MA563 Exercise III – Calculus of Variations

Q1. Find the Euler-Lagrange equation for the functional

$$\mathcal{L}[y] = \int_a^b (y''^2 + 2y'^2 + y^2)\mathrm{d}x$$

and show that the solution of the equation is

$$y(x) = \exp(x)(A + Bx) + \exp(-x)(C + Dx),$$

where the A, B, C and D are constants which can be determined from the endpoint conditions.

Q2. Find the extremals of the intergral

$$\mathcal{L}[u, v] = \int_0^1 (u_x^2 + v_x^2 + u_x v_x)\mathrm{d}x$$

with the endpoint conditions $u(0) = v(0) = 0$, $u(1) = 1$ and $v(1) = -1$.

Q3. Consider the extremising of the integral

$$\mathcal{L}[y] = \int_0^b y^2 y'^2 \mathrm{d}x \ ,$$

where the left-hand endpoint lies at $(0, 0)$ and the right-hand endpoint lies on the curve $y^2 - x^2 = 9$. Find the extremal and give the extremising value of the integral.

Q4. Extremise the integral

$$\mathcal{L}[y] = \int_5^{x_1} \frac{\sqrt{1 + y'^2}}{y}\mathrm{d}x \ , \qquad x_1 > 5 \ ,$$

over all curves joining the point $(5, 5)$ and the line $g(x) = x - 5$.

Q5. Find and solve the Euler-Lagrange eqation for the integral

$$\mathcal{L}[u, v] = \int_0^\pi (u_x^2 + v_x^2 + 2uv + 2v)\mathrm{d}x$$

subject to $u(0) = v(0) = 0$ and the natural boundary condition at $x = \pi$.