MA563 Exercise I – Calculus of Variations

Q1. Use the method of Lagrange multipliers to find maximum and minimum values of $x + 2y - 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Q2. The area A of a triangle with sides a, b,c is given by

$$A = (s(s-a)(s-b)(s-c))^{1/2},$$

where $S = \frac{1}{2}(a + b + c)$. Show that of all triangles of given perimeter $2s$, the triangle of largest area is equilateral.

Q3. Find the general solution of the following differential equations, and the particular solution with any given boundary conditions.

1. $u_{xx} + 1 = 0, \quad u(0) = 3, \quad u(2) = 9$
2. $u_{xx} + 5u = x + 9, \quad u(0) = 0, \quad u(1) = 3$
3. $u_{xx} - 2u_x + 2u = \exp(x)\sin(x), \quad u(0) = 1, \quad u(1) = 3$
4. $u_{xxxx} - u = 0$
5. $u_{xxxx} + u_{xx} = x^2$

Q4. Find and solve the Euler-Lagrange equations for the following variational problems.

1. $$\mathcal{L}[u] = \int_0^1 (u^2 + u_x^2 + 2u\exp(3x))\mathrm{d}x$$
   subject to the boundary conditions $u(0) = \frac{1}{8}$ and $u(1) = 0$.
2. $$\mathcal{L}[u] = \int_0^{\pi/2} (u_x^2 + 5u^2 - \sin(x)u)\mathrm{d}x$$
   subject to the boundary conditions $u(0) = 0$ and $u(\pi) = 1$.

Q5. Let

$$\mathcal{L}[u] = \int_2^3 u^2(1 - u_x)^2\mathrm{d}x.$$

Find a smooth extremal for $\mathcal{L}$ satisfying the boundary conditions $u(2) = 1$ and $u(3) = \sqrt{3}$.