Please hand in Q3, Q5 on Exercise III and Q1 on Exercise IV to the General Office by 12:00 on Wednesday December 16th.

MA563 Exercise IV – Calculus of Variations

Q1. Find the extremals of

$$\mathcal{L}[u] = \int_0^{\pi/2} (u_x^2 - u^2)\mathrm{d}x$$

subject to $u(0) = 0$, $u(\pi/2) = 12$ and

$$\int_0^{\pi/2} xu \ \mathrm{d}x = \pi^2.$$

Q2. Minimize the integral

$$\mathcal{L}[y] = \int_0^1 y'^2 \mathrm{d}x$$

subject to $y(0) = y(1) = 0$ and

$$\int_0^1 y^2 \ \mathrm{d}x = 1.$$

Hence compute the minimum value of $\mathcal{L}[u]$.

Q3. Let $\mathcal{I}$ and $\mathcal{J}$ be defined by

$$\mathcal{I}[u, v] = \int_{x_0}^{x_1} (1 - \frac{u}{\sqrt{u^2 + v^2}})u_x \mathrm{d}x$$

and

$$\mathcal{J}[u, v] = \int_{x_0}^{x_1} v^2 u_x \mathrm{d}x.$$

Suppose $u(x)$ and $v(x)$ are the extremals for $\mathcal{I}$ subject to the constraint $\mathcal{J} = K$, where K is a positive constant. Prove that neither $u(x)$ nor $v(x)$ can be identically zero and there exists a constant Λ such that

$$u = \Lambda(u^2 + v^2)^{3/2} \ .$$

Q4. (Geodesics on a cylinder) Find the extremals of

$$\mathcal{I}[x, y, z] = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2 + z'^2} \mathrm{d}t.$$

subject to constraint $x^2 + y^2 = 1$ and the boundary conditions

$$(x(t_0), y(t_0), z(t_0)) = (1, 0, 0) \quad \text{and} \quad (x(t_1), y(t_1), z(t_1)) = (0, 1, \pi/2).$$