Please hand in Q5 on Exercise I and Q1, Q3, Q4 on Exercise II to the General Office by 12:00 on Monday November 9th.

MA563 Exercise II – Calculus of Variations

Q1. Show that, if $u(x)$ satisfies the Euler-Lagrange equation associated with the integral

$$\mathcal{L}[u] = \int_a^b (p^2 u_x^2 + q^2 u^2)\mathrm{d}x$$

where $p(x)$ and $q(x)$ are given functions, then $\mathcal{L}$ takes the value

$$\mathcal{L}[u] = p^2 u u_x|_a^b \ .$$

Q2. (The Catenary) Consider a thin heavy uniform flexible cable suspended from the top of two poles at point (x_0, y_0) and (x_1, y_1). The problem is to determine the shape of the cable that makes the potential energy minimum. The essence of the problem is to extremise the integral

$$\mathcal{L}[y] = \int_0^1 y\sqrt{1+y'^2}\mathrm{d}x$$

with the boundary conditions $y(0) = 1$ and $y(1) = h$.

1. Write out the Euler-Lagrange equation for above $\mathcal{L}[y]$.
2. Find $H(y,\ y')$ such that $\frac{\mathrm{d}}{\mathrm{d}x}H = 0$ along the Euler-Lagrange equation. (Challenge: Solve for y(x)).
3. Check $y = A\cosh(x/A + B)$, where A and B are constant, is a solution of the Euler-Lagrange equation.
4. Using the boundary conditions, show that $h = \frac{\cosh(\cosh(B)+B)}{\cosh B}$.
5. Using Maple to plot the graph for B from -3 to 3 and h from 0 to 10.
6. Using the graph, investigate for what value of h there are no, one or two solutions for $y(x)$.

Q3. Find the Euler-Lagrange equation associated with the integral

$$\mathcal{L}[y] = \int_0^1 (yy'^2 + \sin(x)y^2)\mathrm{d}x \ .$$

By adapting the code on the help page for Maple's odeplot command, plot solutions satisfying $y(0) = 1$. By varying the value of $y'(0)$ find a solution satisfying both $y(0) = 1$ and $y(10) = 4$.

Q4. The functional for the Brachistrochrone is

$$\mathcal{L}[y] = \int_0^{x_1} \sqrt{\frac{1+y'^2}{y}}\mathrm{d}x \ .$$

Find an extremal for $\mathcal{L}$ subject to the condition that $y(0) = 0$ and $(x_1,\ y(x_1))$ lies on the curve $y = x - 1$.