha \cdot x) = (\alpha + \beta) \cdot x$ *is simply "if you flow for time* α *and then flow for time* β*, you arrive at the same spot as if you flow for time* $\alpha + \beta$*".*

Q2. Consider the scaling action

$$\alpha \cdot x = e^{\alpha}x, \qquad \alpha \cdot u = e^{3\alpha}u.$$

Using the recursive definition

$$\alpha \cdot \frac{\mathrm{d}^n u}{\mathrm{d}x^n} = \frac{\mathrm{d}}{\mathrm{d}\left(\alpha \cdot x\right)}\left(\alpha \cdot \frac{\mathrm{d}^{n-1}u}{\mathrm{d}x^{n-1}}\right)$$

find the induced actions on u_x, u_{xx}, u_{xxx} and find and prove the general formula for the action on the nth derivative of u.

Find invariants involving u, u_x, u_{xx} and then the nth derivative of u.
Show that

$$(\alpha \cdot x)\frac{\mathrm{d}}{\mathrm{d}(\alpha \cdot x)} = x\frac{\mathrm{d}}{\mathrm{d}x}$$

so that $\mathcal{D} = x\,\mathrm{d}/\mathrm{d}x$ is an invariant operator. Denoting your zeroth order invariant by $\mathcal{I}$ (say), show that $\mathcal{D}^n\mathcal{I}$ yields an nth order invariant for all n (but not necessarily the one you found above by inspection).

Q3. The Lagrangian $L = u_x^3/u^2\mathrm{d}x$ is invariant under two different one-parameter group actions, translation in x and the scaling action,

$$\alpha \cdot x = e^{\alpha}x, \qquad \alpha \cdot u = e^{2\alpha}u.$$

Find the first integrals for $E(L) = 0$ guaranteed by Noether's Theorem.
Show that the actions induce an action on the solution space.
Show that the action induces an action on the differential expression $E(L)$. (Use the definition in Q2 above to obtain the induced action on u_{xx}.) What do you observe?

Q4. In lectures we proved that

$$\phi_{[x]} = \frac{\mathrm{d}\phi}{\mathrm{d}x} - u_x\frac{\mathrm{d}\xi}{\mathrm{d}x}.$$

Show by induction that if

$$\phi_{[nx]} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0}\frac{\mathrm{d}^n u}{\mathrm{d}x^n}$$

then

$$\phi_{[nx]} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\phi - u_x\xi\right) + \xi\frac{\mathrm{d}^{n+1}u}{\mathrm{d}x^{n+1}}.$$

Verify the result on the scaling action given in Q2 above.
This result is used to obtain the first integral for higher order invariant Lagrangians.

Q5. For the one-parameter action in Q1, find the induced action on u_x and the induced action on $\mathrm{d}x$.
What change of variable makes the one-parameter group action a simple scaling (in your new variables)?
Using your new variables, have a go at finding the invariants involving u_x and $\mathrm{d}x$.